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A TREATISE

ON

ALGEBRA

Alexander Ziwet

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BY

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PREFACE TO THE FIRST EDITION.

THE following work is designed for the use of the higher classes of Schools and the junior students in the Universities. Although the book is complete in itself, in the sense that it begins at the beginning, it is expected that students who use it will have previously read some more elementary work on Algebra: the simpler parts of the subject are therefore treated somewhat briefly.

I have ventured to make one important change from the usual order adopted in English text-books on Algebra, namely by considering some of the tests of the convergency of infinite series before making any use of such series: this change will, I feel sure, be generally approved. The order in which the different chapters of the book may be read is, however, to a great extent optional.

A knowledge of the elementary properties of Determinants is of great and increasing practical utility; and I have therefore introduced a short discussion of their fundamental properties, founded on the Treatises of Dostor and Muir.

No pains have been spared to ensure variety and interest in the examples. With this end in view, hundreds of examination papers have been consulted; including, with

very few exceptions, every paper which has been set in Cambridge for many years past. Amongst the examples will also be found many interesting theorems which have been taken from the different Mathematical Journals.

I am indebted to many friends for their kindness in looking over the proof-sheets, for help in the verification of the examples, and for valuable suggestions. My especial thanks are due to the following members of Sidney Sussex College: Mr S. R. Wilson, M.A., Mr J. Edwards, M.A., Mr S. L. Loney, M.A., and Mr J. Owen, B.A.

CHARLES SMITH.

CAMBRIDGE,
December 12th, 1887.

PREFACE TO THE THIRD EDITION.

A Chapter on Theory of Equations has been added, which it is hoped will increase the value of the book.

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THEORY OF EQUATIONS.

CHAPTER I.

DEFINITIONS.

1. **ALGEBRA**, like Arithmetic, is a science which treats of numbers.

In Arithmetic numbers are represented by figures which have determinate values. In Algebra the letters of the alphabet are used to represent numbers, and each letter can stand for any number whatever, except that in any connected series of operations each letter must throughout be supposed to represent the same number.

Since the letters employed in Algebra represent any numbers whatever, the results arrived at must be equally true of all numbers.

2. The numbers treated of may be either whole numbers or fractions.

All concrete quantities such as values, lengths, areas, periods of time, &c., with which we have to do in Algebra, must be measured by the *number of times* each contains some unit of its own kind. Thus we have lengths of 4, $\frac{2}{3}$, $5\frac{1}{4}$, the unit being an inch, a yard, a mile, or any other fixed length. It is only these *numbers* with which we are concerned, and our symbols of quantity, whether figures or letters, always represent numbers. On this account the word *quantity* is often used instead of *number*.

3. The sign $+$, which is read 'plus,' is placed before a number to indicate that it is to be *added* to what has gone

before. Thus $6+3$ means that 3 is to be added to 6; $6+3+2$ means that 3 is to be added to 6 and then 2 added to the result. So also $a+b$ means that the number which is represented by b is to be added to the number which is represented by a; or, expressed more briefly, it means that b is to be added to a; again $a+b+c$ means that b is to be added to a and then c added to the result.

4. The sign $-$, which is read 'minus,' is placed before a number to indicate that it is to be *subtracted* from what has gone before. Thus $a-b$ means that b is to be subtracted from a; $a-b-c$ means that b is to be subtracted from a, and then c subtracted from the result; and $a-b+c$ means that b is to be subtracted from a, and then c added to the result.

Thus in additions and subtractions the order of the operations is from left to right.

5. The sign $\times$, which is read 'into,' is placed between two numbers to indicate that the first number is to be *multiplied* by the second. Thus $a \times b$ means that a is to be multiplied by b; also $a \times b \times c$ means that a is to be multiplied by b, and the result multiplied by c.

The sign $\times$ is however generally omitted between two letters, or between a figure and a letter, and the letters are placed consecutively. Thus ab means the same as $a \times b$, and $5ab$ the same as $5 \times a \times b$.

The sign of multiplication cannot be omitted between figures: 63 for example does not stand for 6×3 but for *sixty-three*, as in Arithmetic.

Sometimes the $\times$ is replaced by a point, which is placed *on* the line, to distinguish it from the decimal point which is placed *above* the line. Thus $a \times b \times c$, $a.b.c$ and abc all mean the same, namely that a is to be multiplied by b and the result multiplied by c.

6. The sign $\div$, which is read 'divided by' or 'by,' is placed between two numbers to indicate that the first

number, called the dividend, is to be *divided* by the second number, called the divisor. Thus $a \div b$ means that a is to be divided by b; also $a \div b \div c$ means that a is to be divided by b, and the result divided by c; and $a \div b \times c$ means that a is to be divided by b and the result multiplied by c.

Thus in multiplications and divisions the order of the operations is from left to right.

7. When two or more numbers are multiplied together the result is called the *continued product*, or simply the *product*; and each number is called a *factor* of the product.

When the factors are considered as divided into two sets, each is called the *co-efficient*, that is the *co-factor* of the other. Thus in $3abx$, 3 is the coefficient of abx, $3a$ is the coefficient of bx, and $3ab$ is the coefficient of x.

When one of the factors of a product is a number expressed in figures, it is called the *numerical coefficient* of the product of the other factors.

8. When a product consists of the same factor repeated any number of times it is called a *power* of that factor. Thus aa is called the second power of a, aaa is called the third power of a, $aaaa$ is called the fourth power of a, and so on. Sometimes a is called the first power of a.

Special names are also given to aa and to aaa; they are called respectively the *square* and the *cube* of a.

9. Instead of writing aa, aaa, &c., a more convenient notation is adopted as follows: a^2 is used instead of aa, a^3 is used instead of aaa, and a^n is used instead of $aaaa......$, the factor a being taken n times; the small figure placed above and to the right of a shewing the number of times the factor a is to be taken. So also a^3b^2 is written instead of $aaabb$, and similarly in other cases.

The small figure, or letter, placed above a symbol to

indicate the number of times that symbol is to be taken as a factor is called the *index* or the *exponent*. Thus a^n means that the factor a is to be taken n times, or that the nth *power* of a is to be taken, and n is called the *index*.

When the factor a is only to be taken once, we do not write it a^1, but simply a.

10. A number which when squared is equal to any number a is called a *square root* of a, and is represented by the symbol $\sqrt[2]{a}$, or more often by $\sqrt{a}$: thus 2 is $\sqrt{4}$, since $2^2 = 4$.

A number which when cubed is equal to any number a is called a *cube root* of a, and is represented by the symbol $\sqrt[3]{a}$: thus 3 is $\sqrt[3]{27}$, since $3^3 = 27$.

In general, a number which when raised to the nth power, where n is any whole number, is equal to a, is called an nth root of a, and is represented by the symbol $\sqrt[n]{a}$.

The sign $\sqrt{\ }$ was originally the initial letter of the word *radix*. It is often called the *radical* sign.

11. A root which cannot be obtained exactly is called a *surd*, or an *irrational* quantity: thus $\sqrt{7}$ and $\sqrt[3]{4}$ are surds.

The approximate value of a surd, for example of $\sqrt{7}$, can be found, to any degree of accuracy which may be desired, by the ordinary arithmetical process; but we are not required to find these approximate values in Algebra: for us $\sqrt{7}$ is simply that quantity which when squared will become 7.

12. A collection of algebraical symbols, that is of letters, figures, and signs, is called an *algebraical expression*.

The parts of an algebraical expression which are connected by the signs + or − are called the *terms*.

Thus $2a - 3bx + 5cy^2$ is an algebraical expression containing the three terms $2a$, $-3bx$, and $+5cy^2$.

13. When two terms only differ in their numerical coefficients they are called *like* terms. Thus a and $3a$ are like terms; also $5a^3b^2c$ and $3a^3b^2c$ are like terms.

14. An expression which contains only one term is called a *monomial* expression, and expressions which contain two or more terms are called *multinomial* expressions; expressions which contain two terms, and those which contain three terms are, however, generally called *binomial* and *trinomial* expressions respectively. Thus $3ab^3c$ is a monomial, $a^2 + 3b^2$ is a binomial, and $ax^2 + bx + c$ is a trinomial expression.

15. The sign $=$, which is read 'equals,' or 'is equal to,' is placed between two algebraical expressions to denote that they are equal to one another.

The sign $>$ indicates that the number which precedes the sign is *greater than* that which follows it. Thus $a > b$ means that a is greater than b.

The sign $<$ indicates that the number which precedes the sign is *less than* that which follows it. Thus $a < b$ means that a is less than b.

The signs $\neq$, $\ngtr$ and $\nless$ are used respectively for is *not equal to,* is *not greater than,* and is *not less than.*

The sign $\because$ is written for the word *because* or *since.*

The sign $\therefore$ is written for the word *therefore* or *hence.*

16. To denote that an algebraical expression is to be treated *as a whole,* it is put between brackets. Thus $(a + b)c$ means that b is to be added to a and that the result is to be multiplied by c; again $(a - b)(c + d)$ means that b is to be subtracted from a, and that d is to be added to c, and that then the first result is to be multiplied by the second; so also $(a + b)^3(c + d)^2$ means that the cube of the sum of a and b is to be multiplied by the square of the sum of c and d.

Brackets are of various shapes: thus, ( ), { }, [ ]. Instead of a pair of brackets a line, called a *vinculum,* is often drawn over the expression which is to be treated as

a whole: thus $a-\overline{b-c}$ is equivalent to $a-(b-c)$, and $\sqrt{\overline{a+b}}$ is equivalent to $\sqrt{(a+b)}$. It should be noticed that where no vinculum or bracket is used, a radical sign refers only to the number or letter which immediately follows it: thus $\sqrt{2}a$ means that the square root of 2 is to be multiplied by a, whereas $\sqrt{2a}$ means the square root of $2a$; also $\sqrt{a}+x$ means that x is to be added to the square root of a, whereas $\sqrt{a+x}$ means that x is to be added to a and that the square root of the whole is to be taken.

The line between the numerator and denominator of a fraction acts as a vinculum, for $\dfrac{a+b}{3}$ is the same as $\frac{1}{3}(a+b)$.

Note. It is important for the student to notice that every *term* of an algebraical expression must be added or subtracted *as a whole*, as if it were enclosed in brackets. Thus, in the expression $a+bc-d\div e+f$, b must be multiplied by c before addition, and d must be divided by e before subtraction, just as if the expression were written $a+(bc)-(d\div e)+f$.

EXAMPLES.

1. Find the numerical values of the following expressions in each of which $a=1$, $b=2$, $c=3$, and $d=4$.

(i) $5a+3c-3b-2d$, (ii) $26a-3bc+d$,

(iii) $ab+3bc-5d$, (iv) $bc-ca-ab$,

(v) $a+bc+d$ and (vi) $bcd+cda+dab+abc$.

Ans. 0, 12, 0, 1, 11, 50.

2. If $a=3$, $b=1$ and $c=2$, find the numerical values of

(i) $2a^3-3b^2-4c^3$, (ii) $2a^2b-3b^3c^2$,

(iii) $\dfrac{1}{16}c^3-\dfrac{1}{2}b^3$, (iv) $a^3+3ac^2-3a^2c-c^3$,

and (v) $2a^4b^2c-3b^4c^2a-2c^4a^2b$.

Ans. 19, 6, 0, 1, 0.

3. Find the values of the following expressions in each of which $a=3$, $b=2$, $c=1$ and $d=0$.

(i) $(3a+4d)(2b-3c)$,

(ii) $2a^2-(b^2-3c^2)d$,

(iii) $a^3-b^3-2(a-b+c)^3$,

(iv) $a(b^2-c^2)+b(c^2-d^2)+d(a^2-c^2)$,

(v) $3(a+b)^2(c+d)-2(b+c)^2(a+d)$,

and (vi) $\dfrac{2a^2}{b+c}-\dfrac{2b^2}{c+a}-\dfrac{2c^2}{b+d}+\dfrac{2d^2}{a+b}$.

Ans. 9, 18, 3, 11, 21, 3.

4. Find the values of

$$\sqrt{a^2-b^2},\ \sqrt{5ab}+c,\ \sqrt{(b^4c^2+b^2c^4)} \text{ and } \sqrt[3]{a^2+4b^2+4c^2},$$

when $a=5,\ b=4,\ c=3$.

Ans. 3, 13, 60, 5.

5. Shew that a^2-b^2 and $(a+b)(a-b)$ are equal to one another (i) when $a=2$, $b=1$; (ii) when $a=5$, $b=3$; and (iii) when $a=12$, $b=5$.

6. Shew that the expressions

$$a^3-b^3,\ (a-b)(a^2+ab+b^2),\ (a-b)^3+3ab(a-b),$$

and

$$(a+b)^3-3ab(a+b)-2b^3$$

are all equal to one another (i) when $a=3$, $b=2$; (ii) when $a=5$, $b=1$; and (ii) when $a=6$, $b=3$.

CHAPTER II.

FUNDAMENTAL LAWS.

17. WE have said that all concrete quantities m be measured by the number of times each contains so unit of its own kind. Now a sum of money may be eit a *receipt* or a *payment*, it may be either a *gain* or a *lo* motion along a given straight line may be in either of opposite directions; time may be either *before* or *after* so particular epoch; and so in very many other cases. Tl many concrete magnitudes are capable of existing in t diametrically opposite states: the question then ari whether these magnitudes can be conveniently dist guished from one another by special signs.

18. Now whatever kind of quantity we are consid ing $+4$ will stand for what *increases* that quantity 4 units, and -4 will stand for whatever *decreases* t quantity by 4 units.

If we are calculating the amount of a man's prope (estimated in pounds), $+4$ will stand for whatever increa his property by £4, that is $+4$ stands for £4 that possesses, or that is owing to him; so also -4 will sta for whatever decreases his property by £4, that is, -4 w stand for £4 that he owes.

If, on the other hand, we are calculating the amou of a man's debts, $+4$ will stand for whatever increases l

debts, that is, $+4$ will now stand for a debt of £4; so also -4 will now stand for whatever decreases his debts, that is, -4 will stand for £4 that he has, or that is owing to him.

If we are considering the amount of a man's gains, $+4$ will stand for what increases his total gain, that is, $+4$ will stand for a *gain* of 4; so also -4 will stand for what decreases his total gain, that is, -4 will stand for a loss of 4. If however we are calculating the amount of a man's losses, $+4$ will stand for a *loss* of 4, and -4 will stand for a *gain* of 4.

Again, if the magnitude to be increased or diminished is the distance from any particular place, measured in any particular direction, $+4$ will stand for a distance of 4 units in that direction, and -4 will stand for a distance of 4 units in the *opposite* direction.

19. From the last article it will be seen that it is not necessary to invent any new signs to distinguish between quantities of directly opposite kinds, for this can be done by means of the old signs $+$ and $-$.

The signs $+$ and $-$ are therefore used in Algebra with two entirely different meanings. In addition to their original meaning as signs of the *operations* of addition and subtraction respectively, they are also used as *marks of distinction* between magnitudes of diametrically opposite kinds.

The signs $+$ and $-$ are sometimes called *signs of affection* when they are thus used to indicate a *quality* of the quantities before whose symbols they are placed.

The sign $+$, as a sign of affection, is frequently omitted; and when neither the $+$ nor the $-$ sign is prefixed to a term the $+$ sign is to be understood.

20. A quantity to which the sign $+$ is prefixed is called a *positive* quantity, and a quantity to which the sign $-$ is prefixed is called a *negative* quantity.

The signs $+$ and $-$ are called respectively the *positive* and *negative* signs.

Note. Although there are many signs used in alge the name *sign* is often used to denote the two signs + – exclusively. Thus, when *the sign* of a quantity spoken of, it means the + or – sign which is prefixed it; and when we are directed to *change the signs* of expression, it means that we are to change the + o before every term into – or + respectively.

21. The magnitude of a quantity considered ind pendently of its quality, or of its sign, is called its *absol* magnitude. Thus a rise of 4 feet and a fall of 4 feet a equal in absolute magnitude; so also + 4 and – 4 a equal in absolute magnitude, whatever the unit may be.

Addition.

22. The process of finding the result when two or mo quantities are taken together is called *addition*, and th result is called the *sum*.

Since a positive quantity produces an increase, and negative quantity produces a decrease, to add a positiv quantity we must add its absolute value, and to add negative quantity we must subtract its absolute valu Thus, when we add + 4 to + 6, we get + 6 + 4; and whe we add – 4 to + 10, we get + 10 – 4.

Hence $\qquad +\ 6 + (+4) = +\ 6 + 4,$

and $\qquad +10 + (-4) = +10 - 4.$

So also, when we add $+b$ to $+a$, we get $+a+b$; an when we add $-b$ to $+a$, we get $+a-b$. Hence

$$+a + (+b) = +a + b,$$

and $$+a + (-b) = +a - b.$$

We therefore have the following rule for the additio of any term: *to add any term affix it to the expression t which it is to be added, with its sign unchanged.*

When numerical values are given to a and to b, th numerical values of $a+b$ and $a-b$ can be found; bu

until it is known what numbers a and b stand for, no further step can be taken, and the process is considered to be algebraically complete.

23. When b is greater than a, the arithmetical operation denoted by $a - b$ is impossible. For example, if $a = 3$ and $b = 5$, $a - b$ will be $3 - 5$, and we cannot take 5 from 3. But to subtract 5 is the same as to subtract 3 and 2 in succession, so that

$$3 - 5 = 3 - 3 - 2 = 0 - 2 = -2.$$

We then consider that -2 is 2 which is to be subtracted from some other algebraical expression, or that -2 is two units of the kind opposite to that represented by 2; and if -2 is a final result, the latter is the only view that can be taken.

In some particular cases the quantities under consideration may be such that a negative result is without meaning; for instance, if we have to find the population of a town from certain given conditions; in this case the occurrence of a negative result would shew that the given conditions could not be satisfied, and so also in this case would the occurrence of a fractional result.

Subtraction.

24. Since subtraction is the inverse operation to that of addition, to subtract a positive quantity produces a *decrease*, and to subtract a negative quantity produces an *increase*. Hence to subtract a positive quantity we must subtract its absolute value, and to subtract a negative quantity we must add its absolute value. Thus, to subtract $+4$ from $+10$, we must decrease the amount by 4; we then get $+10 - 4$.

Also to subtract -4 from $+6$, we must increase the amount by 4; we then get $+6 + 4$.

Hence $\quad +10 - (+4) = +10 - 4 = +6,$

and $\quad +6 - (-4) = +6 + 4 = +10.$

So also, in all cases

$$a-(+b)=a-b,$$

and

$$a-(-b)=a+b.$$

We therefore have the following rule for the subtraction of any term:—*to subtract any term affix it to the expression from which it is to be subtracted but with its sign changed.*

25. We have hitherto supposed that the letters used to represent quantities were restricted to positive values; it would however be very inconvenient to retain this restriction. In what follows therefore it must always be understood, unless the contrary is expressly stated, that each letter may have any positive or negative value.

Since any letter may stand for either a positive or for a negative quantity, a term preceded by the sign + is not necessarily a positive quantity in reality; such terms are however still called *positive terms*, because they are so in appearance; and the terms preceded by the sign − are similarly called *negative terms*.

26. On the supposition that b was a *positive* quantity, it was proved in Articles 22 and 24, that

$$\left.\begin{array}{ll} a+(+b)=a+b \dots\dots\dots (\text{i}) \\ a+(-b)=a-b \dots\dots\dots (\text{ii}) \\ a-(+b)=a-b \dots\dots\dots (\text{iii}) \\ \text{and} \quad a-(-b)=a+b \dots\dots\dots (\text{iv}) \end{array}\right\} \dots\dots\dots (\text{A}).$$

We have now to prove that the above laws being true for all positive values of b must be true also for negative values.

Let b be negative and equal to $-c$, where c is any positive quantity; then

$$+b=+(-c)=-c \qquad \text{from (ii)},$$

and

$$-b=-(-c)=+c \qquad \text{from (iv)}.$$

Hence, putting $-c$ for $+b$, and $+c$ for $-b$ in (i), (ii),

(iii), (iv), it follows that these relations are true for all negative values of b, provided

$$a+(-c)=a-c,$$
$$a+(+c)=a+c,$$
$$a-(-c)=a+c,$$
and
$$a-(+c)=a-c,$$

are true for all positive values of c; and this we know to be the case.

Hence the laws expressed in (A) are true for *all values* of b.

27. **Def.** The *difference* between any two quantities a and b is the result obtained by subtracting the *second* from the *first*.

The algebraical difference may therefore not be the same as the arithmetical difference, which is the result obtained by subtracting the *less* from the *greater*. The symbol $a \sim b$ is sometimes used to denote the *arithmetical difference* of a and b.

Def. One quantity a is said to be *greater* than another quantity b when the *algebraical difference* $a-b$ is positive.

From the definition it is easy to see that in the series 1, 2, 3, 4, &c., each number is *greater* than the one before it; and that, in the series -1, -2, -3, -4, &c., each number is *less* than the one before it.

Thus 7, 5, 0, -5, -7 are in descending order of magnitude.

EXAMPLES.

Ex. 1. Find the sum of (i) 5 and -4, (ii) -5 and 4, (iii) 5, -3 and -6 and (iv) -3, 4, -6 and 5. *Ans.* 1, -1, -4, 0.

Ex. 2. Subtract (i) 3 from -4, (ii) -4 from 3, and (iii) $-a$ from $-b$. *Ans.* -7, 7, $-b+a$.

Ex. 3. A barometer fell ·01 inches one day, it rose ·015 inches on the next day, and fell again ·01 inches on the third day. How much higher was it at the end than at the beginning? *Ans.* $-$·005 inches.

Ex. 4. A thermometer which stood at 10 degrees centigrade, fell 20 degrees when it was put into a freezing mixture: what was the final reading? *Ans.* -10.

Ex. 5. Find the value of $a-b+c$ and of $-a+b-c$, when a
$b=-2$ and $c=3$. *Ans.* 6,

Ex. 6. Find the value of $-a+b-c$ when
$a=1,\ b=-2,\ c=-1$; also when
$a=-2,\ b=-1,\ c=-3$. *Ans.* $-$

Ex. 7. Find the value of $a-(-b)+(-c)$ when
$a=-3,\ b=-2,\ c=-1$. *Ans.*

Ex. 8. Find the value of $-a+(-b)-(-c)$ when
$a=-2,\ b=-3,\ c=-5$. *An*

Ex. 9. Find the value of $-(-a)+b-(-c)$ when
$a=-1,\ b=-2,\ c=-3$. *Ans.*

Multiplication.

28. In Arithmetic, multiplication is first defined to the taking one number as many times as there are units another. Thus, to multiply 5 by 4 is to take as ma fives as there are units in four. As soon, however, fractional numbers are considered, it is found necessary modify somewhat the meaning of multiplication, for by t original definition we can only multiply by *whole* numbe The following is therefore taken as the definition multiplication: "*To multiply one number by a second is do to the first what is done to unity to obtain the second.*"

Thus $$4 \text{ is } 1+1+1+1;$$
$$\therefore\ 5\times 4 \text{ is } 5+5+5+5.$$

Again, to multiply $\frac{5}{7}$ by $\frac{3}{4}$, we must do to $\frac{5}{7}$ what done to unity to obtain $\frac{3}{4}$; that is, we must divide $\frac{5}{7}$ in four equal parts and take three of those parts. Each the parts into which $\frac{5}{7}$ is to be divided will be $\frac{5}{7\times 4}$, an by taking three of these parts we get $\frac{5\times 3}{7\times 4}$. Thus $\frac{5}{7}\times$ $=\frac{5\times 3}{7\times 4}$.

So also,
$$\begin{aligned}(-5)\times 4 &= (-5)+(-5)+(-5)+(-5)\\ &= -5-5-5-5\\ &= -20.\end{aligned}$$

With the above definition, multiplication by a negative quantity presents no difficulty.

For example, to multiply 4 by -5. Since to subtract 5 by one subtraction is the same as to subtract 5 units successively,

$$\begin{aligned}-5 &= -1-1-1-1-1;\\ \therefore 4\times(-5) &= -4-4-4-4-4\\ &= -20.\end{aligned}$$

Again, to multiply -5 by -4. Since

$$\begin{aligned}-4 &= -1-1-1-1;\\ \therefore (-5)\times(-4) &= -(-5)-(-5)-(-5)-(-5)\\ &= +5+5+5+5 \qquad \text{[Art. 26]}\\ &= +20.\end{aligned}$$

We can proceed in a similar manner for any other numbers, whether integral or fractional, positive or negative.

Hence we have the following rule:

To find the product of any two quantities, multiply their absolute values, and prefix the sign + if both factors be positive or both negative, and the sign − if one factor be positive and the other negative.

Thus we have

$$\left.\begin{aligned}(+a)\times(+b) &= +ab \text{.........(i)}\\ (-a)\times(+b) &= -ab \text{.........(ii)}\\ (+a)\times(-b) &= -ab \text{.........(iii)}\\ (-a)\times(-b) &= +ab \text{.........(iv)}\end{aligned}\right\}\text{.........(B).}$$

The rule by which the sign of the product is determined is called the **Law of Signs.** This law is sometimes enunciated briefly as follows: *Like signs give +, and unlike signs give −.*

29. The factors of a product may be taken in any order. It is proved in Arithmetic that when one number, whether integral or fractional, is multiplied by a second, the result is the same as when the second is multiplied by the first.

The proof is as follows: when the numbers are integers, a and b suppose, write down a series of rows of dots, putting a dots in each row; and take b rows, writing the dots under one another as in the following scheme:

```
*  *  *  *  *  * ......... a in a row
*  *  *  *  *  * .........
*  *  *  *  *  * .........
*  *  *  *  *  *
......................
......................
        b rows.
```

Then the whole number of the dots is a repeated b times, that is $a \times b$. Now consider the columns instead of the rows: there are clearly b dots in each column, and there are a columns; thus the whole number of dots is b repeated a times, that is $b \times a$. Hence, *when a and b are integers, $ab = ba$.*

When the numbers are fractions, for example $\frac{5}{7}$ and $\frac{3}{4}$, we prove as in Art. 28 that $\frac{5}{7} \times \frac{3}{4} = \frac{5 \times 3}{7 \times 4}$. And, by the above proof for integers, $\frac{5 \times 3}{7 \times 4} = \frac{3 \times 5}{4 \times 7}$; hence $\frac{5}{7} \times \frac{3}{4} = \frac{3}{4} \times \frac{5}{7}$.

Hence we have $ab = ba$, for all *positive* values of a and b; and the proposition being true for any positive values of a and b, it must be true for all values, whether positive or negative; for from the preceding Article the absolute value of the product is independent of the signs, and the sign of the product is independent of the order of the factors.

Hence for *all values* of a and b we have

$$ab = ba \text{..........................(i).}$$

If in the above scheme we put c in place of each of the

dots; the whole number of the c's will be ab; also the number of c's in the first row will be a, and this is repeated b times. Hence, when a and b are integers, c repeated ab times gives the same result as c repeated a times and this repeated b times. So that to multiply by any two *whole numbers* in succession gives the same result as to multiply at once by their product; and the proposition can, as before, be then proved to be true without restriction to whole numbers or to positive values. Thus, for all values of a, b and c, we have

$$a \times b \times c = a \times (bc) \quad\text{.................}\quad \text{(ii)}.$$

By continued application of (i) and (ii) it is easy to shew that the factors of a product may be taken in any order, however many factors there may be. Thus

$$abc = cab = cba, \text{ \&c.} \quad\text{...............}\quad \text{(C)}.$$

30. Since the factors of a product may be taken in any order, we are able to simplify many products. For example:

$$3a \times 4a = 3 \times 4 \times a \times a = 12a^2,$$
$$(-3a) \times (-4b) = +3a \times 4b = 3 \times 4 \times a \times b = 12ab,$$
$$(ab)^2 = ab \times ab = a \times a \times b \times b = a^2b^2,$$
$$(\sqrt{2a})^2 = \sqrt{2a} \times \sqrt{2a} = \sqrt{2} \times \sqrt{2} \times aa = 2a^2.$$

Although the order of the factors in a product is indifferent, a factor expressed in figures is always put first, and the letters are usually arranged in alphabetical order.

31. Since $a^2 = aa$, and $a^3 = aaa$; we have

$$a^2 \times a^3 = aa \times aaa = a^5 = a^{2+3}.$$

So also

$$a^3 \times a^4 = aaa \times aaaa = a^7 = a^{3+4},$$

and $$a^4 \times a = aaaa \times a = a^5 = a^{4+1}.$$

In the above examples we see that *the index of the product of two powers of the same letter is equal to the sum of the indices of the factors.* We can prove in the following

manner that this is true whenever the indices are positive integers:
since by definition

$$a^m = aaaa \ldots \text{ to } m \text{ factors},$$

and $$a^n = aaaa \ldots \text{ to } n \text{ factors};$$

$$\therefore\ a^m \times a^n = (aaa \ldots \text{ to } m \text{ factors}) \times (aaa \ldots \text{ to } n \text{ factors})$$
$$= aaa \ldots \text{ to } (m+n) \text{ factors},$$
$$= a^{m+n}, \qquad \text{by definition;}$$

hence $a^m \times a^n = a^{m+n}$ ....(D).

The law expressed in (D) is called the **Index Law.**

32. Since $(-a) \times (-a) = +a^2 = (+a)(+a)$ [Art. 28], it follows conversely that the square root of a^2 is either $+a$ or $-a$: this is written $\sqrt{a^2} = \pm a$, the double sign being read 'plus or minus.'

Thus there are *two square roots of any algebraical quantity*, which are equal in absolute magnitude but opposite in sign.

EXAMPLES.

1. Multiply $2a$ by $-4b$, a^2 by $-a^3$ and $-2a^3b$ by $-3ab^3$.
Ans. $-8ab$, $-a^5$, $6a^4b^4$.

2. Multiply
$-2xy^2$ by $-3y^2z$, $3ax^2y$ by $-5a^2xy^2$, and $3a^2bc^2x$ by $12ab^2cx^3$.
Ans. $6xy^4z$, $-15a^3x^3y^3$, $36a^3b^3c^3x^4$.

3. Multiply
$7a^4b^3c^2$ by $-3a^3b^5c^7$, and $-2ab^3x^5y^2$ by $-4a^3b^2x^4y^6$.
Ans. $-21a^7b^8c^9$, $8a^4b^5x^9y^8$.

4. Find the values of $(-a)^2$, $(-a)^3$, $(-a)^4$ and $(-a)^5$.
Ans. a^2, $-a^3$, a^4, $-a^5$.

5. Find the values of $(-ab)^2$, $(a^2b)^4$ and $(-3ab^2c^3)^3$.
Ans. a^2b^2, a^8b^4, $-27a^3b^6c^9$.

6. Shew that the successive powers of a negative quantity are alternately positive and negative.

7. Find the cubes of $2a^2b$, $-3ab^2c^3$, and $-2a^2bx^3y^2$.
Ans. $8a^6b^3$, $-27a^3b^6c^9$ and $-8a^6b^3x^9y^6$.

8. Find the value of $(-a)^2 \times (-b)^3$, of $(-2ab^2)^3 \times (-3a^2b)^2$, and of $(-3abc)^2 \times (2a^3b)^3$.

Ans. $-a^2b^3$, $216a^7b^8$, $72a^{11}b^5c^2$.

9. Find the value of $3abc - 2a^2bc^3 + 4c^4$, when $a=2$, $b=-1$, and $c=-2$.

Ans. 12.

10. Find the value of $2a^2bc - 3b^2cd + 4c^2da - 5d^2ab$, when $a=-1$, $b=-2$, $c=-3$ and $d=-4$.

Ans. -148.

Division.

33. Division is the inverse operation to that of multiplication; so that to divide a by b is to find a quantity c such that $c \times b = a$.

Since division is the inverse of multiplication and multiplications can be performed in any order [Art. 29], it follows that successive divisions can be performed in any order. Thus $a \div b \div c = a \div c \div b$.

It also follows from Art. 29 that to divide by two quantities in succession gives the same result as to divide at once by their product. Thus $a \div b \div c = a \div (bc)$, which is usually written $a \div bc$.

Not only may a succession of divisions be performed in any order, but divisions and multiplications together may be performed in any order. For example

$$a \times b \div c = a \div c \times b.$$

For $$a = a \div c \times c;$$

$$\therefore\ a \times b = a \div c \times c \times b$$
$$= a \div c \times b \times c; \qquad \text{[by Art. 29]}$$

therefore, dividing each by c, we have

$$a \times b \div c = a \div c \times b.$$

Hence we get the same result whether we divide the product of a and b by c, or divide a by c and then multiply by b, or divide b by c and then multiply by a.

34. The operation of division is often indicated by placing the dividend over the divisor with a line between

them: thus $\frac{a}{b}$ means $a \div b$. Sometimes a/b is written for $\frac{a}{b}$. When $a \div b$ is written in the fractional form $\frac{a}{b}$, a is called the numerator, and b the denominator.

Since $$\frac{1}{c} = 1 \div c,$$

$$\frac{1}{c} \times c = 1 \div c \times c = 1.$$

Also $$a \times \frac{1}{c} \times c = a \times \left(\frac{1}{c} \times c\right) = a \times 1 = a.$$

Therefore, dividing by c,

$$a \times \frac{1}{c} = a \div c,$$

so that to divide by any quantity c is the same as to multiply by the quantity $\frac{1}{c}$.

Hence $$a \times b \div c = a \div c \times b,$$
can be written,

$$a \times b \times \frac{1}{c} = a \times \frac{1}{c} \times b,$$

in which form it is seen to be included in Art. 29 (C).

35. Since $a^3 \times a^2 = a^5$, and $a^7 \times a^3 = a^{10}$; we have conversely $a^5 \div a^3 = a^2$, and $a^{10} \div a^3 = a^7$.

And, in general, when m and n are any positive integers and $m > n$, we have

$$a^m \div a^n = a^{m-n},$$

for by Art. 31

$$a^{m-n} \times a^n = a^m.$$

Hence if one power of any quantity be divided by a lower power of the same quantity, the index of the quotient is equal to the *difference* of the indices of the dividend and the divisor.

Hence $a^5b^2 \div a^2b = a^5 \times b^2 \div a^2 \div b = a^5 \div a^2 \times b^2 \div b = a^3b$,

and $$a^7b^6c^4 \div a^2b^3c^4 = a^5b^3.$$

36. We have proved in Art. 28 that

$$a \times (-b) = -ab;$$

$$\therefore\ (-ab) \div (-b) = a, \text{ and } (-ab) \div a = -b;$$

we have also proved that

$$(-a)(-b) = +ab = (+a)(+b);$$

$$\therefore\ (+ab) \div (-a) = -b, \text{ and } (+ab) \div (+a) = +b.$$

Hence if the signs of the dividend and divisor are alike, the sign of the quotient is $+$; and if the signs of the dividend and divisor are unlike, the sign of the quotient is $-$; we therefore have the same **Law of Signs** in division as in multiplication.

Thus $$-a^3b^6 \div ab^2 = -a^2b^4,$$

and $$-2a^5bc^7 \div -3a^4bc^2 = \frac{2}{3}ac^5.$$

EXAMPLES.

1. Divide $10a$ by $-2a$, $3a^2b^3$ by $-2ab^3$, and $-7a^5b^3c^4$ by $-3a^2b^2c^2$.

Ans. -5, $-\frac{3}{2}a$, $\frac{7}{3}a^3bc^2$.

2. Divide $-2a^5b^7c^8$ by $4a^3bc^7$, $-6x^5y^4$ by $3x^3y$, and $-5a^2b^4x^7y^8$ by $-2ab^4x^2y^5$.

Ans. $-\frac{1}{2}a^2b^6c$, $-2x^2y^3$, $\frac{5}{2}ax^5y^3$.

3. Multiply $-2a^3bc^5$ by $-3ab^7c^2$ and divide the result by $8a^3b^6c^6$.

Ans. $\frac{3}{4}ab^2c$.

37. The fundamental laws of Algebra, so far as monomial expressions are concerned, are those which were

marked A, B, C, D in the preceding articles, and which a collected below:

$$\left.\begin{aligned} +(+a) &= +a \\ +(-a) &= -a \\ -(+a) &= -a \\ -(-a) &= +a \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(\text{A}),$$

$$\left.\begin{aligned} (+a)(+b) &= +ab \\ (+a)(-b) &= -ab \\ (-a)(+b) &= -ab \\ (-a)(-b) &= +ab \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(\text{B}),$$

$$abc = cba = cab = \&\text{c.} \ldots\ldots\ldots\ldots\ldots(\text{C}),$$

$$a^m a^n = a^{m+n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{D}).$$

It should be remarked that the laws expressed in (A (B), (C) have been proved to be true for *all values* of a an b; but both m and n are supposed in (D) to be positiv integers.

Multinomial Expressions.

38. We now proceed to the consideration of mult nomial expressions.

We first observe that any multinomial expression ca be put in the form

$$a + b + c + \&\text{c.},$$

where a, b, c, &c. may be *any* quantities, positive or nega tive.

For example, the expression $3x^2y - \frac{5}{3}xy^2 - 7xyz$, whic by (A) is the same as $3x^2y + (-\frac{5}{3}xy^2) + (-7xyz)$, tak the required form if we put a for $3x^2y$, b for $-\frac{5}{3}xy^2$, and for $(-7xyz)$.

It therefore follows that in order to prove any theore to be true for *any* algebraical expression, it is only necessa

to prove it for the expression $a+b+c+$ &c., where $a,b,c,$ &c. are supposed to have *any* values, positive or negative.

39. It follows at once from the meaning of addition that the sum of two or more algebraical quantities is the same in whatever order they are added. For example, to find how much a man is worth, we can take the different items of property, considering debts as negative, in any order.

Thus $a+b+c=c+a+b=b+c+a=$ &c.......(E).

The laws [C] and [E] are together called the **Commutative Law**, which may be enunciated in the following form: *Additions or Multiplications may be made in any order.*

40. Since additions may be made in any order, we have

$$\begin{aligned} a+(b+c+d+\ldots) &= (b+c+d+\ldots)+a \text{ (from E)} \\ &= b+c+d+\ldots+a \\ &= a+b+c+d+\ldots \text{ (from E).} \end{aligned}$$

Hence, to add any algebraical expression as a whole is the same as to add its terms in succession.

Since the expression $+a-b+c-d$ may be written in the form $+a+(-b)+c+(-d)$, we have

$$\begin{aligned} +\{+a-b+c-d\} &= +\{+a+(-b)+c+(-d)\} \\ &= +a+(-b)+c+(-d). \end{aligned}$$

When we say that we can add the *terms* of an expression in succession, it must be borne in mind that the *terms include the prefixed signs.*

41. Since subtraction and addition are inverse operations, it follows from the preceding that to subtract an expression as a whole is the same as to subtract the terms in succession. Thus

$$a-(b+c+d+\ldots)=a-b-c-d-\ldots$$

42. If c be any positive integer, a and b having any values whatever, then

$$\begin{aligned}(a+b)c &= (a+b)+(a+b)+(a+b)+ \dots \text{ repeated } c \text{ times}\\ &= a+b+a+b+a+b+\dots \text{ [Art. 40]}\\ &= a+a+a+\dots \text{ repeated } c \text{ times}\\ &\quad + b+b+b+\dots \text{ repeated } c \text{ times}\\ &= ac+bc.\end{aligned}$$

Hence, when c is a positive integer, we have

$$(a+b)c = ac+bc \dots\dots\dots\dots\dots\dots (\text{F}).$$

Since division is the inverse of multiplication, it follows that when d is any positive integer

$$(a+b) \div d = a \div d + b \div d.$$

And hence

$$\begin{aligned}(a+b) \times c \div d &= \{(a+b) \times c\} \div d\\ &= (ac+bc) \div d = ac \div d + bc \div d,\end{aligned}$$

that is $$(a+b) \times \frac{c}{d} = a \times \frac{c}{d} + b \times \frac{c}{d}.$$

Thus the law expressed in (F) is true for all positive values of c; and being true for any positive value of c, it must also be true for any negative value. For, if

$$(a+b)c = ac+bc,$$

then $$\begin{aligned}(a+b)(-c) &= -(a+b)c = -ac-bc\\ &= a(-c)+b(-c).\end{aligned}$$

Hence for *all values* of a, b and c we have

$$(a+b)c = ac+bc \dots\dots\dots\dots\dots\dots (\text{F}).$$

Thus the product of the sum of any two algebraical quantities by a third is the sum of the products obtained by multiplying the quantities separately by the third.

The above is generally called the **Distributive Law.**

43. Since $(a+b) \div c = (a+b) \times \frac{1}{c}$

$$= a \times \frac{1}{c} + b \times \frac{1}{c} = a \div c + b \div c,$$

we see that the quotient obtained by dividing the sum of any two algebraical quantities by a third is the sum of the quotients obtained by dividing the quantities separately by the third.

44. From Art. 40 it follows that

$$a + b + c + d + e + \ldots = (a+b) + c + (d+e) + \ldots$$
$$= a + (b + c + d) + e + \ldots = \&c.,$$

so that the terms of an expression may be grouped in any manner.

Again, from Art. 29, it follows that

$$abcde \ldots = a\,(bc)\,(de) \ldots = a\,(bcd)\,e \ldots = \&c.,$$

so that the factors of a product may be grouped in any manner.

These two results are called the **Associative Law.**

45. We have now considered all the fundamental laws of Algebra, and in the succeeding chapters we have only to develope the consequences of these laws.

CHAPTER III.

ADDITION. SUBTRACTION. BRACKETS.

Addition.

46. WE have already seen that any term is added b writing it down, with its sign unchanged, after the expres sion to which it is to be added; and we have also see that to add any expression as a whole gives the sam result as to add all its terms in succession. We therefor have the following rule:—*to add two or more algebraica expressions, write down all the terms in succession wit their signs unchanged.*

Thus the sum of $a-2b+3c$ and $-4d-5e+6f$ is

$$a-2b+3c-4d-5e+6f.$$

47. If some of the terms which are to be added ar 'like' terms, the result can, and must, be simplified befor the process is considered to be complete.

Now two 'like' terms which have the same sign ar added by taking the arithmetical sum of their numerica coefficients with the common sign, and affixing the com mon letters.

For example, to add $2a$ and $5a$ in succession gives the same result whatever a may be, as to add $7a$; that is, $+2a+5a=+7a$. Also, to subtract $2a$ and $5a$ in succession gives the same result as to subtrac $7a$; that is, $-2a-5a=-7a$.

Also two 'like' terms whose signs are different are added by taking the arithmetical difference of their numerical coefficients with the sign of the greater, and affixing the common letters.

For example, $+5a-3a=+2a+3a-3a=+2a$,

also $+3a-5a=+3a-3a-2a=-2a$.

Thus, when there are several 'like' terms some of which are positive and some negative, they can all be reduced to one term.

Ex. 1. Add $2a+5b$ to $a-6b$.

The sum is

$$\begin{aligned}&a-6b+2a+5b\\=&a+2a-6b+5b\\=&3a-b.\end{aligned}$$

Ex. 2. Add $3a^2-5ab+7b^2$, $-4a^2-2ab+3b^2$,

and $2a^2+5ab-8b^2$.

The sum is $3a^2-5ab+7b^2-4a^2-2ab+3b^2+2a^2+5ab-8b^2$.

The terms $3a^2$, $-4a^2$, and $+2a^2$ can be combined *mentally;* and we have a^2. Similarly we have $-2ab$ and $+2b^2$.

Thus the required sum is $a^2-2ab+2b^2$.

The beginner will find it desirable to put like terms under one another.

Subtraction.

48. We have already seen that any term may be subtracted by writing it down, with its sign changed, after the expression from which it is to be subtracted; and we have also seen that to subtract any expression as a whole gives the same result as to subtract its terms in succession. We therefore have the following rule: *To subtract any algebraical expression, write down its terms in succession with all the signs changed.*

Thus, if $a-2b+3c$ be subtracted from $2a-3b-4c$, the result will be $2a-3b-4c-a+2b-3c=a-b-7c$.

49. The expression which is to be subtracted is sometimes placed under that from which it is to be taken, 'like' terms being for convenience placed under one another;

and the signs of the lower line are changed *mentally* befo[r]e combining the 'like' terms.

Thus the previous example would be written down as under:

$$\begin{array}{r} 2a-3b-4c \\ a-2b+3c \\ \hline a-b-7c \end{array}$$

As another example, if we have to subtract $3ab-5ac+c^2$ fro[m] $a^2-5ab+2ac-2b^2$, the process is written

$$\begin{array}{rrrrr} a^2-5ab & +2ac & -2b^2 & \\ 3ab & -5ac & & +c^2 \\ \hline a^2-8ab & +7ac & -2b^2 & -c^2 \end{array}.$$

Brackets.

50. To indicate that an expression is to be added as [a] whole, it is put in a bracket with the + sign prefixed[.] But, as we have seen in Art. 46, to add any algebraica[l] expression we have only to write down the terms in succession with their signs unchanged.

Hence, when a bracket is preceded by a + sign, the bracket may be omitted.

Thus $\quad +(2a-5b+7c)=+2a-5b+7c.$

Hence also, any number of terms of an expression may be enclosed in brackets with the sign + placed before each bracket. Thus

$$\begin{aligned} 3a-2b+4c-d+e-f &= 3a-2b+(4c-d+e-f) \\ &= 3a+(-2b+4c)-d+(e-f). \end{aligned}$$

When the sign of the *first* term in a bracket is + it is generally omitted for shortness, as in the preceding example.

51. To indicate that an expression is to be subtracted as a whole, it is put in a bracket with the − sign prefixed. But, as we have seen in Art. 48, to subtract any algebraical expression we have only to write down the terms in succession with all their signs changed.

Hence, when a bracket is preceded by a $-$ sign, the bracket may be omitted, provided that the signs of all the terms within the bracket are changed. Thus

$$a-(2b-c+d)=a-2b+c-d.$$

Hence also, any number of terms of an expression may be enclosed in a bracket with the sign $-$ prefixed, provided that the signs of all the terms which are placed in the bracket are changed. Thus

$$a-2b+3c-d=a-(2b-3c+d)=a-2b-(-3c+d).$$

52. Sometimes brackets are put within brackets: in this case the different brackets must be of different shapes to prevent confusion.

Thus $a-[2b-\{3c-(2d-e)\}]$; which means that we are to subtract from $2b$ the whole quantity within the bracket marked $\{\,\}$, and then subtract the result from a; and, to find the quantity within the bracket marked $\{\,\}$, we must subtract e from $2d$, and then subtract the result from $3c$.

When there are several pairs of brackets they may be removed *one at a time* by the rules of Arts. 50 and 51.

Thus
$$\begin{aligned} &a-[b+\{c-(d-e)\}] \\ =\;&a-[b+\{c-d+e\}] \\ =\;&a-[b+c-d+e] \\ =\;&a-b-c+d-e. \end{aligned}$$

EXAMPLES I.

1. Add $3x-5y$, $5x-2y$ and $7y-4x$.
2. Add $3x-5y+2z$, $5x-7y-5z$ and $6y-z-10x$.
3. Add $\frac{1}{2}a-\frac{1}{3}b+\frac{1}{4}c$, $\frac{1}{2}b-\frac{1}{3}c+\frac{1}{4}a$ and $\frac{1}{2}c-\frac{1}{3}a+\frac{1}{4}b$.
4. Add a^3-a^2+a, a^2-a+1 and a^4-a^3-1.
5. Add $x^2-5xy-7y^2$ and $3y^2+4xy-x^2$.

6. Add $m^2 - 3mn + 2n^2$, $3n^2 - m^2$ and $5mn - 3n^2 + 2m^2$.

7. Add $3a^2 - 2ac - 2ab$, $2b^2 + 3bc + 3ab$ and $c^2 - 2ac - 2bc$

8. Add $\frac{3}{2}a^2b - 5ab^2 + 7b^3$, $2a^3 - \frac{1}{2}a^2b + 5ab^2$ and $3b^3 - 2a^3$.

9. Subtract $3a - 4b + 2c$ from $a + b - 2c$.

10. Subtract $\frac{a}{2} + \frac{3}{2}b - \frac{5}{3}c$ from $c - \frac{1}{2}a - \frac{2}{3}b$.

11. Subtract $3x^2 - 4x + 2$ from $4x^2 - 5x - 7$.

12. Subtract $5a^4 - 3a^3b + 4a^2b^2$ from $5b^4 - 3ab^3 + 4a^2b^2$.

13. What is the difference between $-3x^2 - 5xy + 4y^2$ and $-5x^2 + 2xy - 3y^2$?

14. What must be added to $2bc - 3ca - 4ab$ in order that the sum may be $bc + ca$?

15. What must be added to $3a^2 - 2b^2 + 3c^2$ in order that the sum may be $bc + ca + ab$?

16. Simplify $3x - \{2y + (5x - \overline{3x + y})\}$.

17. Simplify $x - [3y + \{3z - \overline{x - 2y}\} + 2x]$.

18. Simplify $y - 2x - \{z - x - \overline{y - x + z}\}$.

19. Simplify $a - [a - b - \{a - b + c - \overline{a - b + c - d}\}]$.

20. Simplify $2x - [3x - 9y - \{2x - 3y - (x + 5y)\}]$.

21. Simplify $a - [3a + c - \{4a - (3b - c) + 3b\} - 2a]$.

22. Subtract $x - (3y - z)$ from $y - \{2x - \overline{z - y}\}$.

23. Subtract $2m - (3m - \overline{2n - m})$ from $2n - (3n - \overline{2m - n})$.

24. Find the value of

$$\{a - (b - c)\}^2 + \{b - (c - a)\}^2 + \{c - (a - b)\}^2 \text{ when } a = -1,\ b = -2,\ c = -3.$$

25. Find the value of

$$\{a^2 - (b - c)^2\} - \{b^2 - (c - a)^2\} - \{c^2 - (a - b)^2\} \text{ when } a = 1,\ b = 2,\ c = -3.$$

CHAPTER IV.

MULTIPLICATION.

53. **Product of monomial expressions.** The multiplication of monomial expressions was considered in Chapter II., and the results arrived at were:

(i) The factors of a product may be taken in any order.

(ii) The sign of the product of two quantities is + when both the factors are positive or both negative; and the sign of the product is − when one factor is positive and the other negative.

(iii) The index of the product of any two powers of the same quantity is the sum of the indices of the factors.

From (i), (ii) and (iii) we can find the continued product of any number of monomial expressions.

$$\begin{aligned} \text{Thus } (-2a^2bc^3)\times(-3a^3b^2c) &= +2a^2bc^3\times 3a^3b^2c, && \text{from (ii)},\\ &= 2\times3\times a^2 . a^3 . b . b^2 . c^3 . c, && \text{from (i)},\\ &= 6a^5b^3c^4, && \text{from (iii)}. \end{aligned}$$

$$\text{Again, } (-3a^2b)(-5ab^3)(-7a^4b^2) = \{+3a^2b . 5ab^3\}(-7a^4b^2)$$
$$= -3 . 5 . 7 . a^2 . a . a^4 . b . b^3 . b^2 = -105a^7b^6.$$

54. **Product of a multinomial expression and a monomial.** It was proved in Art. 42 that the product of the sum of *any* two algebraical quantities by a third is equal to the sum of the products obtained by multiplying the two quantities separately by the third.

Thus $$(x + y)\,z = xz + yz \quad\text{.................(i)}.$$

Since (i) is true for *all values* of x, y and z, it will be true when we put $(a + b)$ in place of x; hence

$$\{(a + b) + y\}\,z = (a + b)\,z + yz$$
$$= az + bz + yz.$$
$$\therefore\ (a + b + y)\,z = az + bz + yz.$$

And similarly

$$(a + b + c + d + \dots)\,z = az + bz + cz + dz + \dots,$$

however many terms there may be in the expression

$$a + b + c + d + \dots$$

Thus *the product of any multinomial expression by a monomial is the sum of the products obtained by multiplying the separate terms of the multinomial expression by the monomial.*

55. **Product of two multinomial expressions.** We now consider the most general case of multiplication, namely the multiplication of any two multinomial expressions.

We have to find

$$(a + b + c + \dots) \times (x + y + z + \dots);$$

and, from Art. 38, this includes all possible cases.

Put M for $x + y + z + \dots$; then, by the last article, we have

$$(a + b + c + \dots)\,M = aM + bM + cM + \dots$$
$$= Ma + Mb + Mc + \dots$$
$$= (x + y + z + \dots)\,a + (x + y + z + \dots)\,b$$
$$+ (x + y + z + \dots)\,c + \dots$$
$$= ax + ay + az + \dots + bx + by + bz + \dots + cx + cy + cz + \dots$$

Hence $(a + b + c + \dots)(x + y + z + \dots)$

$$= ax + ay + az + \dots + bx + by + bz + \dots + cx + cy + cz + \dots$$

Thus, *the product of any two algebraical expressions is equal to the sum of the products obtained by multiplying every term of the one by every term of the other.*

For example

$$(a+b)(c+d)=ac+ad+bc+bd;$$

also

$$\begin{aligned}(3a+5b)&(2a+3b)\\ &=(3a)(2a)+(3a)(3b)+(5b)(2a)+(5b)(3b)\\ &=6a^2+9ab+10ab+15b^2=6a^2+19ab+15b^2.\end{aligned}$$

Again, to find $(a-b)(c-d)$, we first write this in the form $\{a+(-b)\}\{c+(-d)\}$, and we then have for the product

$$\begin{aligned}ac+a(-d)&+(-b)c+(-b)(-d)\\ &=ac-ad-bc+bd.\end{aligned}$$

In the rule given above for the multiplication of two algebraical expressions it must be borne in mind that the *terms include the prefixed signs.*

56. The following are important examples:—

I. $(a+b)^2=(a+b)(a+b)=aa+ab+ba+bb;$

$$\therefore\ (a+b)^2=a^2+2ab+b^2.$$

Thus, *the square of the sum of any two quantities is equal to the sum of their squares plus twice their product.*

II. $(a-b)^2=(a-b)(a-b)=aa+a(-b)+(-b)a$

$$+(-b)(-b)=a^2-ab-ab+b^2;$$

$$\therefore\ (a-b)^2=a^2-2ab+b^2.$$

Thus, *the square of the difference of any two quantities is equal to the sum of their squares minus twice their product.*

III. $(a+b)(a-b)=aa+a(-b)+ba+b(-b)$

$$=a^2-ab+ab-b^2;$$

$$\therefore\ (a+b)(a-b)=a^2-b^2.$$

Thus, *the product of the sum and difference of any two quantities is equal to the difference of their squares.*

57. It is usual to exhibit the process of multiplication in the following convenient form:

$$
\begin{array}{rrrrr}
a^2 & +2ab & -b^2 & & \\
a^2 & -2ab & +b^2 & & \\
\hline
a^4 & +2a^3b & -a^2b^2 & & \\
 & -2a^3b & -4a^2b^2 & +2ab^3 & \\
 & & a^2b^2 & +2ab^3 & -b^4 \\
\hline
a^4 & & -4a^2b^2 & +4ab^3 & -b^4.
\end{array}
$$

The multiplier is placed under the multiplicand and a line is drawn. The successive terms of the multiplicand, namely a^2, $+2ab$, and $-b^2$, are multiplied by a^2, the first term on the left of the multiplier, and the products a^4, $+2a^3b$ and $-a^2b^2$ which are thus obtained are put in a horizontal row. The terms of the multiplicand are then multiplied by $-2ab$, the second term of the multiplier, and the products thus obtained are put in another horizontal row, the terms being so placed that 'like' terms are under one another. And similarly for all the other terms of the multiplier. The final result is then obtained by adding the rows of partial products; and this final sum can be readily written down, since the different sets of 'like' terms are in vertical columns.

The following are examples of multiplication arranged as above described:

$$
\begin{array}{rrr}
a & +b & \\
a & +b & \\
\hline
a^2 & +ab & \\
 & +ab & +b^2 \\
\hline
a^2 & +2ab & +b^2
\end{array}
\qquad
\begin{array}{rrr}
a & +b & \\
a & -b & \\
\hline
a^2 & +ab & \\
 & -ab & -b^2 \\
\hline
a^2 & & -b^2
\end{array}
\qquad
\begin{array}{rrrr}
a^2 & +ab & +b^2 & \\
a & -b & & \\
\hline
a^3 & +a^2b & +ab^2 & \\
 & -a^2b & -ab^2 & -b^3 \\
\hline
a^3 & & & -b^3
\end{array}
$$

$$
\begin{array}{rrrrrr}
a & +b & +c & & & \\
a & +b & +c & & & \\
\hline
a^2 & +ab & +ac & & & \\
 & +ab & & +b^2 & +bc & \\
 & & +ac & & +bc & +c^2 \\
\hline
a^2 & +2ab & +2ac & +b^2 & +2bc & +c^2
\end{array}
\qquad
\begin{array}{rrrrr}
3x^2 & -xy & +2y^2 & & \\
3x^2 & +xy & -2y^2 & & \\
\hline
9x^4 & -3x^3y & +6x^2y^2 & & \\
 & +3x^3y & -x^2y^2 & +2xy^3 & \\
 & & -6x^2y^2 & +2xy^3 & -4y^4 \\
\hline
9x^4 & & -x^2y^2 & +4xy^3 & -4y^4
\end{array}
$$

58. If in an expression consisting of several terms which contain different powers of the same letter, the

term which contains the highest power of that letter be put first on the left, the term which contains the next highest power be put next, and so on; the terms, if any, which do not contain the letter being put last; then the whole expression is said to be *arranged according to descending powers* of that letter. Thus the expression

$$a^3 + a^2b + ab^2 + b^3$$

is arranged according to descending powers of a. In like manner we say that the expression is arranged according to *ascending* powers of b.

59. Although it is not *necessary* to arrange the terms either of the multiplicand or of the multiplier in any particular order, it will be found convenient to arrange *both* expressions according to descending or *both* according to ascending powers of the same letter: some trouble in the arrangement of the different sets of 'like' terms in vertical columns will thus be avoided.

60. **Definitions.** A term which is the product of n letters is said to be of *n dimensions*, or of the nth *degree*. Thus $3abc$ is of three dimensions, or of the third degree; and $5a^3b^2c$, that is $5aaabbc$, is of six dimensions, or of the sixth degree. Thus the degree of a term is found by taking the sum of the indices of its factors.

The *degree of an expression* is the degree of that term of it which is of highest dimensions.

In estimating the degree of a term, or of an expression, we sometimes take into account only a particular letter, or particular letters: thus $ax^2 + bx + c$ is of the second degree *in x*, and is often called a *quadratic expression* in x; also $ax^2y + bxy + cx^2$ is of the third degree *in x and y*, and is often called a *cubic expression* in x and y. An expression, or a term, which does not contain x is said to be of *no degree* in x, or to be *independent* of x.

When all the terms of an expression are of the same dimensions, the expression is said to be *homogeneous*. Thus $a^3 + 3a^2b - 5b^3$ is a homogeneous expression, every

term being of the third degree; also $ax^2 + bxy + cy^2$ is homogeneous expression of the second degree in x and y.

61. **Product of homogeneous expressions.** Th product of any two homogeneous expressions must b homogeneous; for the different terms of the product ar obtained by multiplying any term of the multiplicand b any term of the multiplier, and the number of dimensior in the product of any two monomials is clearly the sur of the number of dimensions in the separate quantities hence if all the terms of the multiplicand are of th same degree, as also all the terms of the multiplier, i follows that all the terms of the product are of the sam degree; and it also follows that the degree of the produc is the *sum* of the degrees of the factors.

The fact that two expressions which are to be multi plied are homogeneous should in all cases be noticed; an if the product obtained is not homogeneous, it is clea that there is an error.

62. It is of importance to notice that, in the produc of two algebraical expressions, the term which is of *highe* degree in a particular letter is the product of the term in the factors which are of highest degree in that lette and the term of *lowest* degree is the product of the term which are of lowest degree in the factors: thus there is onl *one* term of highest degree and *one* term of lowest degree

63. **Detached Coefficients.** When two expression are both arranged according to descending, or to ascending powers of some letter, much of the labour of multiplicatio can be saved by writing down the coefficients only.

Thus, to multiply $3x^2 - x + 2$ by $3x^2 + 2x - 2$, we writ

$$\begin{array}{rrrrr}
3 & -1 & +2 & & \\
3 & +2 & -2 & & \\
\hline
9 & -3 & +6 & & \\
 & 6 & -2 & +4 & \\
 & & -6 & +2 & -4 \\
\hline
9 & +3 & -2 & +6 & -4
\end{array}$$

The highest power of x in the product is clearly x^4, and the rest follow in order. Hence the required product is

$$9x^4 + 3x^3 - 2x^2 + 6x - 4.$$

When some of the powers are absent their places must be supplied by 0's.

Thus, to multiply $x^4 - 2x^2 + x - 3$ by $x^4 + x^3 - x - 3$, we write

$$\begin{array}{l}
1+0-2+1-3 \\
\underline{1+1+0-1-3} \\
1+0-2+1-3 \\
\quad\;\; 1+0-2+1-3 \\
\qquad\qquad -1-0+2-1+3 \\
\qquad\qquad\quad\;\; -3-0+6-3+9 \\
\hline
1+1-2-2-5-1+5+0+9
\end{array}$$

Hence the product is

$$x^8 + x^7 - 2x^6 - 2x^5 - 5x^4 - x^3 + 5x^2 + 9.$$

This is generally called the **method of detached coefficients.**

64. We now return to the three important cases of multiplication considered in Art. 56, namely,

$$(a+b)^2 = a^2 + 2ab + b^2 \;\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)},$$
$$(a-b)^2 = a^2 - 2ab + b^2 \;\ldots\ldots\ldots\ldots\ldots \text{(ii)},$$
$$(a+b)(a-b) = a^2 - b^2 \;\ldots\ldots\ldots\ldots\ldots \text{(iii)}.$$

A general result expressed by means of symbols is called a *formula.*

Since the laws from which the above formulae were deduced were proved to be true for all algebraical quantities whatever, we may substitute for a and for b any other algebraical quantities, or algebraical expressions, and the results will still hold good.

We give some examples of results obtained by substitution.

Put $-b$ in the place of b in (i); we then have

$$\{a+(-b)\}^2 = a^2 + 2a(-b) + (-b)^2,$$

that is $$(a-b)^2 = a^2 - 2ab + b^2.$$

Thus (ii) is seen to be really included in (i).

Put $\sqrt{2}$ in the place of b in (iii); we then have

$$(a+\sqrt{2})(a-\sqrt{2}) = a^2 - (\sqrt{2})^2 = a^2 - 2.$$

[We here, however, *assume* that all the fundamental laws are true for surds: this will be considered in a subsequent chapter.]

Put $b+c$ in the place of b in (i); we then have

$$\{a+(b+c)\}^2 = a^2 + 2a(b+c) + (b+c)^2;$$

$$\therefore (a+b+c)^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2 \ldots\ldots \text{(iv)}.$$

Now put $-c$ for c in (iv), and we have

$$\{a+b+(-c)\}^2 = a^2 + 2ab + 2a(-c) + b^2 + 2b(-c) + (-c)^2;$$

$$\therefore (a+b-c)^2 = a^2 + 2ab - 2ac + b^2 - 2bc + c^2.$$

Put $b+c$ in the place of b in (iii); we then have

$$\{a+(b+c)\}\{a-(b+c)\} = a^2 - (b+c)^2 = a^2 - (b^2 + 2bc + c^2);$$

$$\therefore (a+b+c)(a-b-c) = a^2 - b^2 - 2bc - c^2.$$

The following are additional examples of products which can be written down at once.

$$(a^2+2b^2)(a^2-2b^2) = (a^2)^2 - (2b^2)^2 = a^4 - 4b^4.$$

$$(a^2+\sqrt{3}b^2)(a^2-\sqrt{3}b^2) = (a^2)^2 - (\sqrt{3}b^2)^2 = a^4 - 3b^4.$$

$$(a-b+c)(a+b-c) = \{a-(b-c)\}\{a+(b-c)\} = a^2 - (b-c)^2.$$

$$(a^2+ab+b^2)(a^2-ab+b^2) = \{(a^2+b^2)+ab\}\{(a^2+b^2)-ab\}$$
$$= (a^2+b^2)^2 - (ab)^2 = a^4 + a^2b^2 + b^4.$$

$$(x^3+x^2+x+1)(x^3-x^2+x-1) = \{(x^3+x)+(x^2+1)\}\{(x^3+x)-(x^2+1)\}$$
$$= (x^3+x)^2 - (x^2+1)^2 = x^6 + 2x^4 + x^2 - (x^4+2x^2+1) = x^6 + x^4 - x^2 - 1.$$

65. Square of a multinomial expression. We have found in the preceding Article, and also by direct multiplication in Art. 57, the square of the sum of *three* algebraical quantities; and the square of the sum of more than three quantities can be obtained by the same methods. The square of any multinomial expression can however best be found in the following manner.

We have to find

$$(a+b+c+d+\ldots)(a+b+c+d+\ldots).$$

Now we know that the product of any two algebraical expressions is equal to the sum of the partial products obtained by multiplying every term of one expression by every term of the other. If we multiply the term a of the multiplicand by the term a of the multiplier, we obtain the term a^2 of the product: we similarly obtain the terms b^2, c^2, &c. We can multiply any term, say b, of the multiplicand by any different term, say d, of the multiplier; and we thus obtain the term bd of the product. But we also obtain the term bd by multiplying the term d of the multiplicand by the term b of the multiplier, and the term bd can be obtained in no other way, so that every such term as bd, in which the letters are different, occurs *twice* in the product. The required product is therefore the sum of the squares of all the quantities a, b, c, &c. together with twice the product of every pair.

Thus, *the square of the sum of any number of algebraical quantities is equal to the sum of their squares together with twice the product of every pair.*

For example, to find $(a+b+c)^2$.

The squares of the separate terms are a^2, b^2, c^2.

The products of the different pairs of terms are ab, ac and bc.

Hence $\qquad (a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$

Similarly,

$$\begin{aligned}(a+2b-3c)^2&=a^2+(2b)^2+(-3c)^2+2a(2b)+2a(-3c)+2(2b)(-3c)\\&=a^2+4b^2+9c^2+4ab-6ac-12bc.\end{aligned}$$

And

$$(a-b+c-d)^2 = a^2 + (-b)^2 + c^2 + (-d)^2 + 2a(-b) + 2ac + 2a(-d)$$
$$+ 2(-b)c + 2(-b)(-d) + 2c(-d) = a^2 + b^2 + c^2 + d^2 - 2ab + 2ac$$
$$- 2ad - 2bc + 2bd - 2cd.$$

After a little practice the intermediate steps should be omitted and the final result written down at once. To ensure taking twice the product of *every* pair it is best to take twice the product of each term and of every term which *follows* it.

66. **Continued Products.** The continued product of several algebraical expressions is obtained by finding the product of any two of the expressions, and then multiplying this product by a third expression, and so on.

For example, to find $(x+a)(x+b)(x+c)$, we have

$$\begin{array}{l}
x+a \\
\underline{x+b} \\
x^2+ax \\
\quad\underline{+bx+ab} \\
x^2+(a+b)x+ab \\
\underline{x+c} \\
x^3+(a+b)x^2+abx \\
\quad\underline{+cx^2+(a+b)cx+abc} \\
x^3+(a+b+c)x^2+(ab+ac-bc)x+abc
\end{array}$$

In the above all the terms which contain the same powers of x are collected together: it is frequently necessary to arrange expressions in this way.

Again, to find $\qquad (x^2+a^2)^2(x+a)^2(x-a)^2.$

The factors can be taken in any order; hence the required product $= [(x-a)(x+a)(x^2+a^2)]^2 = [(x^2-a^2)(x^2+a^2)]^2 = (x^4-a^4)^2 = x^8 - 2a^4x^4 + a^8.$

67. We have proved in Art. 55 that the product of any two multinomial expressions is the sum of all the partial products obtained by multiplying any term of one expression by any term of the other.

To find the continued product of three expressions we must therefore multiply each of the terms in the product of the first two expressions by each of the terms in the third; hence the continued product is the sum of all the partial products which can be obtained by multiplying together any term of the first, any term of the second, and any term of the third.

And similarly, the continued product of any number of expressions is the sum of all the partial products which can be obtained by multiplying together any term of the first, any term of the second, any term of the third, &c.

For example, if we take a letter from each of the three factors of

$$(a+b)\ (a+b)\ (a+b),$$

and multiply the three together, we shall obtain a term of the continued product; and if we do this in every possible way we shall obtain all the terms of the continued product.

Now we can take a every time, and we can do this in only one way; hence a^3 is a term of the continued product.

We can take a twice and b once, and this can be done in *three* ways, for the b can be taken from either of the three binomial factors; hence we have $3a^2b$.

We can take a once and b twice, and we can do this also in *three* ways; hence we have $3ab^2$.

Finally, we can take b every time, and this can be done in only one way; hence we have b^3.

Thus the continued product is

$$a^3 + 3a^2b + 3ab^2 + b^3,$$

that is $$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

The continued product $(x+a)\ (x+b)\ (x+c)$ can similarly be written down at once.

For we can take x every time: we thus get x^3.

We can take *two* x's and either a or b or c: we thus have x^2a, x^2b and x^2c.

We can take *one* x and any two of a, b, c: we thus have xab, xac, and xbc.

Finally, if we take no x's, we have the term abc.

Thus, arranging the result according to powers of x, we have

$$(x+a)(x+b)(x+c)=x^3+x^2(a+b+c)+x(ab+ac+bc)+abc.$$

68. **Powers of a binomial.** We have already found the square and the cube of a binomial expression; and higher powers can be obtained in succession by actual multiplication. The method of detached coefficients should be used to shorten the work.

The following should be remembered:

$$(a+b)^2=a^2+2ab+b^2,$$

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3,$$

and $$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

To find any power, higher than the fourth, of a binomial expression a formula called the Binomial Theorem should be employed: this theorem will be considered in a subsequent chapter.

EXAMPLES II.

1. Multiply $2x-a$ by $x-2a$.
2. Multiply $3x-\frac{1}{3}$ by $\frac{1}{3}x-3$.
3. Multiply x^2+x+1 by $x-1$.
4. Multiply x^2-xy+y^2 by $x+y$.
5. Multiply $1+x+x^2+x^3$ by $x-1$.
6. Multiply $x^4+x^3y+x^2y^2+xy^3+y^4$ by $y-x$.
7. Multiply x^2-x+2 by x^2+x-2.
8. Multiply $1+ax+a^2x^2$ by $1-ax+a^2x^2$.
9. Multiply x^4+x^2+1 by x^4-x^2+1.

10. Multiply $3x^2 - xy + 2y^2$ by $3y^2 - xy + 2x^2$.

11. Multiply $x^3 - 5x^2 + 1$ by $2x^3 + 5x + 1$.

12. Multiply $2x^3 - 5x^2y + y^3$ by $y^3 + 5xy^2 + 2x^3$.

13. Multiply $3a^3 - 2a^2b + 3ab^2 - 3b^3$ by $2a^3 + 5a^2b - 4ab^2 + b^3$.

14. Multiply $2a^3x^3 - 3a^2x^2y^2 + 5y^6$ by $a^3x^3 + 4axy^4 - 2y^6$.

15. Multiply $2a - 3a^2 + 5a^3 - 7a^5$ by $1 - 2a^2 + 6a^4$.

16. Multiply $a^2 - ab - ac + b^2 - bc + c^2$ by $a + b + c$.

17. Multiply $x^2 + y^2 + z^2 - yz - zx - xy$ by $x + y + z$.

18. Multiply $4a^2 + 9b^2 + c^2 + 3bc + 2ca - 6ab$ by $2a + 3b - c$.

19. Multiply together $x^4 + 1$, $x^2 + 1$ and $x^2 - 1$.

20. Multiply together $x^4 + 16y^4$, $x^2 + 4y^2$, $x + 2y$ and $x - 2y$.

21. Multiply together $(x - y)^2$, $(x + y)^2$ and $(x^2 + y^2)^2$.

22. Multiply together $(x^2 + 1)^3$, $(x + 1)^3$ and $(x - 1)^3$.

23. Multiply together $x^2 - x + 1$, $x^2 + x + 1$ and $x^4 - x^2 + 1$.

24. Multiply together $a^2 - 2ab + 4b^2$, $a^2 + 2ab + 4b^2$ and $a^4 - 4a^2b^2 + 16b^4$.

25. Find the squares of (i) $a + 2b - 3c$, (ii) $a^2 - ab + b^2$, (iii) $bc + ca + ab$, (iv) $1 - 2x + 3x^2$, and (v) $x^3 + x^2 + x + 1$.

26. Find the cubes of (i) $a + b + c$, (2) $2a - 3b - 2c$ and (iii) $1 + x + x^2$.

27. Simplify

$$(x + y + z)^2 - (-x + y + z)^2 + (x - y + z)^2 - (x + y - z)^2.$$

28. Shew that

$$(x + y)(x + z) - x^2 = (y + z)(y + x) - y^2 = (z + x)(z + y) - z^2.$$

29. Shew that

$$(y + z)^2 + (z + x)^2 + (x + y)^2 - x^2 - y^2 - z^2 = (x + y + z)^2.$$

30. Simplify $\{x(x+a)-a(x-a)\}\{x(x-a)-a(x+a)\}$.

31. Shew that
$$(y-z)^3+(z-x)^3+(x-y)^3=3(y-z)(z-x)(x-y).$$

32. Shew that $a^3+b^3=(a+b)^3-3ab(a+b)$, and that
$$a^4+b^4=(a+b)^4-4ab(a+b)^2+2a^2b^2.$$

33. Shew that $(x^2+xy+y^2)^2-4xy(x^2+y^2)=(x^2-xy+y^2)^2$.

34. Shew that
$$(y+z)^2+(z+x)^2+(x+y)^2+2(x+y)(x+z)+2(y+z)(y+x)$$
$$+2(z+x)(z+y)=4(x+y+z)^2.$$

35. Shew that $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$.

36. Shew that, if $x=a+d$, $y=b+d$, and $z=c+d$; then will $x^2+y^2+z^2-yz-zx-xy=a^2+b^2+c^2-bc-ca-ab$.

37. Shew that, if $x=b+c$, $y=c+a$, and $z=a+b$; then will $x^2+y^2+z^2-yz-zx-xy=a^2+b^2+c^2-bc-ca-ab$.

38. Shew that $2(a-b)(a-c)+2(b-c)(b-a)+2(c-a)(c-b)$
$$=(b-c)^2+(c-a)^2+(a-b)^2.$$

39. Shew that $(x^2+y^2+z^2)(a^2+b^2+c^2)-(ax+by+cz)^2$
$$=(bz-cy)^2+(cx-az)^2+(ay-bx)^2.$$

40. Shew that, if $x=a^2-bc$, $y=b^2-ca$, $z=c^2-ab$; then will
$$ax+by+cz=(x+y+z)(a+b+c),$$
and
$$bc(x^2-yz)=ca(y^2-zx)=ab(z^2-xy).$$

41. Find the value of
$$(x-a)^3+(x-b)^3+(x-c)^3-3(x-a)(x-b)(x-c)$$
when $3x=a+b+c$.

42. Shew that $(a^2+b^2+c^2)^2$
$$=(b^2+c^2)^2+(ab+ac)^2+(ab-ac)^2+a^4$$
$$=(bc+ca+ab)^2+(a^2-bc)^2+(b^2-ca)^2+(c^2-ab)^2.$$

43. Shew that $(x^2+xy+y^2)(a^2+ab+b^2)$
$$=(ax-by)^2+(ax-by)(ay+bx+by)+(ay+bx+by)^2.$$

44. Shew that $1+a^2+b^2+c^2+b^2c^2+c^2a^2+a^2b^2+a^2b^2c^2$
$$=(1-bc-ca-ab)^2+(a+b+c-abc)^2.$$

45. Shew that
$$(a^2+b^2+c^2+d^2)^2=(a^2+b^2-c^2-d^2)^2+4(ac+bd)^2+4(ad-bc)^2.$$

46. Shew that

(i) $(a+2)^2-4(a+1)^2+6a^2-4(a-1)^2+(a-2)^2=0.$

(ii) $(a+2)(b+2)-4(a+1)(b+1)+6ab$
$$-4(a-1)(b-1)+(a-2)(b-2)=0.$$

47. Shew that

(i) $(a+2)^3-4(a+1)^3+6a^3-4(a-1)^3+(a-2)^3=0.$

(ii) $(a+2)(b+2)(c+2)-4(a+1)(b+1)(c+1)+6abc$
$$-4(a-1)(b-1)(c-1)+(a-2)(b-2)(c-2)=0.$$

48. Shew that
$$(a+b+c)^3+(b+c-a)(c+a-b)(a+b-c)$$
$$=4a^2(b+c)+4b^2(c+a)+4c^2(a+b)+4abc.$$

49. Shew that
$$x(x-y+z)(x+y-z)+y(x+y-z)(-x+y+z)$$
$$+z(-x+y+z)(x-y+z)+(-x+y+z)(x-y+z)(x+y-z)$$
$$=4xyz.$$

50. Multiply
$a^2+b^2+c^2+d^2-bc-ca-ab-ad-bd-cd$ by $a+b+c+d.$

51. Shew that
$$(x^2+x+1)(x^2-x+1)(x^4-x^2+1)(x^8-x^4+1)\ldots(x^{2^n}-x^{2^{n-1}}+1)$$
$$=x^{2^{n+1}}+x^{2^n}+1.$$

CHAPTER V.

DIVISION.

69. **Division by a monomial expression.** We have already considered the division of one monomial expression by another. We have also seen (Art. 43) that the quotient obtained by dividing the sum of two algebraical quantities by a third is the sum of the quotients obtained by dividing the quantities separately by the third; and we can shew by the method of Art. 54 that when any multinomial expression is divided by a monomial the quotient is the sum of the quotients obtained by dividing the separate terms of the multinomial expression by that monomial.

Thus $(a^2x - 3ax) \div ax = a^2x \div ax - 3ax \div ax = a - 3.$

And $(12x^3 - 5ax^2 - 2a^2x) \div 3x = 12x^3 \div 3x - 5ax^2 \div 3x - 2a^2x \div 3x = 4x^2 - \frac{5}{3}ax - \frac{2}{3}a^2.$

70. **Division by a multinomial expression.** We have now to consider the most general case of division, namely the division of one multinomial expression by another.

Since division is the inverse of multiplication, what we have to do is to find the algebraical expression which, when multiplied by the divisor, will produce the dividend.

Both dividend and divisor are first arranged according

to descending powers of some common letter, a suppose; and the quotient also is considered to be so arranged. Then (Art. 62) the first term of the dividend will be the product of the first term of the divisor and the first term of the quotient; and therefore the *first term of the quotient will be found by dividing the first term of the dividend by the first term of the divisor.* If we now multiply the whole divisor by the first term of the quotient so obtained, and subtract the product from the dividend, the remainder must be the product of the divisor by the sum of all the other terms of the quotient; and, this remainder being also arranged according to descending powers of a, the *second term of the quotient will be found as before by dividing the first term of the remainder by the first term of the divisor.* If we now multiply the whole divisor by the second term of the quotient and subtract the quotient from the remainder, it is clear that the *third and other terms of the quotient can be found in succession in a similar manner.*

For example, to divide $8a^3 + 8a^2b - 4ab^2 + b^3$ by $2a + b$.

The arrangement is the same as in Arithmetic.

$$
\begin{array}{rl}
2a + b\,) & 8a^3 + 8a^2b + 4ab^2 + b^3\ (\,4a^2 + 2ab + b^2 \\
 & \underline{8a^3 + 4a^2b} \\
 & \qquad\quad 4a^2b + 4ab^2 + b^3 \\
 & \qquad\quad \underline{4a^2b + 2ab^2} \\
 & \qquad\qquad\qquad 2ab^2 + b^3 \\
 & \qquad\qquad\qquad \underline{2ab^2 + b^3}
\end{array}
$$

The *first term* of the quotient is $8a^3 \div 2a = 4a^2$. Multiply the divisor by $4a^2$ and subtract the product from the dividend: we then have the remainder $4a^2b + 4ab^2 + b^3$. The *second term* of the quotient is $4a^2b \div 2a = 2ab$. Multiply the divisor by $2ab$, and subtract the product from the remainder: we thus get the second remainder $2ab^2 + b^3$. The *third term* of the quotient is $2ab^2 \div 2a = b^2$. Multiply the divisor by b^2, and subtract the product from

$2ab^2+b^3$, and there is no remainder. Since there is no remainder after the last subtraction, the dividend must be equal to the sum of the different quantities which have been subtracted from it; but we have subtracted in succession the divisor multiplied by $4a^2$, by $+2ab$, and by $+b^2$; we have therefore subtracted altogether the divisor multiplied by $4a^2+2ab+b^2$. And, since the divisor multiplied by $4a^2+2ab+b^2$ is equal to the dividend, the required quotient is $4a^2+2ab+b^2$.

The dividend and divisor may be arranged according to *ascending* instead of according to *descending* powers of the common letter, as in the last example considered with reference to the letter b; but the dividend and the divisor must *both* be arranged in the same way.

71. The following are additional examples:

Ex. 1. Divide $a^4-a^3b+2a^2b^2-ab^3+b^4$ by a^2+b^2.

$$\begin{array}{l}
a^2+b^2\,\big)\,a^4-a^3b+2a^2b^2-ab^3+b^4\,\big(\,a^2-ab+b^2 \\
\qquad\qquad a^4 \qquad\quad +a^2b^2 \\
\hline
\qquad\qquad\quad -a^3b+a^2b^2\quad -ab^3+b^4 \\
\qquad\qquad\quad -a^3b \qquad\qquad -ab^3 \\
\hline
\qquad\qquad\qquad\quad +a^2b^2 \qquad\quad +b^4 \\
\qquad\qquad\qquad\quad +a^2b^2 \qquad\quad +b^4 \\
\hline
\end{array}$$

Ex. 2. Divide $a^4+a^2b^2+b^4$ by a^2-ab+b^2.

$$\begin{array}{l}
a^2-ab+b^2\,\big)\,a^4 \qquad\quad +a^2b^2 \qquad\quad +b^4\,\big(\,a^2+ab+b^2 \\
\qquad\qquad\qquad a^4-a^3b+a^2b^2 \\
\hline
\qquad\qquad\qquad\quad +a^3b \qquad\qquad\quad +b^4 \\
\qquad\qquad\qquad\quad +a^3b-a^2b^2+ab^3 \\
\hline
\qquad\qquad\qquad\qquad\quad +a^2b^2-ab^3+b^4 \\
\qquad\qquad\qquad\qquad\quad +a^2b^2-ab^3+b^4 \\
\hline
\end{array}$$

In this example the terms of the dividend were placed apart, in order that 'like' terms might be placed under one another without altering the order of the terms in descending powers of a. The subtractions can be easily performed without placing 'like' terms under one another; but the arrangement of the terms according to descending (or ascending) powers of the chosen letter should never be departed from.

Ex. 3. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

$$
\begin{array}{l}
a+b+c\,\big)\,a^3-3abc+b^3+c^3\,\big(\,a^2-ab-ac+b^2-bc+c^2\\
\qquad\quad\; a^3+a^2b+a^2c\\
\hline
\qquad\quad\; -a^2b-a^2c-3abc+b^3+c^3\\
\qquad\quad\; -a^2b-ab^2-abc\\
\hline
\qquad\qquad\quad -a^2c+ab^2-2abc+b^3+c^3\\
\qquad\qquad\quad -a^2c\qquad -abc\;\;-ac^2\\
\hline
\qquad\qquad\qquad +ab^2-abc+ac^2+b^3+c^3\\
\qquad\qquad\qquad +ab^2\qquad\qquad +b^3+b^2c\\
\hline
\qquad\qquad\qquad\quad -abc+ac^2-b^2c+c^3\\
\qquad\qquad\qquad\quad -abc\qquad -b^2c-bc^2\\
\hline
\qquad\qquad\qquad\qquad +ac^2+bc^2+c^3\\
\qquad\qquad\qquad\qquad +ac^2+bc^2+c^3\\
\hline
\end{array}
$$

Where, as in the above example, more than two letters are involved, it is not sufficient to arrange the terms according to descending powers of a; but b also is given the precedence over c.

By using brackets, the above process may be shortened. Thus

$$
\begin{array}{l}
a+b+c\,\big)\,a^3-3abc+b^3+c^3\,\big(\,a^2-a(b+c)+(b^2-bc+c^2)\\
\qquad\quad\; a^3+a^2(b+c)\\
\hline
\qquad\quad\; -a^2(b+c)-3abc+b^3+c^3\\
\qquad\quad\; -a^2(b+c)-a(b+c)^2\\
\hline
\qquad\qquad\qquad a(b^2-bc+c^2)+b^3+c^3\\
\qquad\qquad\qquad a(b^2-bc+c^2)+b^3+c^3\\
\hline
\end{array}
$$

72. The method of detached coefficients may often be employed in Division with great advantage. For example, to divide

$2x^6-7x^5+5x^4+3x^3-3x^2+4x-4$ by $2x^3-3x^2+x-2$,

we write—

$$
\begin{array}{l}
2-3+1-2\,\big)\,2-7+5+3-3+4-4\,\big(\,1-2-1+2\\
\qquad\qquad\quad\; 2-3+1-2\\
\hline
\qquad\qquad\qquad -4+4+5-3+4-4\\
\qquad\qquad\qquad -4+6-2+4\\
\hline
\qquad\qquad\qquad\qquad -2+7-7+4-4\\
\qquad\qquad\qquad\qquad -2+3-1+2\\
\hline
\qquad\qquad\qquad\qquad\qquad 4-6+2-4\\
\qquad\qquad\qquad\qquad\qquad 4-6+2-4\\
\hline
\end{array}
$$

The first term of the quotient is x^3 and the other powers follow in order: thus the quotient is

$$x^3-2x^2-x+2.$$

73. **Extended definition of Division.** In the process of division as described in Art. 70, it is clear that the remainder after the first subtraction must be of *lower degree* in a than the dividend; and also that every remainder must be of lower degree than the preceding remainder. Hence by proceeding far enough we must come to a stage where there is no remainder, or else where there is a remainder such that the highest power of a in it is less than the highest power of a in the divisor, and in this latter case the division cannot be *exactly* performed.

It is convenient to extend the definition of division to the following: *To divide A by B is to find an algebraical expression C such that $B \times C$ is either equal to A, or differs from A by an expression which is of lower degree, in some particular letter, than the divisor B.*

For example, if we divide $a^2 + 3ab + 4b^2$ by $a + b$, we have

$$\begin{array}{r|l} a+b\,) & a^2 + 3ab + 4b^2\ (\,a + 2b \\ & \underline{a^2 + \ \ ab\qquad\ \ } \\ & \quad 2ab + 4b^2 \\ & \quad \underline{2ab + 2b^2} \\ & \qquad\ \ \underline{+\ 2b^2} \end{array}$$

Thus $(a^2 + 3ab + 4b^2) \div (a + b) = a + 2b$, with remainder $2b^2$; that is $a^2 + 3ab + 4b^2 = (a + b)(a + 2b) + 2b^2$. We have also, by arranging the dividend and divisor differently,

$$\begin{array}{r|l} b+a\,) & 4b^2 + 3ab + a^2\ (\,4b - a \\ & \underline{4b^2 + 4ab\qquad\ \ } \\ & \quad -\ ab + a^2 \\ & \quad \underline{-\ ab - a^2} \\ & \qquad\ \ \underline{+\ 2a^2} \end{array}$$

Hence a change in the order of the dividend and divisor leads to a result of a different form. This is, however, what might be expected considering that in the first

case we find what the divisor must be multiplied by in order to agree with the dividend so far as certain terms which contain a are concerned, and in the second we find what the divisor must be multiplied by in order to agree with the dividend so far as certain terms which contain b are concerned.

When therefore we have to divide one expression by another, both expressions being arranged in the same way, it must be understood that this arrangement is to be adhered to.

74. **Def.** A relation of equality which is true for *all values* of the letters it contains, is called an *identity*.

The following identities can easily be verified, and should be remembered:

$$(x^2+2ax+a^2)\div(x+a)=x+a.$$
$$(x^2-2ax+a^2)\div(x-a)=x-a.$$
$$(x^2-a^2)\div(x\pm a)=x\mp a.$$
$$(x^3\mp a^3)\div(x\mp a)=x^2\pm ax+a^2.$$
$$(x^4-a^4)\div(x\mp a)=x^3\pm ax^2+a^2x\pm a^3.$$
$$(x^4+a^2x^2+a^4)\div(x^2\mp ax+a^2)=x^2\pm ax+a^2.$$
$$(x^3+y^3+z^3-3xyz)\div(x+y+z)=x^2+y^2+z^2-yz-zx-xy.$$

EXAMPLES III.

1. Divide x^2-9y^2 by $x+3y$.
2. Divide x^4-16y^4 by x^2-4y^2.
3. Divide $27x^3+64y^3$ by $4y+3x$.
4. Divide $3x^2-4xy-4y^2$ by $2y-x$.
5. Divide $1-5x^4+4x^5$ by $1-x$.
6. Divide $x^5-5xy^4+4y^5$ by $x-y$.

7. Divide $1-6x^5+5x^6$ by $1-2x+x^2$.

8. Divide $m^6-6mn^5+5n^6$ by $m^2-2mn+n^2$.

9. Divide $1-7x^6+6x^7$ by $(1-x)^2$.

10. Divide $1-x^8$ by $1-x^2$.

11. Divide $1+x-8x^2+19x^3-15x^4$ by $1+3x-5x^2$.

12. Divide $4-9x^2+12x^3-4x^4$ by $2+3x-2x^2$.

13. Divide $4x^4-9x^2y^2+6xy^3-y^4$ by $2x^2+3xy-y^2$.

14. Divide $x^3-3x^2+3x+y^3-1$ by $x+y-1$.

15. Divide $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$ by x^2+xy+y^2.

16. Divide
$x^5-5x^4y+7x^3y^2-x^2y^3-4xy^4+2y^5$ by $x^3-3x^2y+3xy^2-y^3$.

17. Divide $a^2-2b^2-6c^2+ab-ac+7bc$ by $a-b+2c$.

18. Divide $a^2+2b^2-3c^2+bc+2ac+3ab$ by $a+b-c$.

19. Divide
$6a^4+4b^4-a^3b+13ab^3+2a^2b^2$ by $2a^2+4b^2-3ab$.

20. Divide $x^4+y^4-z^4+2x^2y^2+2z^2-1$ by $x^2+y^2-z^2+1$.

21. Divide $a^3-3a^2b+3ab^2-b^3-c^3$ by $a-b-c$.

22. Divide $a^3+8b^3-c^3+6abc$ by $a+2b-c$.

23. Divide
$a^3+8b^3+27c^3-18abc$ by $a^2+4b^2+9c^2-6bc-3ca-2ab$.

24. Divide $27a^3-8b^3-27c^3-54abc$ by $3a-2b-3c$.

25. Divide $acx^3+(ad-bc)x^2-(ac+bd)x+bc$ by $ax-b$.

26. Divide
$2a^2x^2-2(b-c)(3b-4c)y^2+abxy$ by $ax+2(b-c)y$.

27. Divide
$9a^2b^3-12a^4b+3b^5+2a^3b^2+4a^5-11ab^4$ by $3b^3+4a^3-2ab^2$.

28. Divide x^3+y^3 by $x+y$; and from the result *write down* the quotient of $(x+y)^3+z^3$ by $x+y+z$.

29. Divide x^3-y^3 by $x-y$; and hence *write down* the quotient of $(x+y)^3-8z^3$ by $x+y-2z$.

CHAPTER VI.

FACTORS.

75. **Definitions.** An algebraical expression which does not contain any *letter* in the denominator of any term is said to be an *integral expression:* thus $\frac{1}{2}a^3b - \frac{1}{4}b^3$ is an integral expression.

An expression is said to be *integral with respect to any particular letter*, when that letter does not occur in the denominator of any term: thus $\frac{x^2}{a} + \frac{x}{a+b}$ is integral *with respect to x.*

An expression is said to be *rational* when none of its terms contain square or other roots.

76. In the present chapter we shall shew how factors of algebraical expressions can be found in certain simple cases.

We shall only consider rational and integral expressions; and by the *factors* of an expression will be meant the rational and integral expressions, or the expressions which are rational and integral in some particular letter, which exactly divide it.

77. **Monomial Factors.** When some letter is common to all the terms of an expression, each term, and therefore the whole expression, is divisible by that letter.

Thus $\qquad 2ax + x^2 = x(2a + x),$

$$ax + a^2x^2 = ax(1 + ax),$$

and $\qquad 2a^3b^2x + 3a^2b^3y = a^2b^2(2ax + 3by).$

Such monomial factors, if there be any, are obvious on inspection.

78. Factors found by comparing with known identities. Sometimes an algebraical expression is of the same form as some known result of multiplication: in this case factors can be written down at once.

Thus, from the known identity

$$a^2 - b^2 = (a+b)(a-b),$$

we have

$$a^2 - 4b^2 = a^2 - (2b)^2 = (a+2b)(a-2b),$$

$$a^2 - 2 = a^2 - (\sqrt{2})^2 = (a+\sqrt{2})(a-\sqrt{2}),$$

$$a^4 - 16b^4 = (a^2)^2 - (4b^2)^2 = (a^2+4b^2)(a^2-4b^2)$$
$$= (a^2+4b^2)(a+2b)(a-2b),$$

and $\quad a^3 - 9ab^2 = a(a^2 - 9b^2) = a(a+3b)(a-3b).$

Again, from the identity

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2),$$

we have

$$a^3 + 8b^3 = a^3 + (2b)^3 = (a+2b)\{a^2 - a(2b) + (2b)^2\}$$
$$= (a+2b)(a^2 - 2ab + 4b^2),$$

$$8a^3 + 27b^6 = (2a)^3 + (3b^2)^3 = (2a+3b^2)\{(2a)^2 - (2a)(3b^2) + (3b^2)^2\}$$
$$= (2a+3b^2)(4a^2 - 6ab^2 + 9b^4),$$

and $\quad a^9 + x^9 = (a^3)^3 + (x^3)^3 = (a^3+x^3)(a^6 - a^3x^3 + x^6)$
$$= (a+x)(a^2 - ax + x^2)(a^6 - a^3x^3 + x^6).$$

And, from the identity

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2),$$

we have

$$a^3b^3 - \frac{1}{8}x^3y^3 = \left(ab - \frac{1}{2}xy\right)\left(a^2b^2 + \frac{1}{2}abxy + \frac{1}{4}x^2y^2\right).$$

The following are additional examples of the same principle:

(i) $\quad (a+b)^2 - (c+d)^2 = \{(a+b)+(c+d)\}\{(a+b)-(c+d)\}$
$$= (a+b+c+d)(a+b-c-d).$$

(ii) $4a^2b^2 - (a^2+b^2-c^2)^2 = \{2ab + (a^2+b^2-c^2)\}\{2ab - (a^2+b^2-c^2)\};$

and, since

$$2ab + a^2 + b^2 - c^2 = (a+b)^2 - c^2 = (a+b+c)(a+b-c),$$

and $\quad 2ab - a^2 - b^2 + c^2 = c^2 - (a-b)^2 = (c+a-b)(c-a+b),$

we have finally

$$4a^2b^2 - (a^2+b^2-c^2)^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

(iii) $(a+2b)^3-(2a+b)^3$

$=\{(a+2b)-(2a+b)\}\{(a+2b)^2+(a+2b)(2a+b)+(2a+b)^2\}$

$=(b-a)(7a^2+13ab+7b^2)$.

79. **Factors of x^2+px+q found by inspection.** From the identity

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

it follows conversely that expressions of the form

$$x^2+px+q$$

can sometimes, if not always, be expressed as the product of two factors of the form $x+a$, $x+b$.

We shall presently give a method by which two factors of x^2+px+q of the form $x+a$ and $x+b$ can always be found; but whenever a and b are *rational*, the factors can be more easily found by inspection. For, if $(x+a)(x+b)$, that is $x^2+(a+b)x+ab$, is the same as x^2+px+q, we must have $a+b=p$ and $ab=q$. Hence *a and b are such that their sum is p, and their product is q.*

For example, to find the factors of $x^2+7x+12$. The factors will be $x+a$ and $x+b$, where $a+b=7$ and $ab=12$. Hence we must find two numbers whose product is 12 and whose sum is 7: pairs of numbers whose product is 12 are 12 and 1, 6 and 2, and 4 and 3; and the sum of the last pair is 7. Hence $x^2+7x+12=(x+4)(x+3)$.

Again, to find the factors of $x^2-7x+10$. We have to find two numbers whose product is 10, and whose sum is -7. Since the product is $+10$, the two numbers are *both* positive or *both* negative; and since the sum is -7, they must both be negative. The pairs of negative numbers whose product is 10 are -10 and -1, and -5 and -2; and the sum of the last pair is -7. Hence $x^2-7x+10=(x-5)(x-2)$.

Again, to find the factors of $x^2+3x-18$. We have to find two numbers whose product is -18 and whose sum is 3. The pairs of numbers whose product is -18 are -18 and 1, -9 and 2, -6 and 3, -3 and 6, -2 and 9 and -1 and 18; and the sum of 6 and -3 is 3. Hence $x^2+3x-18=(x+6)(x-3)$.

It should be noticed that if the factors of x^2+px+q be $x+a$ and $x+b$, the factors of $x^2+pxy+qy^2$ will be $x+ay$ and $x+by$; also the factors of $(x+y)^2+p(x+y)z+qz^2$ will be $x+y+az$ and $x+y+bz$.

Hence from the above we have

$$x^2+7xy+12y^2=(x+4y)(x+3y),$$
$$x^2+3xy^2-18y^4=(x+6y^2)(x-3y^2),$$
$$(a+b)^2-7(a+b)x+10x^2=(a+b-5x)(a+b-2x),$$

and
$$x^4-5x^2+4=(x^2)^2-5x^2+4=(x^2-4)(x^2-1)$$
$$=(x+2)(x-2)(x+1)(x-1).$$

EXAMPLES IV.

Find the factors of the following expressions:

1. a^4-16b^4.
2. $16x^4-81a^4b^4$.
3. $16-(3a-2b)^2$.
4. $4y^2-(2z-x)^2$.
5. $20a^3x^3-45axy^2$.
6. $36a^2x^6-4a^2x^2y^4$.
7. $(3a^2-b^2)^2-(a^2-3b^2)^2$.
8. $(5a^2-3b^2)^2-(3a^2-5b^2)^2$
9. $(5x^2+2x-3)^2-(x^2-2x-3)^2$.
10. $(3x^2-4x-2)^2-(3x^2+4x-2)^2$.
11. $32a^3b^3-4b^9$.
12. $(a^2-2bc)^3-8b^3c^3$.
13. a^2-2a-8.
14. $x+12-x^2$.
15. $1-18x-63x^2$.
16. $8a-4a^2-4$.
17. $a^3b-4a^2b^2+3ab^3$.
18. $a^4b+5a^3b^2+4a^2b^3$.
19. $(b+c)^2-6a(b+c)+5a^2$.
20. $9(a+b)^2-6(a+b)(c+d)+(c+d)^2$.
21. x^4-29x^2+100.
22. $100x^4-29x^2y^2+y^4$.
23. $x^4-8x^2y^2z^2+16y^4z^4$.
24. $9a^6-10a^4b^2+a^2b^4$.
25. $x^2-2ax-b^2+2ab$.
26. $x^2+2xy-a^2-2ay$.
27. $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$.
28. $4(xy-ab)^2-(x^2+y^2-a^2-b^2)^2$.

80. Factors of general quadratic expression. We proceed to shew how to find the factors of any expression of the second degree in a particular letter, x suppose.

The most general quadratic expression [Art. 60] in x is $ax^2 + bx + c$, where a, b and c do not contain x.

The problem before us is to find two factors which are rational and integral *with respect to x*, and are therefore each of the first degree in x, but which are not necessarily, and not generally, rational and integral with respect to arithmetical numbers or to any other letters which may be involved in the expression.

The method of finding the factors of $ax^2 + bx + c$ consists in changing it into an equivalent expression which is *the difference of two squares.*

We first note that since $x^2 + 2ax + a^2$ is a perfect square, in order to complete the trinomial square of which x^2 and $2ax$ are the first two terms, we must add the square of a, that is, we must *add the square of half the coefficient of x.*

For example, x^2+5x is made a perfect square, namely $\left(x+\frac{5}{2}\right)^2$, by the addition of $\left(\frac{5}{2}\right)^2$; also x^2-px is made a perfect square, namely $\left(x-\frac{p}{2}\right)^2$, by the addition of $\left(-\frac{p}{2}\right)^2=\frac{p^2}{4}$.

81. To find the factors of $ax^2 + bx + c$.

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right).$$

Now $x^2 + \frac{b}{a}x$ is made a perfect square, namely $\left(x + \frac{b}{2a}\right)^2$, by the addition of $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$. And, by adding and subtracting $\frac{b^2}{4a^2}$ to the expression within brackets, we have

$$a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right)$$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)\right\}.$$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2-\left(\sqrt{\frac{b^2-4ac}{4a^2}}\right)^2\right\}.$$

Hence as the difference of any two squares is equal the product of their sum and difference, we have

$$ax^2+bx+c$$
$$=a\left\{x+\frac{b}{2a}+\sqrt{\frac{b^2-4ac}{4a^2}}\right\}\left\{x+\frac{b}{2a}-\sqrt{\frac{b^2-4ac}{4a^2}}\right\}$$

Thus the required factors have been found*.

Ex. 1. To find the factors of x^2+4x+3.

$$x^2+4x+3=x^2+4x+4-4+3=(x+2)^2-1=(x+2+1)(x+2-1)$$
$$=(x+3)(x+1).$$

Ex. 2. To find the factors of x^2-5x+3.

$$x^2-5x+3=x^2-5x+\left(-\frac{5}{2}\right)^2-\left(-\frac{5}{2}\right)^2+3=\left(x-\frac{5}{2}\right)^2-\frac{13}{4}$$
$$=\left(x-\frac{5}{2}+\sqrt{\frac{13}{4}}\right)\left(x-\frac{5}{2}-\sqrt{\frac{13}{4}}\right).$$

Ex. 3. To find the factors of $3x^2-4x+1$.

$$3x^2-4x+1=3\left(x^2-\frac{4}{3}x+\frac{1}{3}\right)=3\left\{x^2-\frac{4}{3}x+\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^2+\frac{1}{3}\right\}$$
$$=3\left\{\left(x-\frac{2}{3}\right)^2-\frac{1}{9}\right\}=3\left(x-\frac{2}{3}+\frac{1}{3}\right)\left(x-\frac{2}{3}-\frac{1}{3}\right)=3\left(x-\frac{1}{3}\right)(x-1)$$

Ex. 4. To find the factors of $x^2+2ax-b^2-2ab$.

$$x^2+2ax-b^2-2ab=x^2+2ax+a^2-a^2-b^2-2ab=(x+a)^2-(a+b)^2$$
$$=\{x+a+(a+b)\}\{x+a-(a+b)\}=(x+2a+b)(x-b).$$

* It will be proved later on [see Art. 91] that an expression containing x can be resolved into only one set of factors of the first degree in x.

82. Instead of working out every example from the beginning we may use the formula

$$ax^2+bx+c$$

$$=a\left\{x+\frac{b}{2a}+\sqrt{\frac{b^2-4ac}{4a^2}}\right\}\left\{x+\frac{b}{2a}-\sqrt{\frac{b^2-4ac}{4a^2}}\right\},$$

and we should then only have to substitute for a, b and c their values in the particular case under consideration.

Thus to find the factors of $3x^2-4x+1$. Here $a=3$, $b=-4$, $c=1$. Hence $\sqrt{\frac{b^2-4ac}{4a^2}}=\sqrt{\frac{16-12}{36}}=\sqrt{\frac{1}{9}}=\frac{1}{3}$; the expression is therefore equivalent to $3\left(x-\frac{2}{3}+\frac{1}{3}\right)\left(x-\frac{2}{3}-\frac{1}{3}\right)=3\left(x-\frac{1}{3}\right)(x-1)$.

83. We have from Art. 81

$$ax^2+bx+c$$

$$=a\left(x+\frac{b}{2a}+\sqrt{\frac{b^2-4ac}{4a^2}}\right)\left(x+\frac{b}{2a}-\sqrt{\frac{b^2-4ac}{4a^2}}\right).$$

Now, for particular values of a, b, c, $\frac{b^2-4ac}{4a^2}$ may be *positive, zero*, or *negative.*

I. Let $\frac{b^2-4ac}{4a^2}$ be *positive.* Then the two factors of ax^2+bx+c will be rational or irrational according as $\frac{b^2-4ac}{4a^2}$ is or is not a perfect square.

II. Let $\frac{b^2-4ac}{4a^2}$ be *zero.* Then

$$ax^2+bx+c=a\left(x+\frac{b}{2a}\right)\left(x+\frac{b}{2a}\right).$$

Hence ax^2+bx+c is a *perfect square in x*, if $b^2-4ac=0$.

III. Let $\frac{b^2-4ac}{4a^2}$ be *negative.* Then no positive or negative quantity can be found whose square will be equal to $\frac{b^2-4ac}{4a^2}$; for all squares, whether of positive or negative quantities, are *positive.*

Expressions of the form $\sqrt{-a}$, where a is positive, are called *imaginary*, and positive or negative quantities are distinguished from them by being called *real*.

We shall consider imaginary quantities at length in a subsequent chapter: for our present purpose it is sufficient to observe that they obey *all* the fundamental laws of Algebra; and this being the case, the formula of Art. 81 will hold good when $b^2 - 4ac$ is negative.

Note. For some purposes for which the factors of expressions are required, the only useful factors are those which are altogether rational: on this account irrational and imaginary factors are often not shewn. Thus, for example, the factorisation of $x^3 - 8$ is for many purposes complete in the form $(x-2)(x^2+2x+4)$*, the imaginary factors of x^2+2x+4, namely

$$x+1+\sqrt{-3} \text{ and } x+1-\sqrt{-3},$$

not being shewn.

84. We have in Art. 81 shewn how to resolve any expression of the second degree in a particular letter into two factors (real or imaginary) of the first degree in that letter.

It should be noted that the factors of the most general expression of the third degree, or of the fourth degree, can be found, although the methods are beyond the range of this book; expressions of higher degree than the fourth cannot however, except in a few special cases, be resolved into factors.

85. **Factors found by re-arrangement and grouping of terms.** The factors of many expressions can be found by a suitable re-arrangement and grouping of the terms.

For example

$$\begin{aligned} 1+ax-x^2-ax^3 &= 1+ax-x^2(1+ax) = (1+ax)(1-x^2) \\ &= (1+ax)(1+x)(1-x); \end{aligned}$$

* The reason of this will appear from Art. 178 and Art. 192.

or we may write the expression in the form

$$1-x^2+ax-ax^3=1-x^2+ax(1-x^2),$$

and the factors $1-x^2$, $1+ax$ are now obvious.

For the best arrangement or grouping no general rule can be given: the following cases are however of frequent occurrence and of great importance.

I. When one of the letters occurs only in the *first power*, the factors often become obvious when the expression is arranged according to powers of that letter.

Ex. 1. To find the factors of $ab+bc+cd+da$.

Arranged according to powers of a we have $a(b+d)+bc+cd$, which is at once seen to be $a(b+d)+c(b+d)=(a+c)(b+d)$.

Ex. 2. To find the factors of $x^2+(a+b+c)x+ab+ac$.

The expression $=a(x+b+c)+x^2+bx+cx=(a+x)(x+b+c)$.

Ex. 3. To find the factors of $ax^3+x+a+1$.

$$ax^3+x+a+1=a(x^3+1)+x+1=(x+1)\{a(x^2-x+1)+1\}.$$

Ex. 4. To find the factors of $a^2+2ab-2ac-3b^2+2bc$.

The given expression is of the first degree in c; we therefore write it in the form $a^2+2ab-3b^2-2c(a-b)$

$$=(a-b)(a+3b)-2c(a-b)=(a-b)(a+3b-2c).$$

II. When the expression is of the second degree with respect to *any one* of the letters; factors, which are rational and integral in *that letter*, can be found as in Art. 81.

Ex. 1. Find the factors of $a^2+3b^2-c^2+2bc-4ab$.

Arranging according to powers of a, we have

$$\begin{aligned} a^2-4ab+3b^2-c^2+2bc &= a^2-4ab+4b^2-4b^2+3b^2-c^2+2bc \\ &= (a-2b)^2-(b-c)^2=\{(a-2b)+(b-c)\}\{(a-2b)-(b-c)\} \\ &= (a-b-c)(a-3b+c). \end{aligned}$$

Ex. 2. Find the factors of $a^2-b^2-c^2+d^2-2(ad-bc)$.

The expression

$$\begin{aligned} &= a^2-2ad-b^2-c^2+d^2+2bc \\ &= a^2-2ad+d^2-b^2-c^2+2bc=(a-d)^2-(b-c)^2 \\ &= (a-d+b-c)(a-d-b+c). \end{aligned}$$

Ex. 3. Find the factors of $a^2+2ab-ac-3b^2+5bc-2c^2$.

The expression

$$=a^2+a(2b-c)-3b^2+5bc-2c^2$$

$$=a^2+a(2b-c)+\left(\frac{2b-c}{2}\right)^2-\left(\frac{2b-c}{2}\right)^2-3b^2+5bc-2$$

$$=\left(a+\frac{2b-c}{2}\right)^2-\frac{1}{4}\{4b^2-4bc+c^2+12b^2-20bc+8c^2\}$$

$$=\left(a+\frac{2b-c}{2}\right)^2-\frac{1}{4}(4b-3c)^2$$

$$=\left\{a+\frac{2b-c}{2}+\frac{1}{2}(4b-3c)\right\}\left\{a+\frac{2b-c}{2}-\frac{1}{2}(4b-3c)\right\}$$

$$=(a+3b-2c)(a-b+c).$$

Ex. 4. Find the factors of $x^4+x^2-2ax+1-a^2$.

Arranging according to powers of a, we have

$$-\{a^2+2ax-1-x^2-x^4\}=-\{a^2+2ax+x^2-1-2x^2-x^4\}$$

$$=-\{(a+x)^2-(1+x^2)^2\}=-(a+x+1+x^2)(a+x-1-x^2).$$

III. When the expression contains only two p
of a particular letter and one of those powers is the s
of the other, the method of Art. 81 is applicable.

Ex. 1. To find the factors of x^4-10x^2+9.

$$x^4-10x^2+9=x^4-10x^2+25-25+9=(x^2-5)^2-16$$

$$=(x^2-5+4)(x^2-5-4)=(x^2-9)(x^2-1)=(x+3)(x-3)(x+1)(x-$$

or thus:— $\quad x^4-10x^2+9=(x^2+3)^2-16x^2$

$$=(x^2+3+4x)(x^2+3-4x)=(x+3)(x+1)(x-3)(x-1).$$

Ex. 2. To find the factors of x^4+x^2+1.

Two real quadratic factors can be found as follows:

$$x^4+x^2+1=(x^2+1)^2-x^2=(x^2+1+x)(x^2+1-x).$$

Ex. 3. To find the factors of x^6-28x^3+27.

$$x^6-28x^3+27=x^6-28x^3+14^2-14^2+27=(x^3-14)^2-13^2$$

$$=(x^3-1)(x^3-27)=(x-1)(x-3)(x^2+x+1)(x^2+3x+9).$$

In this case, and also in Ex. 1, two factors can be seen
inspection, as in Art. 79.

Ex. 4. To find the factors of $a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2$.

Arranging according to powers of a, we have

$$a^4-2a^2(b^2+c^2)+b^4+c^4-2b^2c^2$$
$$=a^4-2a^2(b^2+c^2)+(b^2+c^2)^2-(b^2+c^2)^2+b^4+c^4-2b^2c^2$$
$$=\{a^2-(b^2+c^2)\}^2-4b^2c^2=(a^2-b^2-c^2-2bc)(a^2-b^2-c^2+2bc)$$
$$=\{a^2-(b+c)^2\}\{a^2-(b-c)^2\}$$
$$=(a+b+c)(a-b-c)(a-b+c)(a+b-c).$$

IV. Two factors of aP^2+bP+c, where P is any expression which contains x, can always be found by the method of Art. 81; for we have

$$aP^2+bP+c$$
$$=a\left(P+\frac{b}{2a}+\sqrt{\frac{b^2-4ac}{4a^2}}\right)\left(P+\frac{b}{2a}-\sqrt{\frac{b^2-4ac}{4a^2}}\right).$$

Ex. 1. To find the factors of $(x^2+x)^2+4(x^2+x)-12$.

Since $\qquad P^2+4P-12=(P-2)(P+6),$

the given expression $\qquad =(x^2+x-2)(x^2+x+6)$

$\qquad =(x+2)(x-1)(x^2+x+6),$

the factors of x^2+x+6 being imaginary [see Art. 83, Note].

Ex. 2. To find the factors of $(x^2+x+4)^2+8x(x^2+x+4)+15x^2$.

The given expression $\qquad =\{(x^2+x+4)+3x\}\{(x^2+x+4)+5x\}$

$\qquad =(x^2+4x+4)(x^2+6x+4)$

$\qquad =(x+2)^2(x^2+6x+4).$

Ex. 3. To find the factors of

$$2(x^2+6x+1)^2+5(x^2+6x+1)(x^2+1)+2(x^2+1)^2.$$

Since $\qquad 2P^2+5PQ+2Q^2=(P+2Q)(2P+Q),$

the given expression

$$=\{(x^2+6x+1)+2(x^2+1)\}\{2(x^2+6x+1)+x^2+1\}$$
$$=(3x^2+6x+3)(3x^2+12x+3)$$
$$=9(x+1)^2(x^2+4x+1).$$

Ex. 4. To find the factors of $(x^2+x+1)(x^2+x+2)-12$.

The given expression $\qquad =(x^2+x)^2+3(x^2+x)-10$

$\qquad =(x^2+x-2)(x^2+x+5)$

$\qquad =(x+2)(x-1)(x^2+x+5).$

EXAMPLES V.

Find the factors of the following expressions:

1. $x^3 + ax^2 - x - a.$
2. $ac - bd - ad + bc.$
3. $ac^2 + bd^2 - ad^2 - bc^2.$
4. $acx^2 + (bc + ad)\,xy + bdy^2.$
5. $acx^3 + bcx^2 + adx + bd.$
6. $(a + b)^2 + (a + c)^2 - (c + d)^2 - (b + d)^2.$
7. $a^4 + a^3b - ab^3 - b^4.$
8. $a^4 - a^3b - ab^3 + b^4.$
9. $a^2b^2 - a^2 - b^2 + 1.$
10. $x^2y^2 - x^2z^2 - y^2z^2 + z^4.$
11. $x^2y^2z^2 - x^2z - y^2z + 1.$
12. $x^4 + x^3y + xz^3 + yz^3.$
13. $x\,(x + z) - y\,(y + z).$
14. $x^4 - 7x^2 - 18.$
15. $x^4 - 23x^2 + 1.$
16. $x^4 - 14x^2y^2 + y^4.$
17. $x^8 + x^4 + 1.$
18. $x^4 - 2\,(a^2 + b^2)\,x^2 + (a^2 - b^2)^2.$
19. $x^4 - 4x^2y^2z^2 + 4y^4z^4.$
20. $x^2 - 2\,(a + b)\,x - ab\,(a - 2)\,(b + 2).$
21. $x^3 + bx^2 + ax + ab.$
22. $(1 + y)^2 - 2x^2\,(1 + y^2) + x^4\,(1 - y)^2.$
23. $x^2 - y^2 - 3z^2 - 2xz + 4yz.$

24. $2y^2 - 5xy + 2x^2 - ay - ax - a^2$.

25. $a^2 - 3b^2 - 3c^2 + 10bc - 2ca - 2ab$.

26. $2a^2 - 7ab - 22b^2 - 5a + 35b - 3$.

27. $1 + (b - a^2)x^2 - abx^3$.

28. $1 - 2ax - (c - a^2)x^2 + acx^3$.

29. $a^2(b - c) + b^2(c - a) + c^2(a - b)$.

30. $b^2c + bc^2 + c^2a + ca^2 + a^2b + ab^2 + 2abc$.

31. $a^2b - ab^2 + a^2c - ac^2 - 2abc + b^2c + bc^2$.

32. $x^3(a + 1) - xy(x - y)(a - b) + y^3(b + 1)$.

33. $ax(y^3 + b^3) + by(bx^3 + a^3y)$.

34. $2x^3 - 4x^2y - x^2z + 2xy^2 + 2xyz - y^2z$.

35. $xyz(x^3 + y^3 + z^3) - y^3z^3 - z^3x^3 - x^3y^3$.

36. $(x^2 + x)^2 - 14(x^2 + x) + 24$.

37. $(x^2 + 4x + 8)^2 + 3x(x^2 + 4x + 8) + 2x^2$.

38. $(x + 1)(x + 2)(x + 3)(x + 4) - 24$.

39. $(x + 1)(x + 3)(x + 5)(x + 7) + 15$.

40. $4(x + 5)(x + 6)(x + 10)(x + 12) - 3x^2$.

86. **Theorem.** *The expression $x^n - a^n$ is divisible by $x - a$, for all positive integral values of n.*

It is known that $x - a$, $x^2 - a^2$ and $x^3 - a^3$ are all divisible by $x - a$.

We have
$$x^n - a^n = x^n - ax^{n-1} + ax^{n-1} - a^n$$
$$= x^{n-1}(x - a) + a(x^{n-1} - a^{n-1}).$$

Now *if* $x - a$ divides $x^{n-1} - a^{n-1}$ it will also divide $x^{n-1}(x - a) + a(x^{n-1} - a^{n-1})$, that is, it will divide $x^n - a^n$.

Hence, *if* $x - a$ divides $x^{n-1} - a^{n-1}$ it will also divide $x^n - a^n$.

But we know that $x-a$ divides x^3-a^3; it will therefc also divide x^4-a^4. And, since $x-a$ divides x^4-a^4 it w also divide x^5-a^5. And so on indefinitely.

Hence x^n-a^n is divisible by $x-a$, when n is a positive integer.

87. Since $x^n+a^n=x^n-a^n+2a^n$ it follows from t last Article that when x^n+a^n is divided by $x-a$ t remainder is $2a^n$, so that x^n+a^n is *never* divisible by $x-$

If we change a into $-a$, $x-a$ becomes $x-(-a)=x+$ also x^n-a^n becomes $x^n-(-a)^n$, and $x^n-(-a)^n$ is x^n+ or x^n-a^n according as n is odd or even.

Hence, when n is *odd*

$$x^n+a^n \text{ is divisible by } x+a,$$

and when n is *even*

$$x^n-a^n \text{ is divisible by } x+a.$$

Thus, n being any positive integer,

$$x-a \text{ divides } x^n-a^n \text{ always,}$$

$$x-a \quad ,, \quad x^n+a^n \text{ never,}$$

$$x+a \quad ,, \quad x^n-a^n \text{ when } n \text{ is even,}$$

and

$$x+a \quad ,, \quad x^n+a^n \text{ when } n \text{ is odd.}$$

The above results may be written so as to shew tl quotients: thus

$$\frac{x^n-a^n}{x-a}=x^{n-1}+x^{n-2}a+x^{n-3}a^2+\ldots\ldots+a^{n-1},$$

$$\frac{x^n \pm a^n}{x+a}=x^{n-1}-x^{n-2}a+x^{n-3}a^2-\ldots\ldots \pm a^{n-1},$$

the upper or lower signs being taken on each side of tl second formula according as n is odd or even.

88. **Theorem.** *If any rational and integral expre sion which contains x vanish when a is put for x, then w $x-a$ be a factor of the expression.*

Let the expression, arranged according to powers of x, be

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots\ldots$$

Then, by supposition,

$$a\alpha^n + b\alpha^{n-1} + c\alpha^{n-2} + \ldots\ldots\ldots \quad = 0.$$

Hence $\quad ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots\ldots$

$$= ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots - (a\alpha^n + b\alpha^{n-1} + c\alpha^{n-2} + \ldots)$$

$$= a(x^n - \alpha^n) + b(x^{n-1} - \alpha^{n-1}) + c(x^{n-2} - \alpha^{n-2}) + \ldots\ldots$$

But, by the last Article, $x^n - \alpha^n$, $x^{n-1} - \alpha^{n-1}$, $x^{n-2} - \alpha^{n-2}$, &c. are all divisible by $x - \alpha$.

Hence also $ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots\ldots$ is divisible by $x - \alpha$.

The proposition may also be proved in the following manner.

Divide the expression $ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots$ by $x - \alpha$, continuing the process until the remainder, if there be any remainder, does not contain x; and let Q be the quotient and R the remainder.

Then, by the nature of division,

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots = Q(x - \alpha) + R,$$

and this relation is true for *all* values of x.

Now since R does not contain x, no change will be made in R by changing the value of x: put then $x = \alpha$, and we have

$$a\alpha^n + b\alpha^{n-1} + c\alpha^{n-2} + \ldots\ldots = Q(\alpha - \alpha) + R = R.$$

Hence, *if any expression rational and integral in x be divided by $x - \alpha$, the remainder is equal to the result obtained by putting α in the place of x in the expression.*

It therefore follows that the necessary and sufficient condition that an expression rational and integral in x may be exactly divisible by $x - \alpha$ is that the expression should vanish when α is substituted for x.

Ex. 1. Find the remainder when $x^3 - 4x^2 + 2$ is divided by $x - 2$.

The remainder $= 2^3 - 4 . 2^2 + 2 = -6$.

Ex. 2. Find the remainder when $x^3 - 2a^2x + a^3$ is divided by $x - a$

The remainder is $a^3 - 2a^3 + a^3 = 0$, so that $x^3 - 2a^2x + a^3$ is divisil by $x - a$.

Ex. 3. Shew by substitution that $x - 1$, $x - 5$, $x + 2$ and $x + 4$ a factors of $x^4 - 23x^2 - 18x + 40$.

Ex. 4. Shew by substitution that $a - b$ is a factor of

$$a^3(b-c) + b^3(c-a) + c^3(a-b).$$

Put $a = b$ and the expression becomes $a^3(a-c) + a^3(c-a)$, whi is clearly zero: this proves that $a - b$ is a factor.

Ex. 5. Shew that a is a factor of

$$(a+b+c)^3 - (-a+b+c)^3 - (a-b+c)^3 - (a+b-c)^3.$$

89. We have proved that $x - a$ is a factor of th expression $ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots$, provided that th expression vanishes when a is put for x.

If the division were actually performed it is clear tha the first term of the quotient, which is the term of th highest degree in x, would be ax^{n-1}. Hence the given ex pression is equivalent to

$$(x - a)(ax^{n-1} + \&c. \ldots\ldots).$$

Now suppose that the given expression also vanishe when $x = \beta$; then the product of $x - a$ and $ax^{n-1} + \ldots\ldots$ will vanish when $x = \beta$; and since $x - a$ does not vanis when $x = \beta$, it follows that $ax^{n-1} + \ldots\ldots$ must vanis when $x = \beta$. Hence $x - \beta$ is a factor of $ax^{n-1} + \&c.$; an if the division were performed, it is clear that the firs term of the quotient would be ax^{n-2}.

Hence the original expression is equivalent to

$$(x - a)(x - \beta)(ax^{n-2} + \&c. \ldots\ldots).$$

Similarly, if the original expression vanishes also fo the values γ, δ, &c. of x, it must be equivalent to

$$(x - a)(x - \beta)(x - \gamma)(x - \delta) \ldots\ldots (ax^{n-r} + \&c. \ldots\ldots),$$

where r is equal to the number of the factors $x-\alpha$, $x-\beta$, &c.

If therefore the given expression vanishes for n values α, β, γ, &c. there will be n factors such as $x-\alpha$, and the remaining factor, $ax^{n-r}+$ &c. will reduce to a; and hence the given expression is equivalent to

$$a(x-\alpha)(x-\beta)(x-\gamma)\ldots\ldots$$

COR. If any of the factors $x-\alpha$, $x-\beta$, ... occur more than once in $ax^n+bx^{n-1}+\ldots$, it can similarly be proved that the expression is equivalent to $a(x-\alpha)^p(x-\beta)^q\ldots$, the factors $x-\alpha$, $x-\beta$, ... occurring respectively $p, q, \ldots$ times, and $p+q+\ldots=n$.

90. **Theorem.** *An expression of the nth degree in x cannot vanish for more than n values of x.*

For if the expression

$$ax^n+bx^{n-1}+cx^{n-2}+\ldots\ldots$$

vanishes for the n values $\alpha, \beta, \gamma\ldots\ldots$, it must be equivalent to

$$a(x-\alpha)(x-\beta)(x-\gamma)\ldots\ldots$$

If now we substitute any value, k suppose, different from each of the values α, β, γ, &c.; then, since no one of the factors $k-\alpha$, $k-\beta$, &c. is zero, their continued product cannot be zero, and therefore the given expression cannot vanish for the value $x=k$, *except a itself is zero.*

But, if a is zero, the original expression reduces to $bx^{n-1}+cx^{n-2}+\ldots\ldots$, and is of the $(n-1)^{\text{th}}$ degree; and hence as before it can only vanish for $n-1$ values of x, except b is zero. And so on.

Thus an expression of the nth degree in x *cannot vanish for more than n values of x, except the coefficients of all the powers of x are zero;* and when all these coefficients are zero, the expression will clearly vanish for *all values* of x.

91. **Theorem.** *If two expressions of the nth de in x be equal to one another for more than n values (they will be equal for all values of x.*

If the two expressions of the nth degree in x

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots,$$

and
$$px^n + qx^{n-1} + rx^{n-2} + \ldots\ldots,$$

be equal to one another for more than n values of follows that their difference, namely the expression

$$(a-p)\,x^n + (b-q)\,x^{n-1} + (c-r)\,x^{n-2} + \ldots\ldots,$$

will vanish for more than n values of x.

Hence, by Art. 90, the coefficients of all the diffe powers of x must be zero.

Thus $\quad a-p=0,\ b-q=0,\ c-r=0,$ &c.

that is, $\quad a=p,\ b=q,\ c=r,$ &c.

Hence, *if two expressions of the nth degree in x equal to one another for more than n values of x coefficient of any power of x in one expression is equal t coefficient of the same power of x in the other expressio*

When any two expressions, which have a lin number of terms, are equal to one another for *all v* of the letters involved, the above condition is cl satisfied, for the number of values must be greater tha index of the highest power of any contained letter.

Hence when any two expressions, which ha limited number of terms, are equal to one another fc *values* of the letters involved in them, we may *equa coefficients of the different powers of any letter.*

92. **Theorem.** *A rational integral expression taining x can be resolved into only one set of factors o first degree in x.*

For, if it be possible, let the expression $ax^n + bx^{n-1}$ be equivalent to

$$a(x-\alpha)^p(x-\beta)^q \ldots, \text{ and also to } a(x-\xi)^l(x-\eta)^m$$

Put $x=\alpha$ in both expressions; then $a(\alpha-\xi)^l(\alpha-\eta)^m\ldots$ must vanish, and therefore one at least of the quantities $\xi, \eta, \ldots$ must be equal to α. Let $\xi=\alpha$; remove one factor $x-\alpha$ from both expressions, and proceed as before. We thus prove that every factor of one expression occurs to as high a power in the other expression; the two expressions must therefore be identical.

93. **Cyclical order.** It is of importance for the student to attend to the way in which expressions are usually arranged. Consider, for example, the arrangement of the expression $bc+ca+ab$. The term which does not contain the letter a is put first, and the other terms can be obtained in succession by a *cyclical change of the letters*, that is by changing a into b, b into c and c into a. In the expression $a^2(b-c)+b^2(c-a)+c^2(a-b)$ the same arrangement is observed; for by making a cyclical change in the letters of $a^2(b-c)$ we obtain $b^2(c-a)$, and another cyclical change will give $c^2(a-b)$. So also the second and third factors of $(b-c)(c-a)(a-b)$ are obtained from the first by cyclical changes.

94. **Symmetrical expressions.** An expression which is unaltered by interchanging any pair of the letters which it contains is said to be a *symmetrical* expression. Thus $a+b+c$, $bc+ca+ab$, $a^3+b^3+c^3-3abc$ are symmetrical expressions.

Expressions which are unaltered by a *cyclical change* of the letters involved in them are called *cyclically symmetrical* expressions. For example, the expression

$$(b-c)(c-a)(a-b)$$

is a cyclically symmetrical expression since it is unaltered by changing a into b, b into c, and c into a.

It is clear that the product, or the quotient, of two symmetrical expressions is symmetrical, for if neither of two expressions is altered by an interchange of two letters their product, or their quotient, cannot be altered by such interchange.

It is also clear that the product, or the quotien two cyclically symmetrical expressions is cyclically metrical.

Ex. 1. Find the factors of $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

If we put $b=c$ in the expression

$$a^2(b-c)+b^2(c-a)+c^2(a-b) \text{ ....}$$

it is easy to see that the result is zero.

Hence $b-c$ is a factor of (i), and we can prove in a si manner that $c-a$ and $a-b$ are factors.

Now (i) is an expression of the *third* degree; it can therefore have *three* factors.

Hence (i) is equal to

$$L(b-c)(c-a)(a-b) \text{ ....(}$$

where L is some *number*, which is always the same for all valu a, b, c.

By comparing the coefficients [See Art. 91] of a^2 in (i) and (i see that $L=-1$.

We can also find L by giving particular values to a, b a Thus, let $a=0$, $b=1$, $c=2$; then (i) is equal to -2, and (ii) is to $2L$, and hence as before $L=-1$.

Ex. 2. Find the factors of $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

As in the preceding example, $(b-c)$, $(c-a)$ and $(a-b)$ ar factors of

$$a^3(b-c)+b^3(c-a)+c^3(a-b) \text{ ....(}$$

Now the given expression is of the *fourth* degree; hence, be the three factors already found, there must be one other fact the *first* degree, and this factor must be *symmetrical* in a, b, must therefore be $a+b+c$.

Hence the given expression must be equal to

$$L(b-c)(c-a)(a-b)(a+b+c) \text{ ....(i}$$

where L is a number.

By comparing the coefficients of a^3 in (i) and in (ii) we see $L=-1$; hence

$$a^3(b-c)+b^3(c-a)+c^3(a-b)=-(b-c)(c-a)(a-b)(a+b+c).$$

We can also find L by giving particular values to a, b, and c.

Thus, let $a=0$, $b=1$, $c=2$; then (i) is equal to -6 and (i equal to $6L$, so that $L=-1$.

We may also proceed as follows:

Arrange the expression according to powers of a; thus

$$a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2).$$

It is now obvious that $b-c$ is a factor, and we have

$$(b-c)\{a^3-a(b^2+bc+c^2)+bc(b+c)\}$$
$$=(b-c)\{b^2(c-a)+b(c^2-ac)+a^3-ac^2\}$$
$$=(b-c)(c-a)\{b^2+bc-a^2-ac\}=-(b-c)(c-a)(a-b)(a+b+c).$$

Ex. 3. Find the factors of $b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$.

By putting $b=c$ in the expression

$$b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)\ldots\ldots\ldots\ldots\text{(i)},$$

it is easy to see that the result is zero; hence $b-c$ is a factor of (i).

So also $c-a$ and $a-b$ are factors.

The given expression being of the *fifth* degree, there must be, besides the three factors $b-c$, $c-a$, $a-b$, another factor of the *second* degree; also, since this factor must be symmetrical in a, b, c, it must be of the form $L(a^2+b^2+c^2)+M(bc+ca+ab)$.

Thus (i) is equal to

$$(b-c)(c-a)(a-b)\{La^2+Lb^2+Lc^2+Mbc+Mca+Mab\}\ldots\text{(ii)}.$$

Equating coefficients of a^4 in (i) and in (ii) we see that $L=0$; and then equating coefficients of b^3c^2 we see that $M=-1$. Hence (i) is equal to

$$-(b-c)(c-a)(a-b)(bc+ca+ab).$$

We may also proceed as follows.

Arranging according to powers of a, the factor $b-c$ which does not contain a becomes obvious; then, arranging according to powers of b, the factor $c-a$ which does not contain b becomes obvious; and so on. Thus

$$b^2c^2(b-c)-a^2(b^3-c^3)+a^3(b^2-c^2)$$
$$=(b-c)\{b^2c^2-a^2(b^2+bc+c^2)+a^3(b+c)\}$$
$$=(b-c)\{b^2(c^2-a^2)+a^2b(a-c)+a^2c(a-c)\}$$
$$=(b-c)(c-a)\{b^2(c+a)-a^2b-a^2c\}$$
$$=(b-c)(c-a)\{(b^2-a^2)c+b^2a-a^2b\}$$
$$=-(b-c)(c-a)(a-b)(bc+ca+ab).$$

EXAMPLES VI.

Find the factors of the following expressions:

1. $(y-z)^3+(z-x)^3+(x-y)^3$.
2. $(y-z)^5+(z-x)^5+(x-y)^5$.
3. $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$.
4. $a(b-c)^3+b(c-a)^3+c(a-b)^3$.

5. $a(b-c)^5+b(c-a)^5+c(a-b)^5.$

6. $bc(b-c)+ca(c-a)+ab(a-b).$

7. $b^3c^3(b-c)+c^3a^3(c-a)+a^3b^3(a-b).$

8. $a^4(b-c)+b^4(c-a)+c^4(a-b).$

9. $a^5(b-c)+b^5(c-a)+c^5(a-b).$

10. $(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3.$

11. $(a+b+c)^5-(b+c-a)^5-(c+a-b)^5-(a+b-c)^5.$

12. $a(b+c-a)^2+b(c+a-b)^2+c(a+b-c)^2$
$+(b+c-a)(c+a-b)(a+b$

13. $a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)$
$-(b+c-a)(c+a-b)(a+b$

14. $(b+c-a)(c+a-b)(a+b-c)+a(a-b+c)(a+b$
$+b(a+b-c)(-a+b+c)+c(-a+b+c)(a-b$

15. $(b-c)(a-b+c)(a+b-c)+(c-a)(a+b-c)(-a+b$
$+(a-b)(-a+b+c)(a-b$

16. $(x+y+z)^3-x^3-y^3-z^3.$

17. $(x+y+z)^5-x^5-y^5-z^5.$

18. $(b-c)(b+c)^2+(c-a)(c+a)^2+(a-b)(a+b)^2.$

19. $(b-c)(b+c)^3+(c-a)(c+a)^3+(a-b)(a+b)^3.$

20. $(b-c)(b+c)^4+(c-a)(c+a)^4+(a-b)(a+b)^4.$

21. $a^3+b^3+c^3+5abc-a(a-b)(a-c)-b(b-c)(b-a)$
$-c(c-a)(c-$

22. $a^2(a+b)(a+c)(b-c)+b^2(b+c)(b+a)(c-a)$
$+c^2(c+a)(c+b)(a-$

23. $(y+z)(z+x)(x+y)+xyz.$

24. $a^2(b+c)^2+b^2(c+a)^2+c^2(a+b)^2+abc(a+b+c)$
$+(a^2+b^2+c^2)(bc+ca+$

25. $(x+y+z)^4-(y+z)^4-(z+x)^4-(x+y)^4+x^4+y^4+z^4.$

26. $a^2(b+c-2a)+b^2(c+a-2b)+c^2(a+b-2c)$
$+2(c^2-a^2)(c-b)+2(a^2-b^2)(a-c)+2(b^2-c^2)(b-$

27. $(b+c-a-d)^4(b-c)(a-d)+(c+a-b-d)^4(c-a)(b-d)$
$+(a+b-c-d)^4(a-b)(c-d).$

28. Shew that

$$12\{(x+y+z)^{2n}-(y+z)^{2n}-(z+x)^{2n}-(x+y)^{2n}+x^{2n}+y^{2n}+z^{2n}\}$$

is divisible by

$$(x+y+z)^4-(y+z)^4-(z+x)^4-(x+y)^4+x^4+y^4+z^4.$$

29. Shew that

$$a^3(b+c-a)^2+b^3(c+a-b)^2+c^3(a+b-c)^2+abc(a^2+b^2+c^2)$$
$$+(a^2+b^2+c^2-bc-ca-ab)(b+c-a)(c+a-b)(a+b-c)$$
$$=2abc(bc+ca+ab).$$

30. Shew that

$$(b-c)^6+(c-a)^6+(a-b)^6-9(b-c)^2(c-a)^2(a-b)^2$$
$$=2(a-b)^3(a-c)^3+2(b-c)^3(b-a)^3+2(c-a)^3(c-b)^3.$$

31. Shew that

$$(b+c)^3+(c+a)^3+(a+b)^3+(a+d)^3+(b+d)^3+(c+d)^3$$
$$=3(a+b+c+d)(a^2+b^2+c^2+d^2).$$

32. Reduce to its simplest form

$$4(a^2+ab+b^2)^3-(a-b)^2(a+2b)^2(2a+b)^2.$$

33. Shew that

$$a^4(b^2+c^2-a^2)^3+b^4(c^2+a^2-b^2)^3+c^4(a^2+b^2-c^2)^3$$

is divisible by

$$a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2.$$

34. Resolve into quadratic factors

$$4\{cd(a^2-b^2)+ab(c^2-d^2)\}^2+\{(a^2-b^2)(c^2-d^2)-4abcd\}^2.$$

35. Shew that

$$(y^2-z^2)(1+xy)(1+xz)+(z^2-x^2)(1+yz)(1+yx)$$
$$+(x^2-y^2)(1+zx)(1+zy)=(y-z)(z-x)(x-y)(xyz+x+y+z).$$

36. Find the factors of

$$a^3(b-c)(c-d)(d-b)-b^3(c-d)(d-a)(a-c)$$
$$+c^3(d-a)(a-b)(b-d)-d^3(a-b)(b-c)(c-a).$$

37. Find the factors of

$$b^2c^2d^2(b-c)(c-d)(d-b)-c^2d^2a^2(c-d)(d-a)(a-c)$$
$$+d^2a^2b^2(d-a)(a-b)(b-d)-a^2b^2c^2(a-b)(b-c)(c-a).$$

CHAPTER VII.

HIGHEST COMMON FACTORS. LOWEST COMMON MULTIPLES.

95. A *Common Factor* of two or more integral alg braical expressions is an integral expression which wi exactly divide each of them.

The *Highest Common Factor* of two or more integra expressions is the integral expression of *highest dimension* which will exactly divide each of them.

It is usual to write H.C.F. instead of Highest Commo Factor.

96. **The highest common factor of monomia expressions.** The highest common factor of two or mor monomial expressions can be found by inspection.

Thus, to find the highest common factor of a^3b^2c and $a^4b^4c^3$.

The highest power of a which will divide both expressions i a^3; the highest power of b is b^2; and the highest power of c is c and no other letters are common. Hence the H.C.F. is a^3b^2c.

Again, to find the highest common factor of $a^3b^4c^4$, a^2b^3 and a^5bc

The highest power of a which will divide all three expression is a^2; the highest power of b which will divide them all is b; and will not divide all the expressions. Hence the H.C.F. is a^2b.

From the above examples it will be seen that th H.C.F. of two or more monomial expressions is *the produc of each letter which is common to all the expressions take to the lowest power in which it occurs.*

97. **Highest common factor of multinomial expressions whose factors are known.** When the factors of two or more multinomial expressions are known, their H.C.F. can be at once written down, as in the preceding case. The H.C.F. will be *the product of each factor which is common to all the expressions taken to the lowest power in which it occurs.*

Thus, to find the H.C.F. of

$$(x-2)^3(x-1)^2(x-3) \text{ and } (x-2)^2(x-1)(x-3)^3.$$

It is clear that both expressions are divisible by $(x-2)^2$, but by no higher power of $x-2$. Also both expressions are divisible by $x-1$, but by no higher power of $x-1$; and both expressions are divisible by $x-3$, but by no higher power of $x-3$. Hence the H.C.F. is $(x-2)^2(x-1)(x-3)$.

Again, the H.C.F. of $a^2b^3(a-b)^2(a+b)^3$ and $a^3b^2(a-b)(a+b)^2$ is $a^2b^2(a-b)(a+b)^2$.

In the following examples the factors can be seen by inspection, and hence the H.C.F. can be written down.

Ex. 1. Find the H.C.F. of $a^4b^2-a^2b^4$ and $a^4b^3+a^3b^4$.

Ans. $a^2b^2(a+b)$.

Ex. 2. Find the H.C.F. of $a^6b^2-4a^4b^4$ and $a^6b^2-16a^2b^6$.

Ans. $a^2b^2(a^2-4b^2)$.

Ex. 3. Find the H.C.F. of $a^3+3a^2b+2ab^2$ and $a^4+6a^3b+8a^2b^2$.

Ans. $a(a+2b)$.

98. Although we cannot, in general, find the factors of a multinomial expression of higher degree than the second [Art. 84], there is no difficulty in finding the highest *common factor* of any *two* multinomial expressions. The process is analogous to that used in arithmetic to find the greatest common measure of two numbers.

If the expressions have monomial factors, they can be seen by inspection; and the highest common factors of these monomial factors can also be seen by inspection: we have therefore only to find the *multinomial* expression of highest dimensions which is common to the two given expressions.

Let A and B stand for the two expressions, which are supposed to be arranged according to descending powers of some common letter, and let A be of not higher degree than B in that letter. Divide B by A, and let the quotient be Q and the remainder R; then

$$B = AQ + R;$$
$$\therefore\ R = B - AQ.$$

Now an expression is exactly divisible by any other if each of its terms is so divisible; and therefore B is divisible by every common factor of A and R, and R is divisible by every common factor of A and B. Hence the common factors of A and B are exactly the same as the common factors of A and R; and therefore the H.C.F. of A and R is the H.C.F. required.

Now divide A by R, and let the remainder be S; then the H.C.F. of R and S will similarly be the same as the H.C.F. of A and R, and will therefore be the H.C.F. required.

And, if this process be continued to any extent, *the* H.C.F. *of any divisor and the corresponding dividend will always be the* H.C.F. *required.*

If at any stage there is no remainder, the divisor must be a factor of the corresponding dividend, and that divisor is clearly the H.C.F. of itself and the corresponding dividend. It must therefore be the H.C.F. required.

It should be remarked that by the nature of division the remainders are successively of lower and lower dimensions; and hence, unless the division leaves no remainder at some stage, we must at last come to a remainder which does not contain the common letter, in which case the given expressions have no H.C.F. containing that letter.

Since the process we are considering is only to be used to find the highest common *multinomial* factor, it is clear that any of the expressions which occur may be divided or multiplied by any *monomial* expression without destroying the validity of the process; for the multinomial factors will not be affected by such multiplication or division.

Thus the H.C.F. of two expressions can be found by the following

Rule :—*Arrange the two expressions according to descending powers of some common letter, and divide the expression which is of the highest degree in the common letter by the other (if both expressions are of the same degree it is immaterial which is used as the divisor). Take the remainder, if any, after the first division for a new divisor, and the former divisor as dividend; and continue the process until there is no remainder. The last divisor will be the* H.C.F. *required. The process is not used for finding common monomial factors, these can be seen by inspection; and any divisor, dividend, or remainder which occurs may be multiplied or divided by any monomial expression.*

Ex. 1. Find the H.C.F. of x^3+x^2-2 and x^3+2x^2-3.

$$\begin{array}{rl}
x^3+x^2-2\,) & x^3+2x^2-3\ (1 \\
 & \underline{x^3+\ x^2-2} \\
x^2-1\,) & x^3+x^2-2\ (x+1 \\
 & \underline{x^3-x} \\
 & x^2+x-2 \\
 & \underline{x^2\quad\ -1} \\
x-1\,) & x^2-1\ (x+1 \\
 & \underline{x^2-x} \\
 & x-1 \\
 & \underline{x-1}
\end{array}$$

Thus the H.C.F. is $x-1$.

The work would be shortened by noticing that the factors of the first remainder, namely x^2-1, are $x-1$ and $x+1$. And by means of Art. 88 it is at once seen that $x-1$ is, and that $x+1$ is not, a factor of x^3+x^2-2.

Ex. 2. Find the H.C.F. of

$$x^3+4x^2y-8xy^2+24y^3 \text{ and } x^5-x^4y+8x^2y^3-8xy^4.$$

The second expression is divisible by x, which is clearly not a *common* factor: we have therefore to find the H.C.F. of the first expression and $x^4-x^3y+8xy^3-8y^4$.

$$\begin{array}{rl}
x^3+4x^2y-8xy^2+24y^3\,) & x^4-\ x^3y+8xy^3-\ 8y^4\ (x-5y \\
 & \underline{x^4+4x^3y-8x^2y^2+24xy^3} \\
 & -5x^3y+8x^2y^2-16xy^3-8y^4 \\
 & \underline{-5x^3y-20x^2y^2+40xy^3-120y^4} \\
 & 28x^2y^2-56xy^3+112y^4
\end{array}$$

The remainder $=28y^2(x^2-2xy+4y^2)$: the factor $28y^2$ is reject and $x^2-2xy+4y^2$ is used as the new divisor.

$$\begin{array}{r|l|l}
x^2-2xy+4y^2 & x^3+4x^2y-8xy^2+24y^3 & x+6y \\
 & \underline{x^3-2x^2y+4xy^2} & \\
 & 6x^2y-12xy^2+24y^3 & \\
 & \underline{6x^2y-12xy^2+24y^3} &
\end{array}$$

Hence $x^2-2xy+4y^2$ is the H.C.F. required.

Ex. 3. Find the H.C.F. of

$$2x^4+9x^3+14x+3 \text{ and } 3x^4+15x^3+5x^2+10x+2.$$

To avoid the inconvenience of fractions, the second expressic is multiplied by 2: this cannot introduce any additional commc factors. The process is generally written down in the followir form:

$$\begin{array}{r|l|l}
2x^4+9x^3+14x+3 & 3x^4+15x^3+5x^2+10x+2 & \\
 & \underline{2} & \\
 & 6x^4+30x^3+10x^2+20x+4 & 3 \\
 & \underline{6x^4+27x^3+42x+9} & \\
 & 3x^3+10x^2-22x-5 &
\end{array}$$

$$\begin{array}{r|l|l}
3x^3+10x^2-22x-5 & 2x^4+9x^3+14x+3 & \\
 & \underline{3} & \\
 & 6x^4+27x^3+42x+9 & 2x \\
 & \underline{6x^4+20x^3-44x^2-10x} & \\
 & 7x^3+44x^2+52x+9 & \\
 & \underline{3} & \\
 & 21x^3+132x^2+156x+27 & \\
 & \underline{21x^3+70x^2-154x-35} & \\
 & 62\,\underline{)\,62x^2+310x+62} & \\
 & x^2+5x+1 &
\end{array}$$

$$\begin{array}{r|l|l}
x^2+5x+1 & 3x^3+10x^2-22x-5 & 3x-5 \\
 & \underline{3x^3+15x^2+3x} & \\
 & -5x^2-25x-5 & \\
 & \underline{-5x^2-25x-5} &
\end{array}$$

Thus x^2+5x+1 is the H.C.F. required.

Detached coefficients should generally be used [Art. 63]

99. The labour of finding the H.C.F. of two expres sions is frequently lessened by a modification of the pro cess, the principle of which depends on the following

Theorem:—*The common factor of highest degree in a particular letter, x suppose, of any two expressions A and B is the same as the* H.C.F. *of $pA+qB$ and $rA+sB$*

where p, q, r, s are any quantities positive or negative which do not contain x.

To prove this, it is in the first place clear that *any* common factor of A and B is also a factor of $pA+qB$ and of $rA+sB$.

So also, *any* common factor of $pA+qB$ and $rA+sB$ is also a factor of $s(pA+qB)-q(rA+sB)$, that is, of $(sp-qr)A$. Hence, as $(sp-qr)$ does not contain x, *any* common factor of $pA+qB$ and $rA+sB$ must be a factor of A, provided only that p, q, r, s are not so related that $sp-qr=0$. Similarly *any* common factor of $pA+qB$ and $rA+sB$ is also a factor of $r(pA+qB)-p(rA+sB)$, that is of $(rq-ps)B$, and therefore of B.

Since every common factor of A and B is a factor of $pA+qB$ and of $rA+sB$, and every common factor of $pA+qB$ and $rA+sB$ is a factor of A and of B, it follows that the H.C.F. of A and B is the same as the H.C.F. of $pA+qB$ and $rA+sB$.

Ex. To find the H.C.F. of $2x^4+x^3-6x^2-2x+3$ and $2x^4-3x^3+2x-3$.

We have, by subtraction,

$$4x^3-6x^2-4x+6 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)};$$

and, by addition,

$$4x^4-2x^3-6x^2=2x^2(2x^2-x-3) \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(II)}.$$

The required H.C.F. is the H.C.F. of (I) and (II), and therefore of (I) and

$$2x^2-x-3 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(III)}.$$

Multiply (III) by 2 and add (I), and we have another expression, namely

$$4x^3-2x^2-6x=2x(2x^2-x-3)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(IV)},$$

such that the H.C.F. of (III) and (IV) is the H.C.F. required. But the H.C.F. of (III) and (IV) is obviously $2x^2-x-3$.

100. If R, S...... be the successive remainders in the process of finding the H.C.F. of the two expressions A and B by the method of Art. 98; then, as we have seen, *every* common factor of A and B is a factor of R, and therefore a common factor of A and R. Similarly *every* common factor of A and R is a common factor of R and S. And so on; so

that *every* common factor of A and B is a factor of eve remainder, and therefore must be a factor of the H.C.F.

Hence every common multinomial factor of two e pressions is a factor of their *highest* common multinomi factor; and this is obviously true also of monomial facto Therefore *every* common factor of two expressions is factor of their H.C.F.

101. The H.C.F. of three or more multinomial expre sions can be found as follows.

Let the expressions be A, B, C, D,....

Find G the H.C.F. of A and B.

Then, since the required H.C.F. will be a comm factor of A and B, it will be a factor of G: we have ther fore to find the H.C.F. of G, C, D....

Hence we first find the H.C.F. of two of the given e pressions, and then find the H.C.F. of this result and the third expression; and so on.

NOTE. The *highest common factor* of algebraical e pressions is sometimes, but very inappropriately, calle their *greatest common measure* (G.C.M.).

If one expression is of higher dimensions than anothe in a particular letter, we have no reason to suppose that is numerically *greater*: for example, a^2 is not necessaril greater than a; in fact, if a is positive and less tha unity, a^2 is *less* than a.

It should also be noticed that if we give particul numerical values to the letters involved in any two e pressions and in their H.C.F., the numerical value of th H.C.F. is by no means necessarily the G.C.M. of the valu of the expressions. This is not the case even when th given expressions are *integral* for the particular valu chosen. For example, the H.C.F. of $14x^2+15x+1$ an $22x^2+23x+1$ will be found to be $x+1$; but if w suppose x to be $\frac{1}{2}$, the numerical values of the expressio will be 12 and 18, which have 6 for G.C.M., whereas th numerical value of the H.C.F. will be $\frac{3}{2}$.

EXAMPLES VII.

Find the H. C. F. of

1. $a^2-5ab+4b^2$ and $a^3-5a^2b+4b^3$.

2. $2x^2-5x+2$ and $12x^3-8x^2-3x+2$.

3. $2x^4-3x^2y^2+y^4$ and $2x^6-3x^4y^2+y^6$.

4. $2x^3+3x^2y-y^3$ and $4x^3+xy^2-y^3$.

5. $x^2-4y^2+12yz-9z^2$ and $x^2+2xz-4y^2+8yz-3z^2$.

6. $20a^4-3a^3b+b^4$ and $64a^4-3ab^3+5b^4$.

7. $a^3-a^2b+ab^2+14b^3$ and $4a^3+3a^2b-9ab^2+2b^3$.

8. $2x^4+x^3-9x^2+8x-2$ and $2x^4-7x^3+11x^2-8x+2$.

9. $11x^4-9ax^3-a^2x^2-a^4$ and $13x^4-10ax^3-2a^2x^2-a^4$.

10. $x^4+x^3-9x^2-3x+18$ and $x^5+6x^2-49x+42$.

11. $x^4-2x^3+5x^2-4x+3$ and $2x^4-x^3+6x^2+2x+3$.

12. $x^4+3x^2+6x+35$ and $x^4+2x^3-5x^2+26x+21$.

Lowest Common Multiple.

102. **Definitions.** A *Common Multiple* of two or more integral expressions, is an expression which is exactly divisible by each of them.

The *Lowest Common Multiple* of two or more integral expressions, is the expression of *lowest dimensions* which is exactly divisible by each of them.

Instead of Lowest Common Multiple it is usual to write L. C. M.

103. When the factors of expressions are know their L.C.M. can be at once written down.

Consider, for example, the expressions

$$a^3b^2(x-a)^2(x-b)^3 \text{ and } ab^4(x-a)^4(x-b).$$

It is clear that any common multiple must contain a^3 a factor; it must also contain b^4, $(x-a)^4$ and $(x-b)^3$. An common multiple must therefore have $a^3b^4(x-a)^4(x-b$ as a factor; and the common multiple which has no un necessary factors, that is to say the *lowest* common multipl must therefore be $a^3b^4(x-a)^4(x-b)^3$.

From the above example it will be seen that the L.C.M of two or more expressions which are expressed as th product of factors of the first degree, is obtained by takin *every different factor which occurs in the expressions to th highest power which it has in any one of them.*

Ex. 1. Find the L.C.M. of $3x^2yz$, $27x^3y^2z^2$ and $6xy^2z^4$.

Ans. $54x^3y^2$

Ex. 2. Find the L.C.M. of $6ab^2(a+b)^2$ and $4a^2b(a^2-b^2)$.

Ans. $12a^2b^2(a+b)^2(a-$

Ex. 3. Find the L.C.M. of $2axy(x-y)^2$, $3ax^2(x^2-y^2)$ and $4y^2(x+y$

Ans. $12ax^2y^2(x^2-y^2$

Ex. 4. Find the L.C.M. of x^2-3x+2, x^2-5x+6 and x^2-4x+3.

Ans. $(x-1)(x-2)(x-$

104. When the factors of the expressions whose L.C.M is required cannot be seen by inspection, their H.C.F. mus be found by the method of Art. 98.

Thus, to find the L.C.M. of x^3+x^2-2 and x^3+2x^2-3.

The H.C.F. will be found to be $x-1$;

and
$$x^3+x^2-2=(x-1)(x^2+2x+2),$$
$$x^3+2x^2-3=(x-1)(x^2+3x+3).$$

Then, since x^2+2x+2 and x^2+3x+3 have no common factor, th required L.C.M. is $(x-1)(x^2+2x+2)(x^2+3x+3)$.

105. Let A and B stand for any two integral expressions, and let H stand for their H.C.F., and L for their L.C.M.

Let a and b be the quotients when A and B respectively are divided by H; so that

$$A = H \,.\, a \text{ and } B = H \,.\, b.$$

Since H is the highest common factor of A and B, a and b can have no common factors. Hence the L.C.M. of A and B must be $H \times a \times b$. Thus

$$L = H \,.\, a \,.\, b.$$

Hence $$L = Ha \times \frac{Hb}{H} = A \times \frac{B}{H} \text{ ....(i)};$$

also $$L \times H = Ha \times Hb = A \times B \text{ ....(ii)}.$$

From (i) we see that *the* L.C.M. *of any two expressions is found by dividing one of the expressions by their* H.C.F., *and multiplying the quotient by the other expression.*

From (ii) we see that *the product of any two expressions is equal to the product of their* H.C.F. *and* L.C.M.

EXAMPLES VIII.

Find the L.C.M. of

1. $6x^2 - 5ax - 6a^2$ and $4x^3 - 2ax^2 - 9a^3$.

2. $4a^2 - 5ab + b^2$ and $3a^3 - 3a^2b + ab^2 - b^3$.

3. $3x^3 - 13x^2 + 23x - 21$ and $6x^3 + x^2 - 44x + 21$.

4. $x^4 - 11x^2 + 49$ and $7x^4 - 40x^3 + 75x^2 - 40x + 7$.

5. $x^3 + 6x^2 + 11x + 6$ and $x^4 + x^3 - 4x^2 - 4x$.

6. $x^4 - x^3 + 8x - 8$ and $x^4 + 4x^3 - 8x^2 + 24x$.

7. $8a^3 - 18ab^2$, $8a^3 + 8a^2b - 6ab^2$ and $4a^2 - 8ab + 3b^2$.

8. $x^2 - 7x + 12$, $3x^2 - 6x - 9$ and $2x^3 - 6x^2 - 8x$.

9. $8x^3 + 27$, $16x^4 + 36x^2 + 81$ and $6x^2 - 5x - 6$.

10. $x^2 - 6xy + 9y^2$, $x^2 - xy - 6y^2$ and $3x^2 - 12y^2$.

11. $x^2 - 7xy + 12y^2$, $x^2 - 6xy + 8y^2$ and $x^2 - 5xy + 6y^2$.

12. Shew that, if $ax^2 + bx + c$ and $a'x^2 + b'x + c'$ have a common factor of the form $x + f$, then will

$$(ac' - a'c)^2 = (bc' - b'c)(ab' - a'b).$$

13. Shew that, if $ax^3 + bx^2 + cx + d$ and $a'x^3 + b'x^2 + c'x +$ have a common quadratic factor in x, then will

$$\frac{ba' - b'a}{ad' - a'd} = \frac{ca' - c'a}{bd' - b'd} = \frac{da' - d'a}{cd' - c'd}.$$

14. Find the condition that $ax^2 + bx + c$ and $a'x^2 + b'x +$ may have a common factor of the form $x + f$.

15. If g_1, g_2, g_3 are the highest common factors, and $l_1, l_2,$ the lowest common multiples of the three quantities a, b, c taken in pairs; prove that $g_1 g_2 g_3 l_1 l_2 l_3 = (abc)^2$.

16. If A, B, C be any three algebraical expressions, and (BC), (CA), (AB) and (ABC) be respectively the highest common factors of B and C, C and A, A and B, and A, B and C; then the L.C.M. of A, B and C will be

$$A \,.\, B \,.\, C \,.\, (ABC) \div \{(BC) \,.\, (CA) \,.\, (AB)\}.$$

CHAPTER VIII.

FRACTIONS.

106. WHEN the operation of division is indicated by placing the dividend over the divisor with a horizontal line between them, the quotient is called an *algebraical fraction*, the dividend and the divisor being called respectively the *numerator* and the *denominator* of the fraction. Thus $\frac{a}{b}$ means $a \div b$.

Since, by definition, $\frac{a}{b} = a \div b$, it follows that $\frac{a}{b} \times b = a$.

107. **Theorem.** *The value of a fraction is not altered by multiplying its numerator and denominator by the same quantity.*

We have to prove that

$$\frac{a}{b} = \frac{am}{bm},$$

for all values of a, b and m.

Let $x = \frac{a}{b}$;

then $$x \times b = \frac{a}{b} \times b = a, \qquad \text{by definition.}$$

Hence $$x \times b \times m = a \times m;$$

$$\therefore\ x \times (bm) = am. \quad \text{[Art. 29, (ii).]}$$

Divide by bm, and we have

$$x = am \div (bm) = \frac{am}{bm}.$$

108. Since the value of a fraction is not altered by *multiplying* both the numerator and the denominator by the same quantity, it follows conversely that the value of a fraction is not altered by *dividing* both the numerator and the denominator by the same quantity.

Hence a fraction may be simplified by the rejection of any factor which is common to its numerator and denominator. For example, the fraction $\frac{a^2x}{b^2x}$ takes the simpler form $\frac{a^2}{b^2}$, when the factor x, which is common to its numerator and denominator, is rejected.

When the numerator and denominator of a fraction have no common factors, the fraction is said to be in its *lowest terms.*

To reduce a fraction to its lowest terms we must divide its numerator and denominator by their H.C.F.; for we thus obtain an equivalent fraction whose numerator and denominator have no common factors.

Ex. 1. Reduce $\frac{3ax^2y}{6a^2xy}$ to its lowest terms.

The H.C.F. of the numerator and denominator is $3axy$; and

$$\frac{3ax^2y}{6a^2xy} = \frac{3ax^2y \div 3axy}{6a^2xy \div 3axy} = \frac{x}{2a}.$$

Ex. 2. Simplify $\frac{x^2 - 7xy + 10y^2}{x^2 - 8xy + 12y^2}$.

$$\frac{x^2 - 7xy + 10y^2}{x^2 - 8xy + 12y^2} = \frac{(x - 2y)(x - 5y)}{(x - 2y)(x - 6y)} = \frac{x - 5y}{x - 6y}.$$

Ex. 3. Simplify $\dfrac{x^2-ax}{a^2-x^2}$.

$$\frac{x^2-ax}{a^2-x^2}=\frac{x(x-a)}{(a-x)(a+x)}.$$

Since $x-a=-(a-x)$, if we divide the numerator and denominator by $a-x$, we have the equivalent fraction $\dfrac{-x}{a+x}$; and if we divide the numerator and denominator by $x-a$, we have $\dfrac{x}{-(a+x)}$. By the Law of Signs in Division $\dfrac{-x}{a+x}=\dfrac{x}{-(a+x)}=-\dfrac{x}{a+x}$, and the last form is the one in which the result is usually left.

Ex. 4. Simplify $\dfrac{x^4+3x^2+6x+35}{x^4+2x^3-5x^2+26x+21}$.

The H.C.F. will be found to be x^2-3x+7; and, dividing the numerator and denominator by x^2-3x+7, we have the equivalent fraction $\dfrac{x^2+3x+5}{x^2+5x+3}$.

109. **Reduction of fractions to a common denominator.** Since the value of a fraction is unaltered by multiplying its numerator and denominator by the same quantity, any number of fractions can be reduced to equivalent fractions all of which have the same denominator.

The process is as follows. First find the L.C.M. of all the denominators; then divide the L.C.M. by the denominator of one of the fractions, and multiply the numerator and denominator of that fraction by the quotient; and deal in a similar manner with all the other fractions: we thus obtain new fractions equal to the given fractions respectively and all of which have the same denominator.

For example, to reduce

$$\frac{a}{x^3y(x+y)},\quad \frac{b}{xy^3(x-y)}\quad \text{and}\quad \frac{c}{x^2y^2(x^2-y^2)},$$

to a common denominator.

The L.C.M. of the denominators is $x^3y^3(x^2-y^2)$. Dividing this L.C.M. by $x^3y(x+y)$, $xy^3(x-y)$ and $x^2y^2(x^2-y^2)$, we have the

quotients $y^2(x-y)$, $x^2(x+y)$ and xy respectively. Hence the required fractions are

$$\frac{a}{x^3y(x+y)}=\frac{a\times y^2(x-y)}{x^3y(x+y)\times y^2(x-y)}=\frac{ay^2(x-y)}{x^3y^3(x^2-y^2)},$$

$$\frac{b}{xy^3(x-y)}=\frac{b\times x^2(x+y)}{xy^3(x-y)\times x^2(x+y)}=\frac{bx^2(x+y)}{x^3y^3(x^2-y^2)},$$

$$\frac{c}{x^2y^2(x^2-y^2)}=\frac{c\times xy}{x^2y^2(x^2-y^2)\times xy}=\frac{cxy}{x^3y^3(x^2-y^2)}.$$

It is not necessary to take the *lowest* common multiple of the denominators, for *any* common multiple would answer the purpose; but by using the L.C.M. there is some saving of labour.

110. **Addition of fractions.** The sum (or difference) of two fractions which have the same denominator is a fraction whose numerator is the sum (or difference) of their numerators, and which has the common denominator. This follows from Art. 43.

When two fractions have not the same denominator, they must first be reduced to equivalent fractions which have the same denominator: their sum, or difference, will then be found by taking the sum, or difference, of their numerators, retaining the common denominator.

When more than two fractions are to be added, or when there are several fractions some of which are to be added and the others subtracted, the process is precisely the same. The fractions must first be reduced to a common denominator, and then the numerators of the reduced fractions are added or subtracted as may be required.

Ex. 1. Find the value of $\frac{1}{a+b}+\frac{1}{a-b}$.

The L.C.M. of the denominators is $(a+b)(a-b)$; and

$$\frac{1}{a+b}+\frac{1}{a-b}=\frac{a-b}{(a+b)(a-b)}+\frac{a+b}{(a-b)(a+b)}$$

$$=\frac{(a-b)+(a+b)}{a^2-b^2}=\frac{2a}{a^2-b^2}.$$

Ex. 2. Find the value of $\frac{a}{a-b}+\frac{ab}{b^2-a^2}$.

The L.C.M. of the denominators is a^2-b^2; and we have

$$\frac{a(a+b)}{a^2-b^2}+\frac{-ab}{a^2-b^2}=\frac{a(a+b)-ab}{a^2-b^2}=\frac{a^2}{a^2-b^2}.$$

Ex. 3. Simplify $\dfrac{a}{a-x}+\dfrac{a}{a+x}+\dfrac{2a^2}{a^2+x^2}+\dfrac{4a^4}{a^4+x^4}$.

In this case it is not desirable to reduce all the fractions to a common denominator at once: the work is simplified by proceeding as under:

$$\frac{a}{a-x}+\frac{a}{a+x}=\frac{a(a+x)+a(a-x)}{a^2-x^2}=\frac{2a^2}{a^2-x^2};$$

then $$\frac{2a^2}{a^2-x^2}+\frac{2a^2}{a^2+x^2}=\frac{2a^2(a^2+x^2)+2a^2(a^2-x^2)}{a^4-x^4}=\frac{4a^4}{a^4-x^4};$$

and finally $$\frac{4a^4}{a^4-x^4}+\frac{4a^4}{a^4+x^4}=\frac{4a^4(a^4+x^4)+4a^4(a^4-x^4)}{a^8-x^8}=\frac{8a^8}{a^8-x^8}.$$

[The above would be shortened by observing that, except for the factor 2, the second addition only differs from the first by having a^2 and x^2 instead of a and x respectively; and hence the result of the addition can be *written down* at once. So also the result of the third addition can be written down from the first or second.]

Ex. 4. Simplify $\dfrac{1}{x-3}-\dfrac{3}{x-1}+\dfrac{3}{x+1}-\dfrac{1}{x+3}$.

Here again it is best not to reduce all the fractions to a common denominator at once: much labour is often saved by a judicious arrangement and grouping of the terms.

$$\frac{1}{x-3}-\frac{1}{x+3}=\frac{(x+3)-(x-3)}{x^2-9}=\frac{6}{x^2-9},$$

and $$-\frac{3}{x-1}+\frac{3}{x+1}=\frac{-3(x+1)+3(x-1)}{x^2-1}=\frac{-6}{x^2-1};$$

then $$\frac{6}{x^2-9}+\frac{-6}{x^2-1}=\frac{6(x^2-1)-6(x^2-9)}{(x^2-9)(x^2-1)}=\frac{48}{(x^2-9)(x^2-1)}.$$

Ex. 5. Simplify $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}$.

The L.C.M. of the denominators is $(b-c)(c-a)(a-b)$ [See Art. 93]. Hence we have

$$\frac{a^2(c-b)+b^2(a-c)+c^2(b-a)}{(b-c)(c-a)(a-b)}.$$

Now we naturally test, by the method of Art. 88, whether either of the factors of the denominator is a factor of the numerator: we are thus led to find that the numerator is the same as the denominator, so that the given expression is equal to unity.

Ex. 6. Simplify

$$\frac{a^2}{(a-b)(a-c)(x+a)}+\frac{b^2}{(b-c)(b-a)(x+b)}+\frac{c^2}{(c-a)(c-b)(x+c)}.$$

The L.C.M. of the denominators is

$$(b-c)(c-a)(a-b)(x+a)(x+b)(x+c).$$

The expression is therefore equal to the fraction whose denominat is this L.C.M., and whose numerator is

$$a^2(c-b)(x+b)(x+c)+b^2(a-c)(x+c)(x+a)+c^2(b-a)(x+a)(x+$$

Arranging the numerator according to powers of x, the coefficie of x^2 is $a^2(c-b)+b^2(a-c)+c^2(b-a)=(b-c)(c-a)(a-b)$.

The coefficient of x is $a^2(c^2-b^2)+b^2(a^2-c^2)+c^2(b^2-a^2)=0$.

The term which does not contain x is

$$abc\{a(c-b)+b(a-c)+c(b-a)\}=0.$$

Hence the numerator is $x^2(b-c)(c-a)(a-b)$, and therefore t given expression

$$=\frac{x^2(b-c)(c-a)(a-b)}{(b-c)(c-a)(a-b)(x+a)(x+b)(x+c)}=\frac{x^2}{(x+a)(x+b)(x+c)}$$

111. **Multiplication of fractions.** We have no to shew how to multiply algebraical fractions.

Let the fractions be $\frac{a}{b}$ and $\frac{c}{d}$.

Let $$x=\frac{a}{b}\times\frac{c}{d};$$

then $$x\times b\times d=\frac{a}{b}\times\frac{c}{d}\times b\times d$$

$$=\frac{a}{b}\times b\times\frac{c}{d}\times d,$$

by the Commutative Law.

But, by definition, $\frac{a}{b}\times b=a$, and $\frac{c}{d}\times d=c$;

$$\therefore\ x\times b\times d=a\times c;$$

$$\therefore\ x=\frac{ac}{bd}.$$

Thus *the product of any two fractions is another fraction whose numerator is the product of their numerators, and whose denominator is the product of their denominators.*

The continued product of any number of fractions is found by the same rule. For

$$\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ac}{bd} \times \frac{e}{f} = \frac{ace}{bdf},$$

and similarly, however many fractions there may be.

Hence

$$\left(\frac{a}{b}\right)^2 = \frac{a}{b} \times \frac{a}{b} = \frac{aa}{bb} = \frac{a^2}{b^2}; \text{ and, in general, } \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

112. **Division of fractions.** Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two fractions; and let $x = \frac{a}{b} \div \frac{c}{d}$.

Then $$x \times \frac{c}{d} = \frac{a}{b} \div \frac{c}{d} \times \frac{c}{d} = \frac{a}{b};$$

$$\therefore\ x \times \frac{c}{d} \times \frac{d}{c} = \frac{a}{b} \times \frac{d}{c}.$$

Hence $$x = \frac{a}{b} \times \frac{d}{c},$$

since $$\frac{c}{d} \times \frac{d}{c} = \frac{cd}{dc} = 1.$$

Thus *to divide by any fraction* $\frac{c}{d}$ *is the same as to multiply by the reciprocal fraction* $\frac{d}{c}$.

As particular cases of multiplication and division, we have

$$\frac{a}{b} \times c = \frac{a}{b} \times \frac{c}{1} = \frac{ac}{b},$$

and $$\frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \times \frac{1}{c} = \frac{a}{bc}.$$

Note. It should be noticed that the rules for the multiplication and division of algebraical fractions are simply rules concerning the *order* in which certain operations of multiplication and division may be performed, and have really been proved in Art. 33.

Thus $$\frac{a}{b} \times \frac{c}{d} = (a \div b) \times (c \div d)$$
$$= a \div b \times c \div d$$
$$= a \times c \div b \div d = (ac) \div (bd) = \frac{ac}{bd}.$$

Ex. 1. Simplify $\frac{x^3+a^3}{x^2-a^2} \times \frac{x-a}{(x+a)^2}$.

$$\frac{x^3+a^3}{x^2-a^2} \times \frac{x-a}{(x+a)^2} = \frac{(x^3+a^3)(x-a)}{(x^2-a^2)(x+a)^2}$$
$$= \frac{(x^2-ax+a^2)(x+a)(x-a)}{(x-a)(x+a)^3} = \frac{x^2-ax+a^2}{(x+a)^2}.$$

Ex. 2. Simplify $\dfrac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}}$.

$$\frac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}} = \frac{\frac{y-x}{xy}}{\frac{y^2-x^2}{x^2y^2}} = \frac{y-x}{xy} \div \frac{y^2-x^2}{x^2y^2} = \frac{y-x}{xy} \times \frac{x^2y^2}{y^2-x^2} = \frac{xy}{x+y}.$$

Ex. 3. Simplify $\dfrac{\frac{a+x}{a-x}-\frac{a-x}{a+x}}{\frac{a+x}{a-x}+\frac{a-x}{a+x}}$.

$$\frac{a+x}{a-x}-\frac{a-x}{a+x} = \frac{(a+x)(a+x)-(a-x)(a-x)}{a^2-x^2} = \frac{4ax}{a^2-x^2},$$

and $$\frac{a+x}{a-x}+\frac{a-x}{a+x} = \frac{(a+x)(a+x)+(a-x)(a-x)}{a^2-x^2} = \frac{2a^2+2x^2}{a^2-x^2}.$$

Hence the given fraction is equal to

$$\frac{4ax}{a^2-x^2} \div \frac{2a^2+2x^2}{a^2-x^2} = \frac{4ax}{a^2-x^2} \times \frac{a^2-x^2}{2a^2+2x^2} = \frac{2ax}{a^2+x^2}.$$

113. The following theorems (the second of which includes the first) are of importance:

Theorem I. *If the fractions $\frac{a_1}{b_1}$, $\frac{a_2}{b_2}$, $\frac{a_3}{b_3}$, &c. be all equal to one another, then will each fraction be equal to*

$$\frac{pa_1+qa_2+ra_3+\ldots\ldots}{pb_1+qb_2+rb_3+\ldots\ldots}$$

Let each of the equal fractions be equal to x.

Then, since $\frac{a_1}{b_1}=x$, $\qquad a_1=b_1x$;

$$\therefore\ pa_1=pb_1x,$$

so also

$$qa_2=qb_2x,$$

$$ra_3=rb_3x,$$

$$\ldots\ldots=\ldots\ldots$$

Hence, by addition,

$$pa_1+qa_2+ra_3+\ldots\ldots=(pb_1+qb_2+rb_3+\ldots)\,x;$$

$$\therefore\ \frac{pa_1+qa_2+ra_3+\ldots\ldots}{pb_1+qb_2+rb_3+\ldots\ldots}=x=\frac{a_1}{b_1}=\&c.$$

Theorem II. *If the fractions $\frac{a_1}{b_1}$, $\frac{a_2}{b_2}$, $\frac{a_3}{b_3}$, &c. be all equal to one another, then will each fraction be equal to $\frac{\sqrt[n]{A}}{\sqrt[n]{B}}$, where A is any homogeneous expression of the nth degree in a_1, a_2, a_3, &c. and B is the same homogeneous expression with b_1 in the place of a_1, b_2 in the place of a_2, &c.*

Let each of the equal fractions be equal to x, so that

$$a_1=b_1x, \quad a_2=b_2x, \quad a_3=b_3x, \quad \&c.$$

Let $\lambda a_1^\alpha a_2^\beta a_3^\gamma \ldots$ be any term of A; then $\lambda b_1^\alpha b_2^\beta b_3^\gamma \ldots$ will be the corresponding term of B; and since the expressions A and B are homogeneous and of the nth degree $\alpha + \beta + \gamma + \ldots = n$.

$$\text{Now } \lambda a_1^\alpha a_2^\beta a_3^\gamma \ldots = \lambda (b_1 x)^\alpha (b_2 x)^\beta (b_3 x)^\gamma \ldots$$
$$= \lambda (b_1^\alpha b_2^\beta b_3^\gamma \ldots) x^{\alpha+\beta+\gamma+\ldots}$$
$$= x^n . \lambda b_1^\alpha b_2^\beta b_3^\gamma \ldots,$$

since $\alpha + \beta + \gamma + \ldots = n$.*

Hence any term of $A = x^n \times$ corresponding term of B
$\therefore$ sum of all the terms of $A = x^n \times$ sum of all the terms of B
that is $$A = x^n . B;$$

$$\therefore \frac{\sqrt[n]{A}}{\sqrt[n]{B}} = x,$$

which proves the theorem.

Theorem III. *If the denominators of the fractions* $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots\ldots$ *be all positive, then will the fraction* $\frac{a_1 + a_2 + a_3 + \ldots\ldots}{b_1 + b_2 + b_3 + \ldots\ldots}$ *be greater than the least and less than the greatest of the fractions* $\frac{a_1}{b_1}, \frac{a_2}{b_2}$, *&c.*

Let $\frac{a_1}{b_1}$ be the greatest of the fractions, and let $\frac{a_1}{b_1} = x$;
then $\frac{a_2}{b_2} < x$, $\frac{a_3}{b_3} < x$, &c.

Hence, $b_1, b_2, \ldots$ being all positive, we have

$$a_1 = x . b_1,$$
$$a_2 < x . b_2,$$
$$a_3 < x . b_3,$$
$$\ldots\ldots\ldots\ldots$$

* We have in the above assumed certain results which will be proved in Chapter XIII.

Hence by addition

$$a_1 + a_2 + a_3 + \ldots\ldots < x(b_1 + b_2 + b_3 + \ldots\ldots);$$

$$\therefore \frac{a_1 + a_2 + a_3 + \ldots\ldots}{b_1 + b_2 + b_3 + \ldots\ldots} < x.$$

Hence $\frac{a_1 + a_2 + a_3 + \ldots}{b_1 + b_2 + b_3 + \ldots}$ is less than the greatest of the fractions; and it can be similarly proved to be greater than the least of the fractions.

Ex. 1. Shew that, if $\frac{a}{b} = \frac{c}{d}$, then will $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

Let $\frac{a}{b} = x$; then $\frac{c}{d} = x$.

Hence $\frac{a+b}{a-b} = \frac{bx+b}{bx-b} = \frac{x+1}{x-1} = \frac{dx+d}{dx-d} = \frac{c+d}{c-d}$.

Or thus:

Since $\frac{a}{b} = \frac{c}{d}$,

$$\frac{a}{b} + 1 = \frac{c}{d} + 1, \text{ that is } \frac{a+b}{b} = \frac{c+d}{d}.$$

Also $$\frac{a}{b} - 1 = \frac{c}{d} - 1, \text{ that is } \frac{a-b}{b} = \frac{c-d}{d}.$$

Hence $$\frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d};$$

$$\therefore \frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

Ex. 2. Shew that, if $\frac{a}{b} = \frac{c}{d}$, then will each fraction be equal to

$$\frac{\sqrt{(a^2 - 2ac + 2c^2)}}{\sqrt{(b^2 - 2bd + 2d^2)}}.$$

Put $\frac{a}{b} = \frac{c}{d} = x$;

then $$\frac{\sqrt{(a^2 - 2ac + 2c^2)}}{\sqrt{(b^2 - 2bd + 2d^2)}} = \frac{\sqrt{(b^2x^2 - 2bxdx + 2d^2x^2)}}{\sqrt{(b^2 - 2bd + 2d^2)}} = \sqrt{x^2} = x.$$

[This is a simple case of Theorem II., Art. 113.]

Ex. 3. Shew that, if $\frac{cy+bz}{l} = \frac{az+cx}{m} = \frac{bx+ay}{n}$, then will

$$\frac{bcx}{-al+bm+cn} = \frac{cay}{al-bm+cn} = \frac{abz}{al+bm-cn}.$$

Each of the given equal fractions

$$=\frac{-a(cy+bz)+b(az+cx)+c(bx+ay)}{-al+bm+cn}=\frac{2bcx}{-al+bm+cn}$$

and similarly $$=\frac{2cay}{al-bm+cn}=\frac{2abz}{al+bm-cn}.$$

EXAMPLES IX.

Simplify the following fractions:

1. $\dfrac{30a^2b^3c^5x^2y^4z^3}{36a^5bc^2x^5yz^6}$.

2. $\dfrac{3a^7b^2c^{10}x^3yz^4}{a^6c^4x^3y^6}$.

3. $\dfrac{a^2-8ab+7b^2}{a^2-3ab-28b^2}$.

4. $\dfrac{7x^4y^4-8x^2y^2+1}{28x^4y^4+3x^2y^2-1}$.

5. $\dfrac{(x^3-y^3)(x+y)}{(x^3+y^3)(x-y)}$.

6. $\dfrac{(x^6-y^6)(x-y)}{(x^3-y^3)(x^4-y^4)}$.

7. $\dfrac{2x^3+3x^2-1}{x^4+2x^3+2x^2+2x+1}$.

8. $\dfrac{x^4-x^3-x+1}{x^4-2x^3-x^2-2x+1}$.

9. $\dfrac{2x^3+5x^2y+xy^2-3y^3}{3x^4+3x^3y-4x^2y^2-xy^3+y^4}$.

10. $\dfrac{54x^5-27x^4-3x^2-4}{36x^5+3x^3+3x-2}$.

11. $\dfrac{(a+b)\{(a+b)^2-c^2\}}{4b^2c^2-(a^2-b^2-c^2)^2}$.

12. $\dfrac{x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)}{x^2(y-z)+y^2(z-x)+z^2(x-y)}$.

13. $\dfrac{x^4(y-z)+y^4(z-x)+z^4(x-y)}{(y+z)^2+(z+x)^2+(x+y)^2}$.

14. $\dfrac{a(b-c)(c-d)-c(d-a)(a-b)}{b(c-d)(d-a)-d(a-b)(b-c)}$.

15. $\dfrac{x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)}{x^2(y-z)+y^2(z-x)+z^2(x-y)}$.

16. $\dfrac{2a}{a+b}+\dfrac{2b}{a-b}+\dfrac{a^2+b^2}{b^2-a^2}$.

17. $\dfrac{3-x}{1-3x}-\dfrac{3+x}{1+3x}-\dfrac{1-16x}{9x^2-1}$.

18. $\dfrac{x}{x+2y}-\dfrac{y}{2y-x}-\dfrac{(x-y)^2}{x^2-4y^2}$.

19. $\dfrac{x-2a}{x+2a}-\dfrac{x+2a}{2a-x}+\dfrac{8ax}{x^2-4a^2}$.

20. $\dfrac{1}{x+2}-\dfrac{3}{x+4}+\dfrac{3}{x+6}-\dfrac{1}{x+8}$.

21. $\dfrac{1}{x+a}-\dfrac{3}{x+3a}+\dfrac{3}{x+5a}-\dfrac{1}{x+7a}$.

22. $\dfrac{1}{x-2a}-\dfrac{4}{x-a}+\dfrac{6}{x}-\dfrac{4}{x+a}+\dfrac{1}{x+2a}$.

23. $\dfrac{1}{x^2-5xy+6y^2}-\dfrac{2}{x^2-4xy+3y^2}+\dfrac{1}{x^2-3xy+2y^2}$.

24. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}$.

25. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}$.

26. $\dfrac{(1+ab)(1+ac)}{(a-b)(a-c)}+\dfrac{(1+bc)(1+ba)}{(b-c)(b-a)}+\dfrac{(1+ca)(1+cb)}{(c-a)(c-b)}$.

27. $\dfrac{bc(a+d)}{(a-b)(a-c)}+\dfrac{ca(b+d)}{(b-c)(b-a)}+\dfrac{ab(c+d)}{(c-a)(c-b)}$.

28. $\dfrac{x^2-yz}{(x+y)(x+z)}+\dfrac{y^2-zx}{(y+z)(y+x)}+\dfrac{z^2-xy}{(z+x)(z+y)}$.

29. $$\frac{(y-x)(z-x)}{(x-2y+z)(x+y-2z)}+\frac{(z-y)(x-y)}{(x+y-2z)(-2x+y+z)}+\frac{(z-x)(z-y)}{(-2x+y+z)(x-2y+z)}.$$

30. $$\frac{\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}+\dfrac{x+c}{x-c}-3\dfrac{(x+a)(x+b)(x+c)}{(x-a)(x-b)(x-c)}}{\dfrac{x}{x-a}+\dfrac{x}{x-b}+\dfrac{x}{x-c}-3\dfrac{x^3+(bc+ca+ab)x}{(x-a)(x-b)(x-c)}}$$

31. $$\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-c)(b-a)}+\frac{c^3}{(c-a)(c-b)}.$$

32. $$\frac{a^4}{(a-b)(a-c)}+\frac{b^4}{(b-c)(b-a)}+\frac{c^4}{(c-a)(c-b)}.$$

33. $$a^2\frac{(a+b)(a+c)}{(a-b)(a-c)}+b^2\frac{(b+c)(b+a)}{(b-c)(b-a)}+c^2\frac{(c+a)(c+b)}{(c-a)(c-b)}.$$

34. $$\frac{a^2\left(\dfrac{1}{b}-\dfrac{1}{c}\right)+b^2\left(\dfrac{1}{c}-\dfrac{1}{a}\right)+c^2\left(\dfrac{1}{a}-\dfrac{1}{b}\right)}{a\left(\dfrac{1}{b}-\dfrac{1}{c}\right)+b\left(\dfrac{1}{c}-\dfrac{1}{a}\right)+c\left(\dfrac{1}{a}-\dfrac{1}{b}\right)}.$$

35. $$\frac{1}{(a-b+c)(a+b-c)}+\frac{1}{(a+b-c)(-a+b+c)}+\frac{1}{(-a+b+c)(a-b+c)}.$$

36. $$\frac{b-c}{a^2-(b-c)^2}+\frac{c-a}{b^2-(c-a)^2}+\frac{a-b}{c^2-(a-b)^2}$$

37. Shew that

$$16+\left\{\frac{x+a}{x-a}+\frac{x-a}{x+a}-2\frac{x^2-a^2}{x^2+a^2}\right\}^2=16\left(\frac{x^4+a^4}{x^4-a^4}\right)^2.$$

38. Shew that

$$\frac{a+b}{ax+by}+\frac{a-b}{ax-by}+2\frac{a^2x+b^2y}{a^2x^2+b^2y^2}=4\frac{a^4x^3-b^4y^3}{a^4x^4-b^4y^4}.$$

39. Shew that

(i) $$\frac{a^2}{(a-b)(a-c)(1+ax)}+\frac{b^2}{(b-c)(b-a)(1+bx)}+\frac{c^2}{(c-a)(c-b)(1+cx)}=\frac{1}{(1+ax)(1+bx)(1+cx)}.$$

(ii) $$\frac{a}{(a-b)(a-c)(1+ax)}+\frac{b}{(b-c)(b-a)(1+bx)}+\frac{c}{(c-a)(c-b)(1+cx)}=\frac{-x}{(1+ax)(1+bx)(1+cx)}.$$

(iii) $$\frac{1}{(a-b)(a-c)(1+ax)}+\frac{1}{(b-c)(b-a)(1+bx)}+\frac{1}{(c-a)(c-b)(1+cx)}=\frac{x^2}{(1+ax)(1+bx)(1+cx)}.$$

40. Simplify

$$\frac{(a+p)(a+q)}{(a-b)(a-c)(x+a)}+\frac{(b+p)(b+q)}{(b-c)(b-a)(x+b)}+\frac{(c+p)(c+q)}{(c-a)(c-b)(x+c)}.$$

41. Simplify

$$\frac{a(b+c-a)}{(a-b)(a-c)}+\frac{b(c+a-b)}{(b-c)(b-a)}+\frac{c(a+b-c)}{(c-a)(c-b)}.$$

42. Simplify

$$\frac{(a-b+c)(a+b-c)}{(a-b)(a-c)}+\frac{(a+b-c)(-a+b+c)}{(b-c)(b-a)}+\frac{(-a+b+c)(a-b+c)}{(c-a)(c-b)}.$$

43. Simplify

$$\frac{a(b+c)}{b+c-a}+\frac{b(c+a)}{c+a-b}+\frac{c(a+b)}{a+b-c}.$$

44. Shew that $\left(m+\frac{1}{m}\right)^2+\left(n+\frac{1}{n}\right)^2+\left(mn+\frac{1}{mn}\right)^2$

$$-\left(m+\frac{1}{m}\right)\left(n+\frac{1}{n}\right)\left(mn+\frac{1}{mn}\right)=$$

45. Shew that

$$\left\{\frac{2bc}{b+c}-b\right\}\div\left\{\frac{1}{c}+\frac{1}{b-2c}\right\}+\left\{\frac{2bc}{b+c}-c\right\}\div\left\{\frac{1}{b}+\frac{1}{c-2b}\right\}=bc.$$

46. Shew that

$$\frac{b-c}{1+bc}+\frac{c-a}{1+ca}+\frac{a-b}{1+ab}=\frac{(b-c)(c-a)(a-b)}{(1+bc)(1+ca)(1+ab)}.$$

47. Simplify

$$(yz+zx+xy)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-xyz\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right).$$

48. Shew that, if

$$\frac{y+z}{b-c}=\frac{z+x}{c-a}=\frac{x+y}{a-b},$$

then will each fraction be equal to

$$\frac{\sqrt{x^2+y^2+z^2}}{\sqrt{\{(b-c)^2+(c-a)^2+(a-b)^2\}}}.$$

49. Shew that, if $\frac{x}{y}=\frac{a}{b}$, then will

$$\frac{x^2+a^2}{x+a}+\frac{y^2+b^2}{y+b}=\frac{(x+y)^2+(a+b)^2}{x+y+a+b}.$$

50. Shew that, if

$$\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c},$$

then will $\qquad (b-c)x+(c-a)y+(a-b)z=0.$

51. Shew that, if $\dfrac{bz-cy}{b-c}=\dfrac{cx-az}{c-a}$,

and c be not zero, then will each equal $\dfrac{ay-bx}{a-b}$,

and $$a(y-z)+b(z-x)+c(x-y)=0.$$

52. Shew that

$$\frac{a^4}{(a-b)(a-c)(a-d)}+\frac{b^4}{(b-c)(b-d)(b-a)}+\frac{c^4}{(c-d)(c-a)(c-b)}$$
$$+\frac{d^4}{(d-a)(d-b)(d-c)}=a+b+c+d.$$

53. Shew that

$$\frac{a_1^{\,r}}{(a_1-a_2)(a_1-a_3)\ldots\ldots(a_1-a_n)}+\frac{a_2^{\,r}}{(a_2-a_1)(a_2-a_3)\ldots\ldots(a_2-a_n)}$$
$$+\ldots+\frac{a_n^{\,r}}{(a_n-a_1)(a_n-a_2)\ldots\ldots(a_n-a_{n-1})}$$

is equal to zero if r be less than $n-1$, to 1 if $r=n-1$, and to

$$a_1+a_2+\ldots+a_n \text{ if } r=n.$$

54. Shew that

$$1+\frac{a_1}{x-a_1}+\frac{a_2x}{(x-a_1)(x-a_2)}+\frac{a_3x^2}{(x-a_1)(x-a_2)(x-a_3)}+\ldots$$
$$\ldots+\frac{a_nx^{n-1}}{(x-a_1)(x-a_2)\ldots(x-a_n)}=\frac{x^n}{(x-a_1)(x-a_2)\ldots(x-a_n)}.$$

55. Shew that

$$\frac{b+c+d+\ldots+k+l}{a(a+b+c+\ldots+k+l)}=\frac{b}{a(a+b)}+\frac{c}{(a+b)(a+b+c)}+\ldots$$
$$\ldots+\frac{l}{(a+b+\ldots+k)(a+b+\ldots+k+l)}.$$

CHAPTER IX.

EQUATIONS. ONE UNKNOWN QUANTITY.

114. A STATEMENT of the equality of two algebraical expressions is called an *equation;* and the two equal expressions are called the *members*, or *sides*, of the equation.

When the equality is true for *all values* of the letters involved the equation is, as we have already said, called an *identity*, the name *equation* being reserved for those cases in which the equality is only true for certain particular values of the letters involved.

For the sake of distinction, a quantity which is supposed to be known, but which is not expressed by any particular arithmetical number, is usually represented by one of the first letters of the alphabet, a, b, c, &c., and a quantity which is unknown, and which is to be found, is usually represented by one of the last letters of the alphabet x, y, z, &c.

115. We shall in the present chapter only consider equations which contain *one* unknown quantity.

To *solve* an equation is to find the value or values of the unknown quantity for which the equation is true; and these values of the unknown quantity are said to *satisfy* the equation, and are called the *roots* of the equation.

Two equations are said to be *equivalent* when they have the same roots.

An equation which contains only one unknown quantity, x suppose, and which is rational and integral in x, is said to be of the *first degree* when x occurs only in the first power; it is said to be of the *second degree* when x^2 is the highest power of x which occurs; and so on.

Equations of the first, second and third degrees are however generally called *simple, quadratic* and *cubic* equations respectively.

116. In the solution of equations frequent use is made of the following principles.

I. An equation is equivalent to that formed by adding the same quantity to both its members.

For it is clear that $A+m=B+m$ when, and only when, $A=B$.

II. Any term may be transformed from one side of an equation to the other, provided its sign be changed.

Let the equation be

$$a+b-c=p-q+r.$$

Add $\quad -p+q-r$ to both sides;

then $\quad a+b-c-p+q-r=p-q+r-p+q-r,$

that is, $\quad a+b-c-p+q-r=0.$

We thus have an equation equivalent to the given equation, but with the terms p, $-q$, $+r$ changed in sign and transposed.

By means of transposition all the terms of any equation may be written on one side of the sign of equality and zero on the other side.

III. An equation is equivalent to that formed by multiplying (or dividing) each of its members by the same quantity which is not equal to zero.

For, if $A=B$, it is clear that $mA=mB$. Conversely, if $mA=mB$, that is $m(A-B)=0$, it follows that $A-B=0$, *since m is not zero.* Hence $mA=mB$ when, and only when, $A=B$.

The case of division requires no separate examination, for to divide by m is the same as to multiply by $\frac{1}{m}$.

117. **Simple Equations.** The method of solving simple equations will be seen from the following examples.

Ex. 1. Solve the equation $13x - 7 = 5x + 9$.

Transpose the terms $5x$ and -7; then $13x - 5x = 7 + 9$.

That is $8x = 16$.

Divide both sides by 8, the coefficient of x; then $x = 2$.

Ex. 2. Solve the equation $\frac{3x}{4} - 2 = \frac{2x}{5} + 5$.

We may get rid of fractions by multiplying both members by 20, the least common multiple of the denominators; we then have

$$15x - 40 = 8x + 100,$$

or transposing $$15x - 8x = 100 + 40;$$

$$\therefore\ 7x = 140.$$

Divide by 7, the coefficient of x; then $x = 20$.

Ex. 3. Solve the equation $a(x - a) = 2ab - b(x - b)$.

Removing the brackets, we have

$$ax - a^2 = 2ab - bx + b^2,$$

or transposing $$ax + bx = 2ab + b^2 + a^2,$$

that is $$x(a + b) = (a + b)^2.$$

Divide by $a + b$, the coefficient of x; then

$$x = \frac{(a+b)^2}{a+b} = a + b.$$

From the above it will be seen that the different steps in the process of solving a simple equation are as follows. First clear the equation of fractions, and perform the algebraical operations which are indicated. Then transpose all the terms which contain the unknown quantity to one side of the equation, and all the other terms to the other side. Next combine all the terms which contain the unknown quantity into one term, and divide by the

coefficient of the unknown quantity: this gives the required root.

118. **Special Cases.** Every simple equation is reducible to the form $ax+b=0$, the solution of which is $x=-\frac{b}{a}$. The following are special cases.

I. If $b=0$, the equation reduces to $ax=0$; whence $x=0$.

II. If $b=0$ and also $a=0$, the equation is clearly satisfied for *all* values of x.

III. If $a=0$, and $b \neq 0$.

Suppose that while b remains constant, a takes in succession the values $\frac{1}{10}, \frac{1}{10^2}, \frac{1}{10^3}, \ldots$; then will x take in succession the values $-10b, -10^2b, -10^3b \ldots$. Thus as a becomes continually smaller and smaller, x will become continually greater and greater in absolute magnitude; moreover, by making a sufficiently small, x will become *greater than any assignable quantity;* for example, in order that the absolute value of x may be greater than 10^{20} it is only necessary to give to a an absolute value less than $\frac{b}{10^{20}}$.

This is expressed by saying that, in the limit, when a becomes zero, the root of the equation $ax+b=0$ is *infinite*.

The symbol for infinity is ∞.

EXAMPLES.

Solve the equations

1. $\frac{1}{2}(x-2)-\frac{1}{3}(x-3)+\frac{1}{4}(x-4)=4$. *Ans.* $x=12$.

2. $\frac{1}{3}(x-3)-\frac{1}{4}(x-8)+\frac{1}{5}(x-5)=0$. *Ans.* $x=0$.

3. $a(x-a)=b(x-b)$. *Ans.* $x=a+b$.

4. $(x+a)(x+b)-(x-a)(x-b)=(a+b)^2$. *Ans.* $x=\frac{1}{2}(a+b)$.

5. $a(2x-a)+b(2x-b)=2ab$. *Ans.* $x=\frac{1}{2}(a+b)$.

6. $(a^2+x)(b^2+x)=(ab+x)^2$. *Ans.* $x=0$.

7. $3(x+3)^2+5(x+5)^2=8(x+8)^2$. *Ans.* $x=-6$.

8. $(x+a)^4-(x-a)^4-8ax^3+8a^4=0$. *Ans.* $x=-a$.

9. $(x-1)^3+x^3+(x+1)^3=3x(x^2-1)$. *Ans.* $x=0$.

10. $(x+a)^3+(x+b)^3+(x+c)^3=3(x+a)(x+b)(x+c)$.
Ans. $x=-\frac{1}{3}(a+b+c)$.

119. **Equations expressed in Factors.** It is clear that a product is zero when one of its factors is zero; and it is also clear that a product cannot be zero unless one of its factors is zero.

Thus $(x-2)(x-3)$ is zero when $x-2$ is zero, or when $x-3$ is zero, and in no other case.

Hence the equation

$$(x-2)(x-3)=0,$$

is satisfied if $x-2=0$, or if $x-3=0$; that is, if $x=2$, or if $x=3$, and in no other case. The *roots* of the equation are therefore 2 and 3.

Again, the continued product $(x-a)(x-b)(x-c)\dots$ is zero when $x-a$ is zero, or when $x-b$ is zero, or when $x-c$ is zero, &c.; and the continued product is not zero except one of the factors $x-a$, $x-b$, $x-c$, &c. is zero.

Hence the equation

$$(x-a)(x-b)(x-c)\dots=0$$

is equivalent to the system of equations

$$x-a=0,\ x-b=0,\ x-c=0,\ \text{\&c.}$$

From the above it will be apparent that the solution

of an equation of any degree can be written down at once, provided the equation is given in the form of a product of factors of the first degree equated to zero.

Now all the terms of any equation can be transposed to one side, so that any equation can be written with all its terms on one side of the sign of equality and zero on the other side.

It follows therefore that *the problem of solving an equation of any degree is the same as the problem of finding the factors of an expression of the same degree.*

Ex. 1. Solve the equation $x^2-5x=6.$

Transposing, we have $x^2-5x-6=0,$

that is $(x-6)(x+1)=0;$

$\therefore\ x-6=0, \text{ or } x+1=0.$

Hence $x=6, \text{ or } x=-1.$

Ex. 2. Solve the equation $x^3-x^2=6x.$

Transposing, we have $x^3-x^2-6x=0,$

that is $x(x-3)(x+2)=0;$

$\therefore\ x=0, \text{ or } x=3, \text{ or } x=-2.$

120. **Quadratic Equations.** When all the terms of a quadratic equation are transposed to one side it must be of the form

$$ax^2+bx+c=0,$$

where a, b, c are supposed to represent known quantities.

We have already [Art. 80] shewn how to resolve a quadratic expression into factors: the same method will therefore enable us to find the roots of a quadratic equation.

Hence to solve the quadratic equation

$$ax^2+bx+c=0,$$

we proceed as follows.

Divide by a, the coefficient of x^2; the equation then becomes

$$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

Now add and subtract *the square of half the coefficient of x*, that is the square of $\frac{1}{2}\frac{b}{a}$. Then we have

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}=0,$$

that is $$\left(x+\frac{b}{2a}\right)^2-\left\{\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)}\right\}^2=0,$$

that is

$$\left\{x+\frac{b}{2a}+\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)}\right\}\left\{x+\frac{b}{2a}-\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)}\right\}=0.$$

Hence $$x+\frac{b}{2a}+\sqrt{\frac{b^2-4ac}{4a^2}}=0,$$

or $$x+\frac{b}{2a}-\sqrt{\frac{b^2-4ac}{4a^2}}=0.$$

Thus there are *two* roots of the quadratic equation, namely

$$-\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}.$$

Ex. 1. Solve $$x^2-13x+42=0.$$

We have $$x^2-13x+\left(\frac{13}{2}\right)^2-\left(\frac{13}{2}\right)^2+42=0,$$

that is $$\left(x-\frac{13}{2}\right)^2-\frac{1}{4}=0,$$

or $$\left(x-\frac{13}{2}+\frac{1}{2}\right)\left(x-\frac{13}{2}-\frac{1}{2}\right)=0;$$

$$\therefore\ x-\frac{13}{2}+\frac{1}{2}=0,\text{ or } x-\frac{13}{2}-\frac{1}{2}=0.$$

$$\therefore\ x=6,\text{ or } x=7.$$

Ex. 2. Solve $$3x^2-10x+6=0.$$

Dividing by 3, we have

$$x^2-\frac{10}{3}x+2=0.$$

Hence $$x^2-\frac{10}{3}x+\left(\frac{5}{3}\right)^2-\left(\frac{5}{3}\right)^2+2=0,$$

that is $$\left(x-\frac{5}{3}\right)^2-\frac{7}{9}=0,$$

or $$\left(x-\frac{5}{3}+\frac{\sqrt{7}}{3}\right)\left(x-\frac{5}{3}-\frac{\sqrt{7}}{3}\right)=0;$$

$$\therefore\ x-\frac{5}{3}+\frac{\sqrt{7}}{3}=0,\ \text{or}\ x-\frac{5}{3}-\frac{\sqrt{7}}{3}=0.$$

Hence $$x=\frac{1}{3}(5-\sqrt{7}),\ \text{or}\ x=\frac{1}{3}(5+\sqrt{7}).$$

Ex. 3. Solve $$a(x^2+1)=x(a^2+1).$$

Divide by a and transpose; then

$$x^2-x\frac{a^2+1}{a}+1=0.$$

Hence $$x^2-x\frac{a^2+1}{a}+\left(\frac{a^2+1}{2a}\right)^2-\left(\frac{a^2+1}{2a}\right)^2+1=0,$$

that is $$\left(x-\frac{a^2+1}{2a}\right)^2-\left(\frac{a^2-1}{2a}\right)^2=0,$$

or $$\left(x-\frac{a^2+1}{2a}+\frac{a^2-1}{2a}\right)\left(x-\frac{a^2+1}{2a}-\frac{a^2-1}{2a}\right)=0;$$

$$\therefore\ x-\frac{a^2+1}{2a}+\frac{a^2-1}{2a}=0,\ \text{or}\ x-\frac{a^2+1}{2a}-\frac{a^2-1}{2a}=0,$$

that is $$x-\frac{1}{a}=0,\ \text{or}\ x-a=0.$$

Thus the roots are a and $\frac{1}{a}$.

Note. In most cases the factors can be written down at once, as in Art. 79, without completing the square; and much labour is thereby saved.

EXAMPLES.

Find the roots of the following equations:

1. $9x^2 - 24x + 16 = 0.$ *Ans.* $\frac{4}{3}$.

2. $5(x^2 + 4) = 4(x^2 + 9).$ *Ans.* ± 4.

3. $3x^2 = 8x + 3.$ *Ans.* $3, -\frac{1}{3}$.

4. $16x^2 + 16x + 3 = 0.$ *Ans.* $-\frac{1}{4}, -\frac{3}{4}$.

5. $x^2 + (a - x)^2 = (a - 2x)^2.$ *Ans.* $0, a$.

6. $x^2 + (a - 2x)^2 = (a - 3x)^2.$ *Ans.* $0, \frac{a}{2}$.

7. $x^2 + x = a^2 + a.$ *Ans.* $a, -a - 1$.

8. $x^2 + 2ax = b^2 + 2ab.$ *Ans.* $b, -2a - b$.

9. $(x - a)^2 + (x - b)^2 = (a - b)^2.$ *Ans.* a, b.

10. $(a - x)^3 + (x - b)^3 = (a - b)^3.$ *Ans.* a, b.

11. $(b - c)x^2 + (c - a)x + (a - b) = 0.$ *Ans.* $1, \frac{a - b}{b - c}$.

12. $(x - a + 2b)^3 - (x - 2a + b)^3 = (a + b)^3.$ *Ans.* $a - 2b, 2a - b$.

121. Discussion of roots of a quadratic equation. In the preceding article we found that the quadratic equation $ax^2 + bx + c = 0$ had *two* roots, namely

$$-\frac{b}{2a} + \sqrt{\frac{b^2 - 4ac}{4a^2}} \text{ and } -\frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{4a^2}}.$$

Since $\sqrt{\frac{b^2 - 4ac}{4a^2}}$ is real or imaginary according as $b^2 - 4ac$ is positive or negative, it follows that the roots of $ax^2 + bx + c = 0$ are real or imaginary according as $b^2 - 4ac$ is positive or negative.

The roots are clearly rational or irrational according as $b^2 - 4ac$ is or is not a perfect square. It should be remarked also that *both* roots are rational or *both* irrational, and that *both* roots are real or *both* imaginary.

If $b^2 - 4ac = 0$, both roots reduce to $-\dfrac{b}{2a}$, and are thus equal to one another. In this case we do not say that the equation has only *one* root, but that it has *two equal* roots.

It is clear that the roots will be unequal unless $b^2 - 4ac = 0$. Hence in order that the two roots of the equation $ax^2 + bx + c = 0$ may be equal, it is necessary and sufficient that $b^2 = 4ac$.

When $b^2 = 4ac$, the expression $ax^2 + bx + c$ is a perfect square *in* x, as we have already seen. [Art. 83.]

122. **Special Forms.** We will now consider some special forms of quadratic equations, in which one or more of the coefficients vanish.

I. If $c = 0$, the equation reduces to

$$ax^2 + bx = 0,$$

or

$$x(ax + b) = 0,$$

the roots of which are 0 and $-\dfrac{b}{a}$.

II. If $c = 0$ and also $b = 0$, the equation reduces to $ax^2 = 0$, *both* roots of which are zero.

III. If $b = 0$, the equation reduces to $ax^2 + c = 0$, the roots of which are $\pm\sqrt{-\dfrac{c}{a}}$. The roots are therefore *equal* and *opposite* when $b = 0$, that is when the coefficient of x is zero.

IV. If a, b and c are all zero, the equation is clearly satisfied for *all values* of x.

V. If a and b be zero but c not zero,

put $x = \dfrac{1}{y}$ in the equation $ax^2 + bx + c = 0$;

then we have, after multiplying by y^2,

$$cy^2 + by + a = 0.$$

Now from I. and II. one root of this quadratic in y is zero if $a = 0$, and both roots are zero if $a = 0$ and also $b = 0$.

But since $x = \frac{1}{y}$, x is infinity [Art. 118] when y is zero. Thus one root of $ax^2 + bx + c = 0$ is infinite if $a = 0$; also both roots are infinite if $a = 0$ and also $b = 0$.

Thus the quadratic equation

$$(a - a')\,x^2 + (b - b')\,x + c - c' = 0$$

has *one root infinite*, if $a = a'$; it has *two roots infinite*, if $a = a'$ and also $b = b'$; and the equation is satisfied for *all values* of x, if $a = a'$, $b = b'$ and $c = c'$.

Again, the equation

$$a\,(x + b)\,(x + c) + b\,(x + c)\,(x + a) = c\,(x + a)\,(x + b),$$

is a quadratic equation for all values of c except only when $c = a + b$, in which case the coefficient of x^2 in the quadratic equation is zero. When $c = a + b$ we may still however consider that the equation is a *quadratic* equation, but with one of its roots *infinite*.

Note. It is however to be remarked that since infinite roots are not often of practical importance in Algebra, they are generally neglected unless specially required.

123. Zero and infinite roots of any equation. The most general form of the equation of the nth degree is

$$ax^n + bx^{n-1} + \ldots + kx + l = 0 \ldots\ldots\ldots\ldots\text{(i)}.$$

If $l = 0$, the equation may be written

$$x\,(ax^{n-1} + bx^{n-2} + \ldots + k) = 0,$$

one root of which is clearly zero.

Similarly two roots will be zero if $l = 0$ and also $k = 0$; and so on, if more of the coefficients from the end vanish.

Put $x=\frac{1}{y}$; then we have, after multiplying by y^n,

$$a+by+\ldots\ldots+ky^{n-1}+ly^n=0.$$

From the above, one root of the equation in y will be zero when $a=0$; and two roots will be zero if $a=0$ and also $b=0$. But when $y=0$, $x=\frac{1}{y}=\infty$.

Thus one root of (i) is infinite when $a=0$, and two roots are infinite when a and b are both zero; and so on, if more of the coefficients from the beginning vanish.

124. **Equations not integral.** When an equation is not integral, the first step to be taken is to reduce it to an equivalent integral equation.

An equation will be reduced to an integral form by multiplying by any common multiple of the denominators of the fractions which it contains, but the legitimacy of this multiplication requires examination. For if we multiply both sides of an *integral* equation by an expression which contains the unknown quantity, the new equation will not only be satisfied by all the values of the unknown quantity which satisfy the original equation, but also by those values which make the expression by which we have multiplied vanish. Thus if each member of the equation $A=B$, be multiplied by P, the resulting equation $PA=PB$, or $P(A-B)=0$, will have the same roots as the equation $A-B=0$ *together with* the roots of the equation $P=0$.

When however an equation contains fractions in whose denominators the unknown quantity occurs, the equation may be multiplied by the *lowest* common multiple of the denominators without introducing any additional roots, for we cannot divide both sides of the resulting equation by any one of the factors of the L.C.M. without reintroducing fractions, which shews that there are no roots of the resulting equation which correspond to the factors of the expression by which we multiply.

Ex. 1. Solve the equation $\dfrac{3}{x-5}+\dfrac{2x}{x-3}=5.$

Multiply by $(x-5)(x-3)$, the L. C. M. of the denominators; then we have

$$3(x-3)+2x(x-5)=5(x-5)(x-3);$$

$$\therefore\ 3x^2-33x+84=0.$$

Whence $$x=4 \text{ or } x=7.$$

Ex. 2. Solve the equation $\dfrac{x^2-3x}{x^2-1}+2+\dfrac{1}{x-1}=0.$

Multiply by x^2-1, the L. C. M. of the denominators; then we have

$$x^2-3x+2(x^2-1)+x+1=0,$$

which reduces to

$$3x^2-2x-1=0,$$

that is $$(3x+1)(x-1)=0.$$

Thus the roots appear to be $-\frac{1}{3}$ and 1; the latter root is however due to the multiplication by x^2-1.

Since $$\frac{x^2-3x}{x^2-1}+\frac{1}{x-1}=\frac{x^2-3x+x+1}{x^2-1}=\frac{(x-1)^2}{x^2-1}=\frac{x-1}{x+1},$$

the equation is equivalent to

$$\frac{x-1}{x+1}+2=0,$$

which has only one root, namely $x=-\frac{1}{3}$.

From the above example it will be seen that when an equation has been made integral by multiplication, some of the roots of the resulting equation may have to be rejected.

Ex. 3. Solve the equation:

$$\frac{x}{x-2}+\frac{x-9}{x-7}=\frac{x+1}{x-1}+\frac{x-8}{x-6}.$$

In this case it is best not to multiply at once by the L. C. M. of the denominators of the fractions; much labour is often saved by a judicious arrangement and grouping of the terms.

By transposition we have

$$\frac{x}{x-2}-\frac{x+1}{x-1}+\frac{x-9}{x-7}-\frac{x-8}{x-6}=0.$$

$$\text{The first two terms}=\frac{2}{(x-2)(x-1)},$$

and the other terms $=\dfrac{-2}{(x-7)(x-6)}$.

Hence the equation is equivalent to

$$\frac{2}{(x-2)(x-1)}-\frac{2}{(x-7)(x-6)}=0.$$

Now multiply by the L. C. M. of the denominators; then

$$2(x-7)(x-6)-2(x-2)(x-1)=0,$$

which reduces to

$$20x-80=0;$$

$$\therefore\ x=4.$$

Or thus:—

The equation is equivalent to

$$\frac{x}{x-2}-1\ +\frac{x-9}{x-7}-1\ =\frac{x+1}{x-1}-1\ +\frac{x-8}{x-6}-1,$$

that is $$\frac{2}{x-2}-\frac{2}{x-7}=\frac{2}{x-1}-\frac{2}{x-6};$$

$$\therefore\ \frac{-10}{(x-2)(x-7)}=\frac{-10}{(x-1)(x-6)},$$

from which we find as before that $x=4$.

Ex. 4. Solve the equation:

$$\frac{a}{x+a}+\frac{b}{x+b}+\frac{c}{x+c}=3.$$

We have $$\frac{a}{x+a}-1\ +\frac{b}{x+b}-1\ +\frac{c}{x+c}-1=0;$$

$$\therefore\ \frac{x}{x+a}+\frac{x}{x+b}+\frac{x}{x+c}=0.$$

Hence $$x=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)},$$

or else $$\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}=0.$$

Multiply by the L. C. M.; then

$$(x+b)(x+c)+(x+c)(x+a)+(x+a)(x+b)=0,$$

that is

$$3x^2+2x(a+b+c)+bc+ca+ab=0,$$

the roots of which are

$$-\frac{1}{3}\{(a+b+c) \pm \sqrt{(a^2+b^2+c^2-bc-ca-ab)}\} \text{ .... (ii).}$$

Thus there are three roots given by (i) and (ii).

Ex. 5. Solve the equation:

$$\frac{b+c}{bc-x}+\frac{c+a}{ca-x}+\frac{a+b}{ab-x}=\frac{a+b+c}{x}.$$

The equation is equivalent to

$$\frac{b+c}{bc-x}-\frac{a}{x}+\frac{c+a}{ca-x}-\frac{b}{x}+\frac{a+b}{ab-x}-\frac{c}{x}=0.$$

Taking the terms in pairs we have

$$\frac{(a+b+c)x-abc}{x(bc-x)}+\frac{(a+b+c)x-abc}{x(ca-x)}+\frac{(a+b+c)x-abc}{x(ab-x)}=0.$$

Hence $\qquad (a+b+c)x-abc=0$ .... I,

or $\qquad \dfrac{1}{x(bc-x)}+\dfrac{1}{x(ca-x)}+\dfrac{1}{x(ab-x)}=0$ .... II.

From I. we have $x=\dfrac{abc}{a+b+c}$.

From II. we have on multiplication by the L. C. M.

$$(ca-x)(ab-x)+(ab-x)(bc-x)+(bc-x)(ca-x)=0,$$

that is $\qquad 3x^2-2x(bc+ca+ab)+abc(a+b+c)=0,$

whence $x=\frac{1}{3}\{bc+ca+ab\pm\sqrt{b^2c^2+c^2a^2+a^2b^2-abc(a+b+c)}\}$.

125. Irrational Equations. An *irrational* equation is one in which square or other roots of expressions containing the unknown quantity occur.

In order to rationalize an equation it is first written with one of the irrational terms standing by itself on one side of the sign of equality: both sides are then raised to the lowest power necessary to rationalize the isolated term; and the process is repeated as often as may be necessary.

Ex. 1. Solve the equation $\sqrt{x+4}+\sqrt{x+20}-2\sqrt{x+11}=0$.

We have $$\sqrt{x+4}+\sqrt{x+20}=2\sqrt{x+11}.$$

Square both members: then

$$2x+24+2\sqrt{x+4}\sqrt{x+20}=4(x+11),$$

which is equivalent to

$$\sqrt{x+4}\sqrt{x+20}=x+10.$$

Square both members: then

$$(x+4)(x+20)=(x+10)^2,$$

whence $$x=5.$$

Ex. 2. Solve the equation $\sqrt{2x+8}-2\sqrt{x+5}=2$.

Square both members: then

$$2x+8+4(x+5)-4\sqrt{2x+8}\sqrt{x+5}=4;$$

$$\therefore\ 3x+12=2\sqrt{2x+8}\sqrt{x+5}.$$

Square both members: then

$$9x^2+72x+144=4(2x+8)(x+5);$$

$$\therefore\ x^2=16,$$

whence $$x=4 \text{ or } x=-4.$$

Ex. 3. Solve the equation $\sqrt{ax+\alpha}+\sqrt{bx+\beta}+\sqrt{cx+\gamma}=0$.

We have $$\sqrt{ax+\alpha}+\sqrt{bx+\beta}=-\sqrt{cx+\gamma}.$$

Square both members: then we have after transposition

$$(a+b-c)x+\alpha+\beta-\gamma=-2\sqrt{ax+\alpha}\sqrt{bx+\beta}.$$

Squaring again, we have

$$\{(a+b-c)x+\alpha+\beta-\gamma\}^2=4(ax+\alpha)(bx+\beta),$$

that is $$x^2(a^2+b^2+c^2-2bc-2ca-2ab)$$
$$+2x(a\alpha+b\beta+c\gamma-b\gamma-c\beta-c\alpha-a\gamma-a\beta-b\alpha)$$
$$+\alpha^2+\beta^2+\gamma^2-2\beta\gamma-2\gamma\alpha-2\alpha\beta=0.$$

Thus the given equation is equivalent to a *quadratic* equation.

It should be observed that it is quite immaterial what sign is put before a radical in the above examples; for there are *two* square roots of every algebraical expression and we have no symbol

which represents one only to the exclusion of the other; so that $+\sqrt{x+1}$ and $-\sqrt{x+1}$ are alike equivalent to $\pm\sqrt{x+1}$; also $x+\sqrt{x+1}$ has the same *two* values as $x\pm\sqrt{x+1}$.

126. By squaring both members of the rational equation $A=B$, we obtain the equation $A^2=B^2$; and the equation $A^2=B^2$, or $A^2-B^2=0$, is not only satisfied when $A-B=0$, but also when $A+B=0$. Hence an equation is not in general equivalent to that obtained by squaring both its members; for the latter equation has the same roots as the original equation together with other roots which are not roots of the original equation. Additional roots are not however always introduced by squaring both sides of an *irrational* equation. For example, the equation $x+1=\sqrt{x+13}$ is really *two* equations since the radical may have either of two values; and by squaring both members we obtain the equation $(x+1)^2=x+13$, which is equivalent to the two. [See Art. 152.]

127. **A quadratic equation can only have two roots.** We have already proved that an expression of the nth degree in x cannot vanish for more than n values of x, unless it vanishes for *all* values of x. This shews that an equation of the nth degree cannot have more than n roots, and in particular that a *quadratic* equation cannot have more than *two* roots.

The following is another proof that a quadratic equation can only have two roots.

We have to prove that ax^2+bx+c cannot vanish for α, β, γ three unequal values of x. That is we have to prove that

$$a\alpha^2+b\alpha+c=0 \quad \text{(i)},$$

$$a\beta^2+b\beta+c=0 \quad \text{(ii)},$$

and

$$a\gamma^2+b\gamma+c=0 \quad \text{(iii)},$$

cannot be simultaneously true, unless a, b, c are all zero.

From (i) and (ii) we have by subtraction

$$a(\alpha^2-\beta^2)+b(\alpha-\beta)=0,$$

that is $$(\alpha-\beta)\{a(\alpha+\beta)+b\}=0.$$

But $$\alpha-\beta\neq 0;\text{ hence}$$

$$a(\alpha+\beta)+b=0 \quad\text{..................(iv)}.$$

Similarly, since $\beta-\gamma\neq 0$, we have from (ii) and (iii)

$$a(\beta+\gamma)+b=0 \quad\text{..................(v)}.$$

From (iv) and (v) we have by subtraction

$$a(\alpha-\gamma)=0 \quad\text{.....................(vi)}.$$

Now (vi) cannot be true unless $a=0$, for $\alpha-\gamma\neq 0$. Also when $a=0$, it follows from (iv) that $b=0$, and then from (i) that $c=0$.

Thus the quadratic equation $ax^2+bx+c=0$ cannot have *more than two different roots, unless* $a=b=c=0$; and when a, b, c are all zero it is clear that the equation $ax^2+bx+c=0$ will be satisfied for *all values* of x, that is to say the equation is an *identity*.

Ex. 1. Solve the equation $a^2\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\dfrac{(x-c)(x-a)}{(b-c)(b-a)}=x^2$.

The equation is clearly satisfied by $x=a$, and also by $x=b$; hence a, b are roots of the equation, and these are the only roots of the quadratic equation. [The equation is not an *identity*, for it is not satisfied by $x=c$.]

Ex. 2. Solve the equation

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\frac{(x-c)(x-a)}{(b-c)(b-a)}+c^2\frac{(x-a)(x-b)}{(c-a)(c-b)}=x^2.$$

The equation is satisfied by $x=a$, by $x=b$, or by $x=c$. Hence, as it is only of the *second* degree in x, it must be an *identity*.

Ex. 3. Solve the equation

$$a^3\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^3\frac{(x-c)(x-a)}{(b-c)(b-a)}+c^3\frac{(x-a)(x-b)}{(c-a)(c-b)}=x^3.$$

The equation is satisfied by $x=a$, by $x=b$, and by $x=c$; and the equation is not an identity, since the coefficient of x^3 is not zero.
Hence the roots of the cubic are a, b, c.

Ex. 4. Shew that, if

$$(a-\alpha)^2x+(a-\beta)^2y+(a-\gamma)^2z=(a-\delta)^2,$$
$$(b-\alpha)^2x+(b-\beta)^2y+(b-\gamma)^2z=(b-\delta)^2,$$
$$(c-\alpha)^2x+(c-\beta)^2y+(c-\gamma)^2z=(c-\delta)^2,$$

then will

$$(d-\alpha)^2x+(d-\beta)^2y+(d-\gamma)^2z=(d-\delta)^2,$$

where d has any value whatever.

The equation

$$(X-\alpha)^2x+(X-\beta)^2y+(X-\gamma)^2z=(X-\delta)^2$$

is a *quadratic* equation in X, and it has the *three* roots a, b, c. It is therefore satisfied when *any* other quantity d is put for X.

128. **Relations between the roots and the coefficients of a quadratic equation.**

If we put α and β for the roots of the equation $ax^2+bx+c=0$, we have

$$\alpha=-\frac{b}{2a}+\sqrt{\frac{b^2-4ac}{4a^2}},$$

and
$$\beta=-\frac{b}{2a}-\sqrt{\frac{b^2-4ac}{4a^2}}.$$

By addition we have

$$\alpha+\beta=-\frac{b}{a}\ \dots\dots\dots\dots\dots\dots\text{(i)}.$$

By multiplication we have

$$\alpha\beta=\frac{b^2}{4a^2}-\frac{b^2-4ac}{4a^2}=\frac{c}{a}\ \dots\dots\dots\dots\text{(ii)}.$$

The formulae (i) and (ii) giving the sum and the product of the roots of a quadratic equation in terms of the coefficients are very important.

129. **Relations between the roots and the coefficients of any equation.** By the following method relations between the roots and the coefficients of an equation of any degree may be obtained.

We have seen that if the expression of the nth degree in x

$$ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \dots,$$

vanish for the n values $x=\alpha$, $x=\beta$, $x=\gamma$, &c., then will

$$ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \dots = a(x-\alpha)(x-\beta)(x-\gamma)\dots$$

We have therefore only to find* the continued product $(x-\alpha)(x-\beta)(x-\gamma)\dots\dots$ and equate the coefficients of the corresponding powers of x on the two sides of the last equation.

For example, if α, β, γ be the roots of the cubic equation $ax^3+bx^2+cx+d=0$, we have

$$ax^3+bx^2+cx+d = a(x-\alpha)(x-\beta)(x-\gamma)$$
$$= a\{x^3 - (\alpha+\beta+\gamma)x^2 + (\beta\gamma+\gamma\alpha+\alpha\beta)x - \alpha\beta\gamma\}.$$

Hence, equating coefficients, we have

$$\left.\begin{aligned} \alpha+\beta+\gamma &= -\frac{b}{a}, \\ \beta\gamma+\gamma\alpha+\alpha\beta &= \frac{c}{a}, \\ \alpha\beta\gamma &= -\frac{d}{a} \end{aligned}\right\}.$$

It should be remarked that the sum of the roots of any equation will be zero provided that the term one degree lower than the highest is absent*.

We may make use of the above to prove certain identical relations between three quantities whose sum is zero. For a, b, c will be the roots of the cubic $x^3+px+q=0$, provided that $a+b+c=0$, and that p and q satisfy the relations

* See Art. 260,

$$bc+ca+ab=p \qquad \text{(i)},$$

$$abc=-q \qquad \text{(ii)}.$$

Then, since $a+b+c=0$ (iii),

we have
$$a^2+b^2+c^2=(a+b+c)^2-2(bc+ca+ab)$$
$$=-2p \qquad \text{(iv)}.$$

Also, since a, b, c are roots of $x^3+px+q=0$,

$$\left.\begin{aligned} a^3+pa+q&=0 \\ b^3+pb+q&=0 \\ c^3+pc+q&=0 \end{aligned}\right\} \qquad \text{(v)}.$$

From (v) by addition

$$a^3+b^3+c^3=-3q \qquad \text{(vi)}.$$

Multiply the equations (v) in order by a^{n-3}, b^{n-3}, c^{n-3}, and add; then

$$a^n+b^n+c^n+p(a^{n-2}+b^{n-2}+c^{n-2})+q(a^{n-3}+b^{n-3}+c^{n-3})=0.$$

Hence we have in succession

$$a^4+b^4+c^4=2p^2,$$
$$a^5+b^5+c^5=5pq,$$
$$a^6+b^6+c^6=3q^2-2p^3,$$
$$a^7+b^7+c^7=-7p^2q.$$

Hence also

$$\frac{a^5+b^5+c^5}{5}=\frac{a^2+b^2+c^2}{2}\cdot\frac{a^3+b^3+c^3}{3},$$

$$\frac{a^7+b^7+c^7}{7}=\frac{a^2+b^2+c^2}{2}\cdot\frac{a^5+b^5+c^5}{5},$$

$$=2\cdot\frac{a^3+b^3+c^3}{3}\cdot\frac{a^4+b^4+c^4}{4}.$$

[See also Art. 308, Ex. 2.]

130. **Equations with given roots.** Although we cannot in all cases find the roots of a *given equation*, it is very easy to solve the converse problem, namely the problem of finding an equation which has *given roots*.

For example, to find the equation whose roots are 4 and 5.

We want to find an equation which is satisfied when $x=4$, or when $x=5$; that is when $x-4=0$, or when $x-5=0$; and in no other cases. The equation required must be

$$(x-4)(x-5)=0,$$

that is,
$$x^2-9x+20=0,$$

for this is an equation which is a true statement when $x-4=0$, or when $x-5=0$, and in no other case*.

Again, to find the equation whose roots are 2, 3, and -4.

We have to find an equation which is satisfied when $x-2=0$, or when $x-3=0$, or when $x+4=0$, and in no other case. The equation must therefore be $(x-2)(x-3)(x+4)=0$,

that is $x^3-x^2-14x+24=0$.

Ex. 1. If α, β are the roots of the equation $ax^2+bx+c=0$, find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

The required equation is

$$\left(x-\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=0,$$

that is
$$x^2-x\frac{\alpha^2+\beta^2}{\alpha\beta}+1=0.$$

Now, by Art. 128, we have

$$\alpha+\beta=-\frac{b}{a}, \quad \alpha\beta=\frac{c}{a};$$

$$\therefore \alpha^2+\beta^2=\frac{b^2}{a^2}-2\frac{c}{a};$$

$$\therefore \frac{\alpha^2+\beta^2}{\alpha\beta}=\left(\frac{b^2}{a^2}-2\frac{c}{a}\right)\div\frac{c}{a}=\frac{b^2-2ac}{ac}.$$

Hence the required equation is

$$x^2-\frac{b^2-2ac}{ac}x+1=0.$$

Ex. 2. If α, β, γ be the roots of the equation $ax^3+bx^2+cx+d=0$, find the equation whose roots are $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$.

The required equation is

$$(x-\beta\gamma)(x-\gamma\alpha)(x-\alpha\beta)=0,$$

that is
$$x^3-x^2(\beta\gamma+\gamma\alpha+\alpha\beta)+x\alpha\beta\gamma(\alpha+\beta+\gamma)-\alpha^2\beta^2\gamma^2=0.$$

* The equation $x^2-9x+20=0$ is certainly *an* equation with the proposed and with no other roots; but to prove that it is the *only* equation with the proposed and with no other roots, it must be assumed that *every equation has a root.*

If, for example, the equation $x^5+7x^2-2=0$ had no roots, then $(x-4)(x-5)(x^5+7x^2-2)=0$ would also be an equation with the proposed roots and with no others.

The proposition that every equation has a root is by no means easy to prove; the proof is given in works on the Theory of Equations.

Now, by Art. 129, we have

$$\alpha+\beta+\gamma=-\frac{b}{a},$$

$$\beta\gamma+\gamma\alpha+\alpha\beta=\frac{c}{a},$$

and

$$\alpha\beta\gamma=-\frac{d}{a}.$$

Hence the required equation is

$$x^3-\frac{c}{a}x^2+x\frac{db}{a^2}-\frac{d^2}{a^2}=0,$$

or

$$a^2x^3-acx^2+bdx-d^2=0.$$

131. **Changes in value of a trinomial expression.** The expression ax^2+bx+c will alter in value as the value of x is changed; but, by giving to x any real value between $-\infty$ and $+\infty$, we cannot make the expression ax^2+bx+c assume any value we please.

We can find the possible values of ax^2+bx+c, for real values of x, as follows.

In order that the *expression* ax^2+bx+c may be equal to λ for some real value of x, it is necessary and sufficient that the roots of the *equation*

$$ax^2+bx+c=\lambda$$

be real, the condition for which is

$$b^2-4a(c-\lambda)>0,$$

that is

$$b^2-4ac+4a\lambda>0 \quad\text{..................(i).}$$

I. If b^2-4ac be positive, the condition (i) is satisfied for all positive values of $4a\lambda$, and also for all negative values of $4a\lambda$ which are not greater than b^2-4ac.

Thus, when b^2-4ac is positive, ax^2+bx+c can, by giving a suitable value to x, be made equal to any quantity of the same sign as a, or to any quantity not absolutely greater than $\dfrac{b^2-4ac}{4a}$ and whose sign is opposite to that of a.

II. If b^2-4ac be negative, the condition (i) can only be satisfied when $4a\lambda$ is positive and not less than $4ac-b^2$.

Thus, when $b^2 - 4ac$ is negative, $ax^2 + bx + c$ must always have the same sign as a, and its absolute magnitude can never be less than $\dfrac{4ac - b^2}{4a}$.

III. If $b^2 - 4ac$ be zero, the condition (i) is satisfied for all positive values of $a\lambda$.

It follows from the above that the *expression* $ax^2 + bx + c$ will keep its sign unchanged, whatever real value be given to x, provided that $b^2 - 4ac$ be negative or zero, that is provided that the roots of the *equation* $ax^2 + bx + c = 0$ be imaginary or equal, and also that the expression can be made to change its sign when the roots of $ax^2 + bx + c = 0$ are real and unequal. We give another proof of this proposition.

If the equation $ax^2 + bx + c = 0$ have real roots, α, β suppose, then $ax^2 + bx + c = a(x - \alpha)(x - \beta)$.

Now $(x - \alpha)(x - \beta)$ is positive when x has any real value greater than both α and β, or less than both α and β; but $(x - \alpha)(x - \beta)$ is negative when x has any real value intermediate to α and β.

Thus for real values of x the *expression* $ax^2 + bx + c$ has always the same sign as a except for values of x which lie between the roots of the corresponding *equation* $ax^2 + bx + c = 0$.

132. We can also prove that the expression $ax^2 + bx + c$ will or will not change sign for different values of x according as $b^2 - 4ac$ is positive or negative, as follows.

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right].$$

I. Let $b^2 - 4ac$ be positive.

The whole expression within square brackets will clearly be *negative* when $x = -\dfrac{b}{2a}$; also, when x is very

great, $\left(x+\frac{b}{2a}\right)^2$ will be greater than $\frac{b^2-4ac}{4a^2}$, and therefore the whole expression within square brackets will be *positive*.

Thus when b^2-4ac is positive the expression ax^2+bx+c can be made to change its sign by giving suitable real values to x.

II. Let b^2-4ac be negative (or zero).

Since $\left(x+\frac{b}{2a}\right)^2$ is positive for all real values of x, and $-\frac{b^2-4ac}{4a^2}$ is also positive (or zero), the whole expression within square brackets must be always positive.

Thus when b^2-4ac is negative or zero, the expression ax^2+bx+c will always have the same sign as a.

133. It follows from Article 131 or 132 that if an expression of the second degree in x can be made to change its sign by giving real values to x, then must the roots of the corresponding equation be real.

Consider, for example, the expression

$$a^2(x-\beta)(x-\gamma)+b^2(x-\gamma)(x-\alpha)+c^2(x-\alpha)(x-\beta),$$

where the quantities are all real, and α, β, γ are supposed to be in order of magnitude. The expression is clearly positive if $x=\alpha$, and is negative if $x=\beta$. Hence the expression can be made to change its sign, and therefore the roots of the equation

$$a^2(x-\beta)(x-\gamma)+b^2(x-\gamma)(x-\alpha)+c^2(x-\alpha)(x-\beta)=0$$

are real for all real values of a, b, c, α, β, γ.

Ex. 1. Shew that $(x-1)(x-3)(x-4)(x-6)+10$ is positive for all real values of x.

Taking the first and last factors together, and also the other two, the given expression becomes

$$\begin{aligned}&(x^2-7x+6)(x^2-7x+12)+10\\ &=(x^2-7x)^2+18(x^2-7x)+82\\ &=\{(x^2-7x)+9\}^2+1,\end{aligned}$$

which is clearly always positive for real values of x.

Ex. 2. Shew that, by giving an appropriate real value to x, $\frac{4x^2+36x+9}{12x^2+8x+1}$ can be made to assume any real value.

Put
$$\frac{4x^2+36x+9}{12x^2+8x+1}=\lambda;$$
then
$$x^2(4-12\lambda)+(36-8\lambda)x+9-\lambda=0.$$
Now in order that x may be real it is necessary and sufficient that
$$(36-8\lambda)^2-4(4-12\lambda)(9-\lambda)>0,$$
or that
$$\lambda^2-8\lambda+72>0,$$
or
$$(\lambda-4)^2+56>0,$$
which is clearly true for *all real values* of λ. Thus we can find real values of x corresponding to any real value whatever of λ.

Ex. 3. Shew that $\frac{x^2-3x+4}{x^2+3x+4}$ can never be greater than 7 nor less than $\frac{1}{7}$ for real values of x.

Put
$$\frac{x^2-3x+4}{x^2+3x+4}=\lambda;$$
then
$$x^2(1-\lambda)-3x(1+\lambda)+4(1-\lambda)=0.$$
In order that x may be real it is necessary and sufficient that
$$9(1+\lambda)^2-16(1-\lambda)^2>0,$$
that is
$$-7\lambda^2+50\lambda-7>0,$$
or
$$-(7\lambda-1)(\lambda-7)>0.$$
Hence $7\lambda-1$ and $\lambda-7$ must be of *different* signs, and therefore λ must lie between $\frac{1}{7}$ and 7, which proves the proposition.

EXAMPLES X.

Solve the following equations:

1. $(x-a+2b)^3-(x-2a+b)^3=(a+b)^3.$

2. $(c+a-2b)x^2+(a+b-2c)x+(b+c-2a)=0.$

3. $\frac{a^2}{(x-a)^2}=\frac{b^2}{(x+b)^2}.$

4. $\frac{a+x}{b+x}+\frac{b+x}{a+x}=2\frac{1}{2}.$

5. $\frac{ax+b}{a+bx}=\frac{cx+d}{c+dx}.$

6. $\frac{a-x}{1-ax}=\frac{1-bx}{b-x}.$

7. $\frac{3x-4}{x+1}=x^2+2x-\frac{7}{x+1}.$

8. $x+1+\frac{x^2}{x^2-1}=\frac{x^2}{x+1}+\frac{5x-4}{x^2-1}.$

9. $\frac{1}{x-8}+\frac{1}{x-6}+\frac{1}{x+6}+\frac{1}{x+8}=0.$

10. $\frac{2}{x+8}+\frac{5}{x+9}=\frac{3}{x+15}+\frac{4}{x+6}.$

11. $\frac{2}{2x-3}+\frac{1}{x-2}=\frac{6}{3x+2}.$

12. $\frac{x-a}{x-b}+\frac{x-b}{x-c}+\frac{x-c}{x-a}=3.$

13. $\frac{x+a}{a-x}+\frac{x+b}{b-x}+\frac{x+c}{c-x}=3.$

14. $\frac{x+a}{x-a}+\frac{x+b}{x-b}+\frac{x+c}{x-c}=3.$

15. $\frac{2x-1}{x+1}+\frac{3x-1}{x+2}=4+\frac{x-7}{x-1}.$

16. $\frac{x}{2}+\frac{2}{x}=\frac{x}{3}+\frac{3}{x}.$

17. $\frac{x+a}{x-a}+\frac{x-a}{x+a}+\frac{x+b}{x-b}+\frac{x-b}{x+b}=0.$

18. $\frac{x-1}{x+1}+\frac{x-4}{x+4}=\frac{x-2}{x+2}+\frac{x-3}{x+3}.$

19. $\frac{1}{x+a+\frac{1}{x+b}}=\frac{1}{x-a+\frac{1}{x-b}}.$

20. $\dfrac{1}{3a-x}+\dfrac{1}{3b-x}+\dfrac{1}{3c-x}=0.$

21. $\dfrac{a+b}{x+b}+\dfrac{a+c}{x+c}=\dfrac{2(a+b+c)}{x+b+c}.$

22. $\dfrac{a+c}{x+2b}+\dfrac{b+c}{x+2a}=\dfrac{a+b+2c}{x+a+b}.$

23. $\dfrac{x-b}{x-a}-\dfrac{x-a}{x-b}=\dfrac{2(a-b)}{x-a-b}.$

24. $\dfrac{(x+a)(x+b)}{x+a+b}=\dfrac{(x+c)(x+d)}{x+c+d}.$

25. $\dfrac{a(c-d)}{x+a}+\dfrac{d(a-b)}{x+d}=\dfrac{b(c-d)}{x+b}+\dfrac{c(a-b)}{x+c}.$

26. $\dfrac{x-a}{b}+\dfrac{x-b}{a}=\dfrac{b}{x-a}+\dfrac{a}{x-b}.$

27. $\dfrac{a-b}{x+a-b}+\dfrac{b-c}{x+b-c}+\dfrac{c-a}{x+c-a}=0.$

28. $\dfrac{1}{1+2x}-\dfrac{2}{2+3x}+\dfrac{3}{3+4x}-\dfrac{4}{4+5x}=0.$

29. $\dfrac{(x-a)(x-b)}{(x-ma)(x-mb)}=\dfrac{(x+a)(x+b)}{(x+ma)(x+mb)}.$

30. $\sqrt{2x+9}-\sqrt{x-4}=\sqrt{x+1}.$

31. $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2}.$

32. $\sqrt{7x-5}+\sqrt{4x-1}=\sqrt{7x-4}+\sqrt{4x-2}.$

33. $\sqrt{a^2-x}+\sqrt{b^2+x}=a+b.$

34. $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b-2x}.$

35. $\sqrt{a-bx}+\sqrt{c-dx}=\sqrt{a+c-(b+d)x}.$

36. $\sqrt{ax+b^2}+\sqrt{bx+a^2}=a-b.$

37. $\sqrt{a+x}+\sqrt{b+x}=\sqrt{a+b+2x}$.

38. $\sqrt{a-x}+\sqrt{b+x}=\sqrt{2a+2b}$.

39. $\sqrt{(a+x)(x+b)}+\sqrt{(a-x)(x-b)}=2\sqrt{ax}$.

40. $\sqrt{a(a+b+x)}-\sqrt{a(a+b-x)}=x$.

41. $\sqrt{x^2+ax+b^2}-\sqrt{x^2-ax+b^2}=2a$.

42. $\sqrt{x^2+ax+a^2}+\sqrt{x^2-ax+a^2}=\sqrt{2a^2-2b^2}$.

43. $\sqrt{ax-b}+\sqrt{cx+b}=\sqrt{ax+b}+\sqrt{cx-b}$.

44. $\sqrt{x(a+b-x)}+\sqrt{a(b+x-a)}+\sqrt{b(a+x-b)}=0$.

45. $\sqrt{x+a}+\sqrt{x+b}+\sqrt{x+c}=0$.

46. $\sqrt{ab(a+b+x)}=\sqrt{a(a+b)(b-x)}+\sqrt{b(a+b)(a-x)}$.

47. $\sqrt{x^2-b^2-c^2}+\sqrt{x^2-c^2-a^2}+\sqrt{x^2-a^2-b^2}=x$.

48. $\sqrt{a^2-x^2}+\sqrt{b^2-x^2}+\sqrt{c^2-x^2}=\sqrt{a^2+b^2+c^2-x^2}$.

49. For what values of x is $\sqrt{14-(3x-2)(x-1)}$ real.

50. Shew that $\dfrac{x^2+34x-71}{x^2+2x-7}$ can have no real value between 5 and 9.

51. Shew that, if x be real $\dfrac{x^2-6x+5}{x^2+2x+1}$ can never be less than $-\frac{1}{3}$.

52. What values are possible for $\dfrac{x^2-x+1}{x^2+x+1}$, x being real.

53. Find the greatest and least real values of x and y which satisfy the equation $x^2+y^2=6x-8y$.

54. Find the greatest and least real values of x and y when

$$x^2+4y^2-8x-16y-4=0.$$

55. When x and y are taken so as to satisfy the equation $(x^2+y^2)^2=2a^2(x^2-y^2)$, find the greatest possible value of y.

56. Shew that if the roots of the equation

$$x^2(b^2+b'^2)+2x(ab+a'b')+a^2+a'^2=0$$

be real, they will be equal.

57. If the roots of the equation $ax^2+bx+c=0$ be in the ratio $m:n$, then will $mnb^2=(m+n)^2 ac$.

58. If $ax^2+2bx+c=0$ and $a'x^2+2b'x+c'=0$ have one and only one root in common, prove that b^2-ac and $b'^2-a'c'$ must both be perfect squares.

59. If x_1, x_2 be the roots of the equation $ax^2+bx+c=0$, find the equation whose roots are (i) x_1^3 and x_2^3, (ii) $\dfrac{x_1^2}{x_2}$ and $\dfrac{x_2^2}{x_1}$, (iii) $b+ax_1$ and $b+ax_2$.

60. If x_1, x_2 be the roots of $ax^2+bx+c=0$, find in terms of a, b, c the values of

$$x_1^2(bx_2+c)+x_2^2(bx_1+c), \text{ and } x_1^2(bx_2+c)^2+x_2^2(bx_1+c)^2.$$

61. Shew that, if x_1, x_2 be the roots of $x^2+mx+m^2+a=0$, then will $x_1^2+x_1x_2+x_2^2+a=0$.

62. If x_1, x_2 be the roots of $(x^2+1)(a^2+1)=max(ax-1)$, then will $(x_1^2+1)(x_2^2+1)=mx_1x_2(x_1x_2-1)$.

63. If x_1, x_2 be the roots of the equation

$$A(x^2+m^2)+Amx+Bm^2x^2=0,$$

then will $$A(x_1^2+x_2^2)+Ax_1x_2+Bx_1^2x_2^2=0.$$

64. Prove that, if x be real, $2(a-x)(x+\sqrt{x^2+b^2})$ cannot exceed a^2+b^2.

65. Find the least possible value of $\dfrac{2x^4-4x^3+9x^2-4x+2}{(x^2+1)^2}$, for real values of x.

Equations of higher degree than the second.

134. We now consider some special forms of equations of higher degree than the second, the solution of the most general forms of such equations being beyond our range.

135. **Equations of the same form as quadratic equations.**

The equation $ax^4 + bx^2 + c = 0$

can be solved in exactly the same way as the quadratic equation $ax^2 + bx + c = 0$;

we therefore have

$$x^2 = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Hence $$x = \pm\sqrt{\left\{-\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}\right\}}.$$

Thus there are *four* real or imaginary roots.

Similarly, whenever an equation only contains the unknown quantity in two terms one of which is the square of the other, the equation can be reduced to two alternative equations: for, whatever P may be,

$$aP^2 + bP + c = 0$$

is equivalent to $$P = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Ex. 1. To solve $x^4 - 10x^2 + 9 = 0.$

We have $(x^2 - 9)(x^2 - 1) = 0;$

$\therefore x^2 = 9$, giving $x = \pm 3$;

or else $x^2 = 1$, giving $x = \pm 1$.

Thus there are four roots, namely $+1, -1, +3, -3$.

Ex. 2. To solve $(x^2+x)^2+4(x^2+x)-12=0$.

The equation may be written $(x^2+x+6)(x^2+x-2)=0$.

Hence $\qquad x^2+x+6=0$, or $x^2+x-2=0$.

The roots of $\qquad x^2+x+6=0$ are $-\frac{1}{2}\pm\frac{1}{2}\sqrt{-23}$.

The roots of $\qquad x^2+x-2=0$ are 1 and -2.

Thus the roots are $\qquad 1,\ -2,\ -\frac{1}{2}\pm\frac{1}{2}\sqrt{-23}$.

Ex. 3. $$(x^2+2)^2+8x(x^2+2)+15x^2=0.$$

The equation is equivalent to

$$(x^2+2+5x)(x^2+2+3x)=0.$$

The roots of $\qquad x^2+3x+2=0$ are -1 and -2.

The roots of $\qquad x^2+5x+2=0$ are $-\frac{5}{2}\pm\frac{\sqrt{17}}{2}$.

Thus the equation has the four roots $-1,\ -2,\ -\frac{5}{2}\pm\frac{1}{2}\sqrt{17}$.

Ex. 4. To solve $\qquad ax^2+bx+c+p\sqrt{ax^2+bx+c}+q=0$.

Put $$y=\sqrt{ax^2+bx+c};$$

then $$y^2+py+q=0,$$

whence we obtain two values of y, α and β suppose.

We then have $$ax^2+bx+c=\alpha^2,$$

or $$ax^2+bx+c=\beta^2,$$

and the four roots of the last two quadratic equations are the roots required.

Ex. 5. To solve $\qquad 2x^2-4x+3\sqrt{x^2-2x+6}=15$.

The equation may be written

$$2(x^2-2x+6)+3\sqrt{(x^2-2x+6)}-27=0.$$

Put $\quad y=\sqrt{(x^2-2x+6)}$; then we have $2y^2+3y-27=0$,

whence $$y=3, \text{ or } y=-\frac{9}{2}.$$

Hence $\qquad x^2-2x+6=9$, giving $x=3$ or -1;

or else $\qquad x^2-2x+6=\frac{81}{4}$, giving $x=1\pm\frac{1}{2}\sqrt{61}$.

Thus the roots are $\qquad 3;\ -1;\ 1\pm\frac{1}{2}\sqrt{61}$.

Ex. 6. To solve $(x+a)(x+2a)(x+3a)(x+4a)=\frac{9}{16}a^4$.

Taking together the first and last of the factors on the left, and also the second and third, the equation becomes of the form we are now considering. We have

$$(x^2+5ax+4a^2)(x^2+5ax+6a^2)=\frac{9}{16}a^4.$$

Hence $$(x^2+5ax)^2+10a^2(x^2+5ax)+24a^4=\frac{9}{16}a^4,$$

$$\therefore x^2+5ax=-\frac{25}{4}a^2, \text{ or else } x^2+5ax=-\frac{15}{4}a^2.$$

Hence $$x+\frac{5}{2}a=0, \text{ or } x+\frac{5}{2}a=\pm\frac{a}{2}\sqrt{10}.$$

Thus the roots are $$-\frac{5}{2}a, \quad -\frac{5}{2}a\pm\frac{a}{2}\sqrt{10}.$$

136. **Reciprocal Equations.** A *reciprocal* equation is one in which the coefficients are the same whether read in order backwards or forwards; or in which all the coefficients when read in order backwards differ in sign from the coefficients read in order forwards. Thus

$$ax^3+bx^2+bx+a=0,$$
$$ax^4+bx^3+cx^2+bx+a=0,$$

and $$ax^5+bx^4+cx^3-cx^2-bx-a=0$$

are reciprocal equations.

Ex. 1. To solve $ax^3+bx^2+bx+a=0$.

We have $a(x^3+1)+bx(x+1)=0$,

that is $(x+1)\{a(x^2-x+1)+bx\}=0$.

Hence $x=-1$,

or else $ax^2+(b-a)x+a=0$.

Ex. 2. To solve $ax^4+bx^3+cx^2+bx+a=0$.

Divide by x^2; then we have

$$a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0.$$

Now put $$x+\frac{1}{x}=y;$$

then $$x^2+\frac{1}{x^2}=y^2-2.$$

Hence $$a(y^2-2)+by+c=0.$$

Let the two roots of the quadratic in y be α and β; then the roots of the original equation will be the four roots of the two equations

$$x+\frac{1}{x}=\alpha \text{ and } x+\frac{1}{x}=\beta.$$

Ex. 3. To solve $$ax^5+bx^4+cx^3-cx^2-bx-a=0.$$

We have $$a(x^5-1)+bx(x^3-1)+cx^2(x-1)=0,$$

that is $$(x-1)\{a(x^4+x^3+x^2+x+1)+bx(x^2+x+1)+cx^2\}=0.$$

Hence $$x=1, \text{ or else}$$

$$ax^4+(b+a)x^3+(a+b+c)x^2+(b+a)x+a=0.$$

The last equation is a reciprocal equation of the fourth degree and is solved as in Ex. 2.

137. Roots found by inspection. When one root of an equation can be found by inspection, the degree of the equation can be lowered by means of the theorem of Art. 88.

Ex. 1. Solve the equation

$$x(x-1)(x-2)=a(a-1)(a-2).$$

One root of the equation is clearly a. Hence $x-a$ is a factor of $x(x-1)(x-2)-a(a-1)(a-2)$, and it will be found that

$$x(x-1)(x-2)-a(a-1)(a-2)=(x-a)\{x^2-(3-a)x+(a-1)(a-2)\}.$$

Hence one root of the equation is a, and the others are given by

$$x^2-(3-a)x+(a-1)(a-2)=0.$$

Ex. 2. Solve the equation

$$x^3+2x^2-11x+6=0.$$

Here we have to try to guess a root of the equation, and in order to do this we take advantage of the following principle:—

If $x=\pm\frac{\alpha}{\beta}$ be a root of the equation $ax^n+bx^{n-1}+\ldots+k=0$, where $a, b, \ldots k$ are integers and $\frac{\alpha}{\beta}$ is in its lowest terms, then α will be a factor of k and β a factor of a. As a particular case, if there are any rational roots of $x^n+\ldots+k=0$, they will be of the form $x=\pm\alpha$, where α is a factor of k.

In the example before us the only possible rational roots are ± 1, ± 2, ± 3, and ± 6. It will be found that $x=2$ satisfies the equation, and we have

$$(x-2)(x^2+4x-3)=x^3+2x^2-11x+6.$$

Hence the other roots of the equation are given by

$$x^2+4x-3=0,$$

and are therefore $$-2\pm\sqrt{7}.$$

Ex. 3. Solve

$$(a-x)^4+(x-b)^4=(a-b)^4.$$

Since $x=a$ and $x=b$ both satisfy the equation, $(x-a)(x-b)$ will divide $(a-x)^4+(x-b)^4-(a-b)^4$, and as the quotient will be of the *second* degree, the equation formed by equating it to zero can be solved.

We may however proceed as follows. The equation may be written

$$\begin{aligned}(a-x)^4+(x-b)^4&=\{(a-x)+(x-b)\}^4\\ &=(a-x)^4+4(a-x)^3(x-b)+6(a-x)^2(x-b)^2\\ &\qquad+4(a-x)(x-b)^3+(x-b)^4;\end{aligned}$$

$$\therefore\ 2(a-x)(x-b)\{2(a-x)^2+3(a-x)(x-b)+2(x-b)^2\}=0.$$

Thus the required roots are a, b and the roots of the quadratic

$$x^2-x(a+b)+2a^2-3ab+2b^2=0.$$

Ex. 4. Solve the equation

$$a^4\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^4\frac{(x-c)(x-a)}{(b-c)(b-a)}+c^4\frac{(x-a)(x-b)}{(c-a)(c-b)}=x^4.$$

The equation is clearly satisfied by $x=a$, by $x=b$, and by $x=c$. Also, since the coefficient of x^3 is zero, the sum of the roots is zero. [Art. 129.] Hence the remaining root must be $-a-b-c$.

Thus the roots are a, b, c, $-(a+b+c)$.

138. **Binomial Equations.** The general form of a binomial equation is $x^m \pm k=0$.

The following are some of the cases of binomial equations which can be solved by methods already given—for the general case De Moivre's theorem in Trigonometry must be employed.

Ex. 1. To solve $x^3-1=0$.

Since $x^3-1=(x-1)(x^2+x+1)$,

we have $x-1=0$;

or else $x^2+x+1=0$, the roots of which are

$$-\frac{1}{2}\pm\frac{\sqrt{-3}}{2}.$$

Hence there are three roots of the equation $x^3=1$; that is there are *three cube roots* of unity, and these roots are

$$1,\ -\frac{1}{2}+\frac{1}{2}\sqrt{-3}\ \text{and}\ -\frac{1}{2}-\frac{1}{2}\sqrt{-3}.$$

Ex. 2. To solve $$x^4-1=0.$$

Since $x^4-1=(x-1)(x+1)(x+\sqrt{-1})(x-\sqrt{-1})$, the *four fourth roots* of unity are

$$1,\ -1,\ \sqrt{-1}\ \text{ and }\ -\sqrt{-1}.$$

Ex. 3. To solve $$x^5-1=0.$$

$$x^5-1=(x-1)(x^4+x^3+x^2+x+1).$$

Hence $$x=1;$$

or else $$x^4+x^3+x^2+x+1=0.$$

The latter equation is a reciprocal equation. Divide by x^2, and we have

$$x^2+\frac{1}{x^2}+x+\frac{1}{x}+1=0.$$

Put $$x+\frac{1}{x}=y;$$

then $$x^2+\frac{1}{x^2}=y^2-2;$$

$$\therefore\ y^2-2+y+1=0;$$

$$\therefore\ y=-\frac{1}{2}\pm\frac{\sqrt{5}}{2}.$$

Hence $$x+\frac{1}{x}=\frac{-1\pm\sqrt{5}}{2},$$

that is $$x^2-x\frac{-1\pm\sqrt{5}}{2}+1=0.$$

Hence $$x=\frac{-1+\sqrt{5}}{4}\pm\frac{1}{4}\sqrt{-10-2\sqrt{5}},$$

or $$x=\frac{-1-\sqrt{5}}{4}\pm\tfrac{1}{4}\sqrt{-10+2\sqrt{5}},$$

or $$x=1.$$

Ex. 4. To solve $$x^4+1=0.$$

$$x^4+1=(x^2+1)^2-2x^2=(x^2+1-\sqrt{2}x)(x^2+1+\sqrt{2}x).$$

Hence $$x^2\mp\sqrt{2}x+1=0;$$

$$\therefore\ x=\frac{\pm1\pm\sqrt{-1}}{\sqrt{2}}.$$

139. **Cube roots of unity.** In the preceding article we found that the three cube roots of unity are

$$1,\ \tfrac{1}{2}(-1+\sqrt{-3}),\ \tfrac{1}{2}(-1-\sqrt{-3}).$$

An imaginary cube root of unity is generally represented by ω; or, when it is necessary to distinguish

between the two imaginary roots, one is called ω_1, and the other ω_2, so that 1, ω_1 and ω_2 are the three roots of the equation $x^3-1=0$.

Taking the above values, we have

$$1+\omega_1+\omega_2=1+\tfrac{1}{2}(-1+\sqrt{-3})+\tfrac{1}{2}(-1-\sqrt{-3})=0,$$

also $$\omega_1\omega_2=\tfrac{1}{4}(-1+\sqrt{-3})(-1-\sqrt{-3})=1.$$

These relations follow at once from Art. 129; for the sum of the three roots of $x^3-1=0$ is zero, and the product is 1.

Again $$\omega_1^2=\tfrac{1}{4}(-1+\sqrt{-3})^2=\tfrac{1}{2}(-1-\sqrt{-3})=\omega_2,$$

and $$\omega_2^2=\tfrac{1}{4}(-1-\sqrt{-3})^2=\tfrac{1}{2}(-1+\sqrt{-3})=\omega_1,$$

so that $\omega_1^2=\omega_2$ and $\omega_2^2=\omega_1$. These relations follow at once from

$$\omega_1\omega_2=1 \text{ and } \omega_1^3=\omega_2^3=1.$$

Thus *if we square either of the imaginary cube roots of unity we obtain the other.*

Hence if ω be either of the imaginary cube roots of unity, the three roots are 1, ω and ω^2.

We know that

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab).$$

Hence $a+b+c$ is a factor of $a^3+b^3+c^3-3abc$, and this is the case for *all values* of a, b, c.

Hence $a+(\omega b)+(\omega^2c)$ is a factor of $a^3+(\omega b)^3+(\omega^2c)^3-3a(\omega b)(\omega^2c)$, that is of $a^3+b^3+c^3-3abc$; and $a+\omega^2b+\omega c$ can similarly be shewn to be a factor.

Hence $a^3+b^3+c^3-3abc=(a+b+c)(a+\omega b+\omega^2c)(a+\omega^2b+\omega c)$.

EXAMPLES XI.

Solve the following examples:

1. $x^4-2x^2-8=0$.

2. $x^6+7a^3x^3-8a^6=0$.

3. $x^6 - 7a^3x^3 - 8a^6 = 0.$

4. $\dfrac{x}{x^2+1} + \dfrac{x^2+1}{x} = \dfrac{5}{2}.$

5. $\dfrac{x^2+2}{x^2+4x+1} + \dfrac{x^2+4x+1}{x^2+2} = \dfrac{5}{2}.$

6. $(x^2+x+1)(x^2+x+2) = 12.$

7. $(x^2+7x+5)^2 - 3x^2 - 21x = 19.$

8. $\sqrt{16-7x-x^2} = x^2 + 7x - \dfrac{1}{4}.$

9. $6\sqrt{x^2-2x+6} = 21 + 2x - x^2.$

10. $(a-1)(1+x+x^2)^2 = (a+1)(1+x^2+x^4).$

11. $(x+1)(x+2)(x+3)(x+4) = 24.$

12. $(x+a)(x+3a)(x+5a)(x+7a) = 384a^4.$

13. $(x-3a)(x-a)(x+2a)(x+4a) = 2376a^4.$

14. $(x+2)(x+3)(x+8)(x+12) = 4x^2.$

15. $2x^2 - 3x - 21 = 2x\sqrt{x^2-3x+4}.$

16. $x^4 - 2(a+b)x^2 + a^2 + 2ab + b^2 = 0.$

17. $x^4 - 2x^2a^2 - 2x^2b^2 + a^4 + b^4 - 2a^2b^2 = 0.$

18. $4x^4 - 4x^3 - 7x^2 - 4x + 4 = 0.$

19. $9x^4 - 24x^3 - 2x^2 - 24x + 9 = 0.$

20. $x^5 + 1 = 0.$

21. $x^8 - 1 = 0.$

22. $3x^3 - 14x^2 + 20x - 8 = 0.$

23. $x^4 - 15x^2 + 10x + 24 = 0.$

24. $x^4 + 7x^3 - 7x - 1 = 0.$

25. $(x-a)^3(b-c)^3 + (x-b)^3(c-a)^3 + (x-c)^3(a-b)^3 = 0.$

26. $x(x-1)(x-2) = 9\,.\,8\,.\,7.$

27. $x(x-1)(x-2)(x-3) = 9 \cdot 8 \cdot 7 \cdot 6.$

28. $(a-x)^3 + (b-x)^3 = (a+b-2x)^3.$

29. $(a-x)^4 + (b-x)^4 = (a+b-2x)^4.$

30. $(a-x)^5 + (b-x)^5 = (a+b-2x)^5.$

31. $\sqrt[3]{a-x} + \sqrt[3]{b-x} = \sqrt[3]{a+b-2x}.$

32. $\sqrt[4]{a-x} + \sqrt[4]{b-x} = \sqrt[4]{a+b-2x}.$

33. $(a-x)^5 + (x-b)^5 = (a-b)^5.$

34. $\sqrt[3]{a-x} + \sqrt[3]{x-b} = \sqrt[3]{a-b}.$

35. $\sqrt[4]{a-x} + \sqrt[4]{x-b} = \sqrt[4]{a-b}.$

36. $x^4 + (a-x)^4 = b^4.$

37. $(x+a)^4 + (x+b)^4 = 17(a-b)^4.$

38. $\sqrt[4]{x} + \sqrt[4]{a-x} = \sqrt[4]{b}.$

39. $abx(x+a+b)^3 - (ax+bx+ab)^3 = 0.$

40. $abcx(x+a+b+c)^2 - (xbc+xca+xab+abc)^2 = 0.$

41. $$\frac{(a-x)^4+(x-b)^4}{(a+b-2x)^2} = \frac{a^4+b^4}{(a+b)^2}.$$

42. $x^4 + b(a+b)x^3 + (ab-2)b^2x^2 - (a+b)b^3x + b^4 = 0.$

43. $(x^2+b^2)^2 = 2ax^3 + 2ab^2x - a^2x^2.$

44. $(x+b+c)(x+c+a)(x+a+b) + abc = 0.$

45. $$\frac{a}{b+c-x} + \frac{b}{c+a-x} + \frac{c}{a+b-x} + 3 = 0.$$

46. $$\frac{(x-a)^2}{(x-a)^2-(b-c)^2} + \frac{(x-b)^2}{(x-b)^2-(c-a)^2} + \frac{(x-c)^2}{(x-c)^2-(a-b)^2} = 1.$$

47. $$\frac{(x+a)(x+b)}{(x-a)(x-b)} + \frac{(x-a)(x-b)}{(x+a)(x+b)} = \frac{(x+c)(x+d)}{(x-c)(x-d)} + \frac{(x-c)(x-d)}{(x+c)(x+d)}.$$

CHAPTER X.

SIMULTANEOUS EQUATIONS.

140. A SINGLE equation which contains two or more unknown quantities can be satisfied by an indefinite number of values of the unknown quantities. For we can give any values whatever to all but one of the unknown quantities, and we shall then have an equation to determine the remaining unknown quantity.

If there are two equations containing two unknown quantities (or as many equations as there are unknown quantities), each equation taken by itself can be satisfied in an indefinite number of ways, but this is not the case when both (or all) the equations are to be satisfied by the *same values* of the unknown quantities.

Two or more equations which are to be satisfied by the same values of the unknown quantities contained in them are called a system of *simultaneous equations*.

The degree of an equation which contains the unknown quantities $x, y, z\ldots$ is the degree of that term which is of the highest dimensions in $x, y, z\ldots$.

Thus the equations

$$ax + a^2y + a^3z = a^4,$$

$$xy + x + y + z = 0,$$

$$x^3 + y^3 + z^3 - 3xyz = 0,$$

are of the first, second and third degrees respectively.

141. Equations of the First Degree. We proceed to consider equations of the first degree, beginning with those which contain only two unknown quantities, x and y.

Every equation of the first degree in $x, y, z, \ldots$ can by transformation be reduced to the form

$$ax + by + cz + \ldots = k,$$

where $a, b, c, \ldots k$ are supposed to represent known quantities.

NOTE. When there are several equations of the same type it is convenient and usual to employ the same letters in all, but with marks of distinction for the different equations.

Thus we use $a, b, c \ldots$ for one equation; $a', b', c' \ldots$ for a second; $a'', b'', c'' \ldots$ for a third; and so on. Or we use a_1, b_1, c_1 for one equation; a_2, b_2, c_2 for a second; and so on.

Hence two equations containing x and y are in their most general forms

$$ax + by = c,$$

and

$$a'x + b'y = c',$$

and similarly in other cases.

142. Equations with two unknown quantities. Suppose that we have the two equations

$$ax + by = c,$$

and

$$a'x + b'y = c'.$$

Multiply both members of the first equation by b', the coefficient of y in the second; and multiply both members of the second equation by b, the coefficient of y in the first. We thus obtain the equivalent system

$$ab'x + bb'y = cb',$$

$$a'bx + bb'y = c'b.$$

Hence, by subtraction, we have

$$(ab' - a'b)\,x = cb' - c'b;$$

whence $$x = \frac{cb' - c'b}{ab' - a'b}.$$

Substitute this value of x in the first of the given equations; then

$$a\,\frac{cb' - c'b}{ab' - a'b} + by = c,$$

$$\therefore\ by = \frac{c\,(ab' - a'b) - a\,(cb' - c'b)}{ab' - a'b},$$

whence $$y = \frac{ac' - a'c}{ab' - a'b}.$$

The value of y may be found independently of x by multiplying the first equation by a' and the second by a; we thus obtain the equivalent system

$$a'ax + a'by = a'c,$$

$$a'ax + ab'y = ac'.$$

Hence, by subtraction, we have

$$(a'b - ab')\,y = a'c - ac';$$

$$\therefore\ y = \frac{a'c - ac'}{a'b - ab'},$$

which is equal to the value of y obtained by substitution.

NOTE. It is important to notice that when the value either of x or of y is obtained, the value of the other can be *written down.*

For a and a' have the same relation to x that b and b' have to y; we may therefore change x into y provided that we at the same time change a into b, b into a, a' into b', and b' into a'. Thus from

$$x = \frac{cb' - c'b}{ab' - a'b} \text{ we have } y = \frac{ca' - c'a}{ba' - b'a}.$$

It will be seen from the above that in order to solve two simultaneous equations of the first degree, we first deduce from the given equations a third equation which contains only *one* of the unknown quantities; and the unknown quantity which is absent is said to have been *eliminated.*

143. From the last article it will be seen that the values of x and y which satisfy the equations

$$ax + by = c,$$

and

$$a'x + b'y = c',$$

can be expressed in the form

$$\frac{x}{bc' - b'c} = \frac{y}{ca' - c'a} = \frac{-1}{cb' - a'b}.$$

So also, from the equations

$$ax + by + c = 0,$$

and

$$a'x + b'y + c' = 0,$$

we have

$$\frac{x}{bc' - b'c} = \frac{y}{ca' - c'a} = \frac{1}{ab' - a'b}.$$

It is important that the student should be able to quote these formulae.

Ex. 1. Solve the equations

$$3x + 2y = 13,$$

and

$$7x + 3y = 27.$$

We have

$$\frac{x}{2 . 27 - 3 . 13} = \frac{y}{13 . 7 - 27 . 3} = \frac{-1}{3 . 3 - 7 . 2},$$

that is

$$\frac{x}{15} = \frac{y}{10} = \frac{1}{5};$$

$$\therefore\ x = \frac{15}{5} = 3,$$

and

$$y = \frac{10}{5} = 2.$$

Ex. 2. Solve the equations

$$\frac{4}{x}+\frac{3}{y}=2,$$

and
$$\frac{2}{x}-\frac{5}{y}=-7.$$

These may be considered as two simultaneous equations of the *first* degree with $\frac{1}{x}$ and $\frac{1}{y}$ as unknown quantities.

We therefore have

$$\frac{\frac{1}{x}}{3(-7)-(-5)2}=\frac{\frac{1}{y}}{2.2-(-7)4}=\frac{-1}{4(-5)-2.3},$$

that is
$$\frac{\frac{1}{x}}{-11}=\frac{\frac{1}{y}}{32}=\frac{1}{26};$$

$$\therefore \frac{1}{x}=-\frac{11}{26}, \text{ or } x=-\frac{26}{11}.$$

Also
$$\frac{1}{y}=\frac{32}{26}, \text{ or } y=\frac{13}{16}.$$

Ex. 3. Solve the equations

$$x-y=a-b,$$
$$ax-by=2(a^2-b^2).$$

We have

$$\frac{x}{-2(a^2-b^2)+b(a-b)}=\frac{y}{a(a-b)-2(a^2-b^2)}=\frac{-1}{-b+a},$$

that is
$$\frac{x}{b^2+ab-2a^2}=\frac{y}{2b^2-ab-a^2}=\frac{1}{b-a};$$

$$\therefore x=\frac{b^2+ab-2a^2}{b-a}=b+2a;$$

and
$$y=\frac{2b^2-ab-a^2}{b-a}=a+2b.$$

Instead of referring to the general formulae of Art. 143, as we have done in the above examples, the unknown quantities may be eliminated in turn, as in Art. 142; and this latter method is frequently the simpler of the two. Thus in this last example we have at once, by multiplying the first equation by a and then subtracting the second,

$$(b-a)y=a(a-b)-2(a^2-b^2);$$

$$\therefore y=\frac{-a^2-ab+2b^2}{b-a}=a+2b.$$

Then $$x=(a+2b)+a-b;$$
$$\therefore\ x=2a+b.$$

144. **Discussion of solution of two simultaneous equations of the first degree.** We have seen that the values of x and y which satisfy the equations

$$ax+by=c\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{i}),$$

and $$a'x+b'y=c'\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii}),$$

are given by

$$(ab'-a'b)\,x=cb'-c'b\ldots\ldots\ldots\ldots\ldots(\text{iii}),$$
$$(ba'-b'a)\,y=ca'-c'a\ldots\ldots\ldots\ldots\ldots(\text{iv}).$$

Thus there is a single finite value of x, and a single finite value of y, provided that $ab'-a'b\neq 0$.

If $ab'-a'b=0$, x will be *infinite* [see Art. 118] unless $cb'-c'b=0$; and, if $ab'-a'b$ and $cb'-c'b$ are both zero, *any value* of x will satisfy equation (iii).

So also, y will be *infinite* if $ab'-a'b=0$, unless $ca'-c'a$ is also zero, in which case *any value* of y will satisfy equation (iv).

If $ab'-a'b=0$, then $\dfrac{a}{a'}=\dfrac{b}{b'}$; and if $ab'-a'b=0$ and also $cb'-c'b=0$, then $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{c}{c'}$.

When equations cannot be satisfied by *finite values* of the unknown quantities, they are often said to be *inconsistent.* Thus the equations $ax+by=c$ and $a'x+b'y=c'$ are *inconsistent* if $\dfrac{a}{a'}=\dfrac{b}{b'}$, unless each fraction is equal to $\dfrac{c}{c'}$, in which case the equations are *indeterminate.* In fact when $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{c}{c'}$, it is clear that by multiplying the terms of equation (i) by $\dfrac{a'}{a}$ we shall obtain equation (ii), so that the two given equations are equivalent to *one* only.

We have hitherto supposed that a, a', b, b' were none of them zero. It will not be necessary to discuss every possible case: consider, for example, the case in which a and a' are both zero.

When a and a' are both zero, we have from (i) $y=\frac{c}{b}$, and from (ii) $y=\frac{c'}{b'}$. These results are inconsistent with one another unless $\frac{c}{b}=\frac{c'}{b'}$.

Hence, if $a=a'=0$, and $\frac{c}{b}=\frac{c'}{b'}$, the equations (i) and (ii) are satisfied by making $y=\frac{c}{b}$, and by giving to x any finite value whatever.

If however $\frac{c}{b}\neq\frac{c'}{b'}$, the equations $by=c$ and $b'y=c'$ cannot both be satisfied, unless they are looked upon as the limiting forms of the equations $ax+by=c$ and $a'x+b'y=c'$, in which a and a' are indefinitely small and ultimately zero. But from (iii) we see that when a and a' diminish without limit, x must increase without limit, $cb'-c'b$ not being zero. Thus, in the equations (i) and (ii), when a and a' diminish without limit, and $cb'\neq c'b$, the value of x must be infinite.

Equations with three unknown quantities.

145. To solve the three equations:

$$ax+by+cz=d \quad \text{..............} \quad \text{(i)},$$
$$a'x+b'y+c'z=d' \quad \text{..............} \quad \text{(ii)},$$
$$a''x+b''y+c''z=d'' \quad \text{..............} \quad \text{(iii)}.$$

Method of successive elimination. Multiply the first equation by c', and the second by c; then we have

$$ac'x+bc'y+cc'z=dc',$$

and $$a'cx + b'cy + c'cz = d'c;$$

therefore, by subtraction,

$$(ac' - a'c)x + (bc' - b'c)y = dc' - d'c \ldots\ldots (\text{iv}).$$

Again, by multiplying the first equation by c'' and the third by c and subtracting, we have

$$(ac'' - a''c)x + (bc'' - b''c)y = dc'' - d''c \ldots\ldots (\text{v}).$$

We now have the two equations (iv) and (v) from which to determine the unknown quantities x and y. Using the general formulae of Art. 143, we have

$$x = \frac{-(bc' - b'c)(dc'' - d''c) + (dc' - d'c)(bc'' - b''c)}{(ac' - a'c)(bc'' - b''c) - (bc' - b'c)(ac'' - a''c)}.$$

Method of undetermined multipliers. Multiply the equations (i) and (ii) by λ and μ, and add to (iii); then we have the equation

$$x(\lambda a + \mu a' + a'') + y(\lambda b + \mu b' + b'') + z(\lambda c + \mu c' + c'') = (\lambda d + \mu d' + d''),$$

which is true for all values of λ and μ.

Now let λ and μ be so chosen that the co-efficients of y and z may both be zero,

then $$x = \frac{\lambda d + \mu d' + d''}{\lambda a + \mu a' + a''},$$

where λ and μ are found from

$$\lambda b + \mu b' + b'' = 0,$$

and $$\lambda c + \mu c' + c'' = 0;$$

$$\therefore \quad \frac{\lambda}{b'c'' - b''c'} = \frac{\mu}{b''c - bc''} = \frac{1}{bc' - b'c}.$$

Hence

$$x = \frac{d(b'c'' - b''c') + d'(b''c - bc'') + d''(bc' - b'c)}{a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c)}.$$

[The numerator and the denominator of the first value of x, which was obtained by eliminating z and y in succes-

sion, can both be divided by c; and the two values of x will then be seen to agree.]

Having found the value of x by either of the above methods, the values of y and z can be *written down*. For the value of y will be obtained from that of x by interchanging a and b, a' and b', and a'' and b''. The value of y can also be obtained from that of x by a *cyclical change* [see Art. 93] of the letters a, b, c; a', b', c'; and a'', b'', c''; and a second cyclical change will give the value of z.

It should be remarked that the denominators of the values of x, y and z are the same, and that there is a single finite value of each of the unknown quantities unless this denominator is zero.

Ex. 1. Solve the equations:

$$x+2y+3z=6 \quad \text{(i)},$$
$$2x+4y+z=7 \quad \text{(ii)},$$
$$3x+2y+9z=14 \quad \text{(iii)}.$$

Multiply (ii) by 3, and subtract (i); then

$$5x+10y=15 \quad \text{(iv)}.$$

Again multiply (i) by 3, and subtract (iii); then

$$4y=4 \quad \text{(v)}.$$

From (v) we have $y=1$; then, knowing y, we have from (iv) $x=1$; and, knowing x and y, we have from (i) $z=1$.

Thus $$x=y=z=1.$$

Ex. 2. Solve the equations:

$$x+y+z=1 \quad \text{(i)},$$
$$ax+by+cz=d \quad \text{(ii)},$$
$$a^2x+b^2y+c^2z=d^2 \quad \text{(iii)}.$$

Multiply (i) by c and subtract (ii); then

$$(c-a)x+(c-b)y=c-d \quad \text{(iv)}.$$

Again multiply (i) by c^2 and subtract (iii); then

$$(c^2-a^2)x+(c^2-b^2)y=c^2-d^2 \quad \text{(v)}.$$

Now multiply (iv) by $c+b$ and subtract (v); then

$$(c-a)(b-a)x=(c-d)(b-d);$$

$$\therefore \ x=\frac{(b-d)(c-d)}{(b-a)(c-a)}.$$

The values of y and z may now be written down: they are

$$y=\frac{(c-d)(a-d)}{(c-b)(a-b)}; \quad z=\frac{(a-d)(b-d)}{(a-c)(b-c)}.$$

Instead of going through the process of elimination, we may at once quote the general formulae. Thus

$$x=\frac{(bc^2-b^2c)+d(b^2-c^2)+d^2(c-b)}{(bc^2-b^2c)+a(b^2-c^2)+a^2(c-b)}$$
$$=\frac{(b-c)\{-bc+d(b+c)-d^2\}}{(b-c)\{-bc+a(b+c)-a^2\}}$$
$$=\frac{(b-d)(c-d)}{(b-a)(c-a)}, \text{ as above.}$$

Ex. 3. Solve the equations:

$$x+y+z=a+b+c \quad \text{.........................(i)},$$
$$ax+by+cz=bc+ca+ab \quad \text{...................(ii)},$$
$$bcx+cay+abz=3abc \quad \text{..........................(iii)}.$$

We have

$$x=\frac{(a+b+c)(ab^2-ac^2)+(bc+ca+ab)(ca-ab)+3abc(c-b)}{ab^2-ac^2+a(ca-ab)+bc(c-b)}$$
$$=\frac{a(b-c)\{(b+c)(a+b+c)-bc-ca-ab-3bc\}}{(b-c)\{ab+ac-a^2-bc\}}$$
$$=-\frac{a(b-c)^2}{(a-b)(a-c)}.$$

The values of y and z can now be written down: they are

$$y=-\frac{b(c-a)^2}{(b-c)(b-a)}; \quad z=-\frac{c(a-b)^2}{(c-a)(c-b)}.$$

Ex. 4. Solve the equations:

$$x+ay+a^2z+a^3=0 \quad \text{.........................(i)},$$
$$x+by+b^2z+b^3=0 \quad \text{.........................(ii)},$$
$$x+cy+c^2z+c^3=0 \quad \text{.........................(iii)}.$$

The equations may be solved as in the preceding examples, or as follows.

It is clear that a, b, c are the three roots of the following cubic in λ

$$\lambda^3+z\lambda^2+y\lambda+x=0.$$

Hence from Art. 129, we have at once

$$z=-(a+b+c),$$
$$y=bc+ca+ab,$$

and $$x=-abc,$$

146. **Equations with more than three unknown quantities.** We shall return to the consideration of simultaneous equations of the first degree in the Chapter on Determinants, and shall then shew how the solution of any number of such equations can be at once written down.

The method of successive elimination or the method of undetermined multipliers can however be extended to the case when there are more than three unknown quantities. For example, to solve the equations

$$ax + by + cz + dw = e \quad\text{..............(i)},$$
$$a'x + b'y + c'z + d'w = e' \quad\text{............(ii)},$$
$$a''x + b''y + c''z + d''w = e'' \quad\text{...........(iii)},$$
$$a'''x + b'''y + c'''z + d'''w = e''' \quad\text{.........(iv)}.$$

Multiply (i) by λ, (ii) by μ, (iii) by ν, and add the products to (iv). Then we have

$$x(a\lambda + a'\mu + a''\nu + a''') + y(b\lambda + b'\mu + b''\nu + b''')$$
$$+ z(c\lambda + c'\mu + c''\nu + c''') + w(d\lambda + d'\mu + d''\nu + d''')$$
$$= e\lambda + e'\mu + e''\nu + e''' \quad\text{.................................(v)}.$$

Now choose λ, μ, ν so as to make the coefficients of y, z and w in the last equation zero; then

$$x = \frac{e\lambda + e'\mu + e''\nu + e'''}{a\lambda + a'\mu + a''\nu + a'''} \quad\text{...............(vi)},$$

where λ, μ, ν are to be found from the equations

$$\left.\begin{aligned} b\lambda + b'\mu + b''\nu + b''' &= 0 \\ c\lambda + c'\mu + c''\nu + c''' &= 0 \\ d\lambda + d'\mu + d''\nu + d''' &= 0 \end{aligned}\right\} \quad\text{...........(vii)}.$$

Hence we have to solve (vii) by Art. 145 and then substitute the values of λ, μ, and ν in (vi); this will give the value of x; and the values of the other unknown quantities can then be found by cyclical changes of the letters, a, b, c, d, &c.

EXAMPLES XII.

Solve the following equations.

1. $\frac{x}{3} - \frac{y}{6} = \frac{1}{2}$,
$\frac{x}{5} - \frac{3y}{10} = \frac{1}{2}$.

2. $\frac{9}{x} - \frac{4}{y} = 2$,
$\frac{18}{x} + \frac{8}{y} = 10$.

3. $x + \frac{3}{y} = \frac{7}{2}$,
$3x - \frac{2}{y} = \frac{26}{3}$.

4. $\frac{4}{x} - \frac{3}{y} + 5 = \frac{6}{x} + \frac{3}{y} = 10$.

5. $ax + by = 2ab$,
$bx - ay = b^2 - a^2$.

6. $x + ay + a^2 = 0$,
$x + by + b^2 = 0$.

7. $x + y = 2a$,
$(a - b)x = (a + b)y$.

8. $(b + c)x + (b - c)y = 2ab$,
$(c + a)x + (c - a)y = 2ac$.

9. $bx + ay = 2ab$,
$a^2x + b^2y = a^3 + b^3$.

10. $(a + b)x + by = ax + (b + a)y = a^3 - b^3$.

11. $x + y + z = 1$,
$2x + 3y + z = 4$,
$4x + 9y + z = 16$.

12. $x + y + z = 1$,
$\frac{x}{2} + \frac{y}{4} + 4z = 1$,
$\frac{5}{3}x + \frac{3}{4}y - \frac{z}{2} = 1$.

13. $x + 2y + 3z = 3x + y + 2z = 2x + 3y + z = 6$.

14. $y + z = 2a$,
$z + x = 2b$,
$x + y = 2c$.

15. $y + z - x = 2a$,
$z + x - y = 2b$,
$x + y - z = 2c$.

16. $y+z-3x=2a,$
$z+x-3y=2b,$
$x+y-3z=2c.$

17. $ax+by+cz=1,$
$bx+cy+az=1,$
$cx+ay+bz=1.$

18. $\dfrac{y+z-x}{b+c}=\dfrac{z+x-y}{c+a}=\dfrac{x+y-z}{a+b}=1.$

19. $x+y+z=0,$
$ax+by+cz=1,$
$a^2x+b^2y+c^2z=a+b+c.$

20. $x+y+z=a+b+c,$
$bx+cy+az=bc+ca+ab,$
$cx+ay+bz=bc+ca+ab.$

21 $x+y+z=a+b+c,$
$bx+cy+az=a^2+b^2+c^2,$
$cx+ay+bz=a^2+b^2+c^2.$

22. $x+y+z=0,$
$(b+c)\,x+(c+a)\,y+(a+b)\,z=(b-c)\,(c-a)\,(a-b),$
$bcx+cay+abz=0.$

23. $ax+by+cz=a,$
$bx+cy+az=b,$
$cx+ay+bz=c.$

24. $x-ay+a^2z-a^3=0,$
$x-by+b^2z-b^3=0,$
$x-cy+c^2z-c^3=0.$

25. $ax+by+cz=m,$
$a^2x+b^2y+c^2z=m^2,$
$a^3x+b^3y+c^3z=m^3.$

26. $ax+cy+bz=a^2+2bc,$
$cx+by+az=b^2+2ca,$
$bx+ay+cz=c^2+2ab.$

27. $x+y+z=2a+2b+2c,$
$ax+by+cz=2bc+2ca+2ab,$
$(b-c)\,x+(c-a)\,y+(a-b)\,z=0.$

28. $ax+by+cz=a+b+c,$
$a^2x+b^2y+c^2z=(a+b+c)^2,$
$bcx+cay+abz=0.$

29. $x + y + z = l + m + n,$

$lx + my + nz = mn + nl + lm,$

$(m - n)\,x + (n - l)\,y + (l - m)\,z = 0.$

30. $lx + ny + mz = nx + my + lz = mx + ly + nz$
$= l^3 + m^3 + n^3 - 3lmn.$

31. $l^2x + m^2y + n^2z = lmx + mny + nlz = nlx + lmy + mnz$
$= l + m + n.$

32. $\dfrac{x}{a+\alpha} + \dfrac{y}{a+\beta} + \dfrac{z}{a+\gamma} = 1,$

$\dfrac{x}{b+\alpha} + \dfrac{y}{b+\beta} + \dfrac{z}{b+\gamma} = 1,$

$\dfrac{x}{c+\alpha} + \dfrac{y}{c+\beta} + \dfrac{z}{c+\gamma} = 1.$

33. $y + z + w = a,$

$z + w + x = b,$

$w + x + y = c,$

$x + y + z = d.$

34. $x + ay + a^2z + a^3w + a^4 = 0,$

$x + by + b^2z + b^3w + b^4 = 0,$

$x + cy + c^2z + c^3w + c^4 = 0,$

$x + dy + d^2z + d^3w + d^4 = 0.$

Simultaneous Equations of the Second Degree.

147. We now proceed to consider simultaneous equations, one at least of which is of the second or of higher degree.

We first take the case of two equations containing two unknown quantities, one of the equations being of the first degree and the other of the second.

For example, to solve the equations:

$$3x+2y=7,$$
$$3x^2-2y^2=25.$$

From the first equation we have

$$x=\frac{7-2y}{3}.$$

Substitute this value of x in the second equation; we then have

$$3\left(\frac{7-2y}{3}\right)^2-2y^2=25,$$

whence $$y^2+14y+13=0,$$

that is $$(y+13)(y+1)=0;$$

$$\therefore\ y=-1,\ \text{or}\ y=-13.$$

If $y=-1$, $$x=\frac{7-2y}{3}=3;$$

and if $y=-13$, $$x=11.$$

Thus $x=3$, $y=-1$; or $x=11$, $y=-13$.

From the above example it will be seen that to solve two equations of which one is of the first degree, and the other of the second degree, we proceed as follows:—

From the equation of the first degree find the value of one of the unknown quantities in terms of the other unknown quantity and the known quantities, and substitute this value in the equation of the second degree; one of the unknown quantities is thus eliminated, and a quadratic equation is obtained the roots of which are the values of the unknown quantity which is retained.

The most general forms of two equations such as we are now considering are

$$lx+my+n=0,$$
$$ax^2+bxy+cy^2+dx+ey+f=0.$$

From the first equation we have

$$x=-\frac{my+n}{l}.$$

Hence on substitution in the second equation we have to determine y from the quadratic equation

$$a(my+n)^2 - lby(my+n) + cl^2y^2$$
$$- dl(my+n) + el^2y + fl^2 = 0.$$

Having found the two values of y, the corresponding values of x are found by substitution in the first equation.

148. It should be remarked that we cannot solve *any* two equations which are both of the second degree; for the elimination of one of the unknown quantities will in general lead to an equation of the *fourth* degree, from which the remaining unknown quantity would have to be found; and we cannot solve an equation of higher degree than the second, except in very special cases.

For example, to solve the equations

$$ax^2 + bx + c = y, \qquad x^2 + y^2 = d.$$

Substitute $ax^2 + bx + c$ for y in the second equation, and we have

$$x^2 + (ax^2 + bx + c)^2 = d,$$

which is an equation of the fourth degree which cannot be solved by any methods given in the previous chapter.

149. There is one important class of equations with two unknown quantities which can always be solved, namely, equations in which all the terms which contain the unknown quantities are of the second degree. The most general forms of two such equations are

$$ax^2 + bxy + cy^2 = d$$

and

$$a'x^2 + b'xy + c'y^2 = d'.$$

Multiply the first equation by d', and the second by d and subtract; we then have

$$(ad' - a'd)\,x^2 + (bd' - b'd)\,xy + (cd' - c'd)\,y^2 = 0.$$

The factors of the above equation can be found either by inspection, or as in Art. 81; we therefore have two equations of the form $lx + my = 0$ either of which combined with the first of the given equations will give, as in Art. 147, two pairs of values of x and y.

Ex. 5. To solve the equations:

$$y^2 - xy = 15 \quad\text{.............................(i)},$$
$$x^2 + xy = 14 \quad\text{.............................(ii)}.$$

We have $\quad 14(y^2 - xy) = 15(x^2 + xy);$

$$\therefore\ 15x^2 + 29xy - 14y^2 = 0,$$

that is $\quad (5x - 2y)(3x + 7y) = 0.$

Hence $\quad 5x - 2y = 0,$

or else $\quad 3x + 7y = 0.$

If $5x - 2y = 0$, we have from (i)

$$y^2 - \frac{2}{5}y^2 = 15,$$

whence $\quad y = \pm 5.$

Hence also $\quad x = \pm 2.$

If $3x + 7y = 0$, we have from (i)

$$y^2 + \frac{7}{3}y^2 = 15,$$

whence $\quad y = \pm \frac{3}{\sqrt{2}},$

and then $\quad x = \mp \frac{7}{\sqrt{2}}.$

Thus $x = \pm 2,\ y = \pm 5$; or $x = \pm \frac{7}{\sqrt{2}},\ y = \mp \frac{3}{\sqrt{2}}.$

150. The following examples will shew how to deal with some other cases of simultaneous equations with two unknown quantities; but no general rules can be given.

Ex. 1. To solve $\quad x - y = 2,$

$$xy = 15.$$

Square the members of the first equation, and add four times the second; then

$$(x + y)^2 = 64.$$

Hence $x+y=\pm 8$,

which with $x-y=2$,

gives $x=5$ or -3,

and $y=3$ or -5.

Thus $x=5,\ y=3$; or $x=-3,\ y=-5$.

Ex. 2. To solve

$$x^2+xy+y^2=a^2 \quad\text{.........(i)},$$

$$x^4+x^2y^2+y^4=b^4 \quad\text{.........(ii)}.$$

Divide the members of the second equation by the corresponding members of the first; then

$$x^2-xy+y^2=\frac{b^4}{a^2} \quad\text{.........(iii)}.$$

From (i) and (iii) by subtraction we have

$$2xy=a^2-\frac{b^4}{a^2} \quad\text{.........(iv)}.$$

From (i) and (iv)

$$x^2+2xy+y^2=\frac{3a^4-b^4}{2a^2};$$

$$\therefore\ x+y=\pm\sqrt{\frac{3a^4-b^4}{2a^2}} \quad\text{.........(v)}.$$

From (iii) and (iv) we have

$$x^2-2xy+y^2=\frac{3b^4-a^4}{2a^2};$$

$$\therefore\ x-y=\pm\sqrt{\frac{3b^4-a^4}{2a^2}} \quad\text{.........(vi)}.$$

Finally, from (v) and (vi) we have

$$x=\frac{1}{2}\left\{\pm\sqrt{\frac{3a^4-b^4}{2a^2}}\pm\sqrt{\frac{3b^4-a^4}{2a^2}}\right\},$$

and

$$y=\frac{1}{2}\left\{\pm\sqrt{\frac{3a^4-b^4}{2a^2}}\mp\sqrt{\frac{3b^4-a^4}{2a^2}}\right\}.$$

Ex. 3. To solve

$$x^2-2y^2=4y,$$

$$3x^2+xy-2y^2=16y.$$

Multiply the first equation by 4, and subtract the second; then

$$x^2-xy-6y^2=0,$$

that is $(x+2y)(x-3y)=0$;

$$\therefore\ x+2y=0,$$

or else $x-3y=0$.

If $x+2y=0$, the first equation gives

$$4y^2-2y^2=4y;$$

$$\therefore\ y=0 \text{ or } y=2,$$

whence $$x=0 \text{ or } x=-4.$$

If $x-3y=0$, the first equation gives

$$9y^2-2y^2=4y;$$

$$\therefore\ y=0 \text{ or } y=\frac{4}{7},$$

whence $$x=0 \text{ or } x=\frac{12}{7}.$$

Thus $$x=0,\quad y=0;\quad x=-4,\quad y=2;$$

or $$x=\frac{12}{7},\quad y=\frac{4}{7}.$$

Ex. 4. To solve $$x^2+y^2=(x+y+1)^2,$$
$$x^2+y^2=(x-y+2)^2.$$

By subtraction we have

$$(x+y+1)^2-(x-y+2)^2=0,$$

that is $$(2x+3)(2y-1)=0.$$

Hence $$2x+3=0, \text{ or } 2y-1=0.$$

If $2x+3=0$, we have

$$\frac{9}{4}+y^2=\left(y-\frac{1}{2}\right)^2,$$

whence $$y=-2.$$

If $2y-1=0$, we have

$$x^2+\frac{1}{4}=\left(x+\frac{3}{2}\right)^2,$$

whence $$x=-\frac{2}{3}.$$

Thus $$x=-\frac{3}{2},\quad y=-2;$$

or $$x=-\frac{2}{3},\quad y=\frac{1}{2}.$$

Ex. 5. To solve $$x+y=2b,$$
$$x^4+y^4=2a^4.$$

Put $x=b+z$; then, from the first equation, $y=b-z$.

Hence $(b+z)^4+(b-z)^4=2a^4,$

whence after reduction

$$z^4+6b^2z^2+b^4-a^4=0;$$

$$\therefore\ z^2=-3b^2\pm\sqrt{8b^4+a^4};$$

$$\therefore\ z=\pm\sqrt{\{-3b^2\pm\sqrt{8b^4+a^4}\}}.$$

Thus $x=b\pm\sqrt{\{-3b^2\pm\sqrt{8b^4+a^4}\}},$

and $y=b\mp\sqrt{\{-3b^2\pm\sqrt{8b^4+a^4}\}}.$

EXAMPLES XIII.

Solve the following equations:—

1. $x+y=x^2-y^2=23.$

2. $x^2-4y^2+x+3y=2x-y=1.$

3. $x^2+xy=12,$
   $xy-2y^2=1.$

4. $x^2+2y^2=22,$
   $3y^2-xy-x^2=17.$

5. $x-y=5,$
   $\dfrac{1}{y}-\dfrac{1}{x}=\dfrac{5}{84}.$

6. $x+y=a+b,$
   $\dfrac{a}{x+b}+\dfrac{b}{y+a}=1.$

7. $a(x+y)=b(x-y)=xy.$

8. $\dfrac{1}{x^2}+\dfrac{1}{xy}=\dfrac{1}{a^2},$
   $\dfrac{1}{y^2}+\dfrac{1}{yx}=\dfrac{1}{b^2}.$

9. $\dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=10,$
   $\dfrac{ab}{xy}=3.$

10. $x+y=2a,$
    $x^3+y^3=2b^3.$

11. $x^2-xy+y^2=109,$
    $x^4+x^2y^2+y^4=4251.$

12. $x^2+xy+y^2=133,$
    $x+\sqrt{xy}+y=19.$

13. $x+y=72,$
    $\sqrt[3]{x}+\sqrt[3]{y}=6.$

14. $\frac{1}{x}+\frac{1}{y}=2,$
$xy+\frac{1}{x}+\frac{1}{y}=8.$

15. $x+y=1,$
$x^5+y^5=31.$

16. $x^2+y^2+3xy-4(x+y)+3=0,$
$xy+2(x+y)-5=0.$

17. $x^2+xy+x=14,$
$y^2+xy+y=28.$

18. $x^3+y^3=9,$
$x^2-xy+y^2=3.$

19. $x(y-b)=y(x-a)=2ab.$

20. $x+\frac{1}{y}=1,$
$y+\frac{1}{x}=4.$

21. $ax+by=2ab,$
$\frac{a}{y}+\frac{b}{x}=2.$

22. $\frac{x^2}{y}+\frac{y^2}{x}=12,$
$\frac{1}{x}+\frac{1}{y}=\frac{1}{3}.$

23. $\frac{x^3}{y}+xy=a^2,$
$\frac{y^3}{x}+xy=b^2.$

24. $xy-\frac{x}{y}=a,$
$xy-\frac{y}{x}=\frac{1}{a}.$

25. $x+y+\frac{y^2}{x}=14,$
$x^2+y^2+\frac{y^4}{x^2}=84.$

26. $x+y=6,$
$(x^2+y^2)(x^3+y^3)=1440.$

27. $x+y=8xy,$
$x^2+y^2=40x^2y^2.$

28. $x^2-xy=8x+3,$
$xy-y^2=8y-6.$

29. $\frac{x+y}{1-xy}=3,$
$\frac{x-y}{1+xy}=\frac{1}{3}.$

30. $x - y = a(x^2 - y^2)$,
$x + y = b(x^2 - y^2)$.

31. $\dfrac{x}{a} + \dfrac{y}{b} = \dfrac{b}{a} + \dfrac{a}{b}$,
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{b^2}{a^2} + \dfrac{a^2}{b^2}$.

32. $\dfrac{x}{a} + \dfrac{a}{x} = \dfrac{y}{b} + \dfrac{b}{y} = \dfrac{x}{y} + \dfrac{y}{x}$.

151. **Equations with more than two unknown quantities.** No general rules can be given for the solution of simultaneous equations of the second degree with more than two unknown quantities: all that can be done is to solve some typical examples.

Ex. 1. Solve the equations:

$$(x+y)(x+z)=a^2 \qquad \text{(i)},$$
$$(y+z)(y+x)=b^2 \qquad \text{(ii)},$$
$$(z+x)(z+y)=c^2 \qquad \text{(iii)}.$$

Multiply (ii) and (iii) and divide by (i);

then $$(y+z)^2=\frac{b^2c^2}{a^2};$$

$$\therefore\ y+z=\pm\frac{bc}{a} \qquad \text{(iv)}.$$

Similarly we have

$$z+x=\pm\frac{ca}{b} \qquad \text{(v)},$$

and $$x+y=\pm\frac{ab}{c} \qquad \text{(vi)}.$$

Also from the original equations it is clear that the signs must all be positive or all be negative.

Add (v) and (vi) and subtract (iv) from the sum; then

$$2x=\pm\left(\frac{ca}{b}+\frac{ab}{c}-\frac{bc}{a}\right);$$

$$\therefore\ x=\pm\frac{c^2a^2+a^2b^2-b^2c^2}{2abc}.$$

So also $$y=\pm\frac{a^2b^2+b^2c^2-c^2a^2}{2abc},$$

and $$z=\pm\frac{b^2c^2+c^2a^2-a^2b^2}{2abc}.$$

Ex. 2. Solve the equations:

$$x(y+z)=a \qquad \text{(i)},$$
$$y(z+x)=b \qquad \text{(ii)},$$
$$z(x+y)=c \qquad \text{(iii)}.$$

We have $y(z+x)+z(x+y)-x(y+z)=b+c-a$,

that is $2yz=b+c-a$.

Similarly $2zx=c+a-b$,

and $2xy=a+b-c$.

Hence $$\frac{(2xy)(2zx)}{2yz}=\frac{(a+b-c)(c+a-b)}{b+c-a};$$

$$\therefore\ 2x^2=\frac{(a+b-c)(c+a-b)}{(b+c-a)}.$$

Hence $$x=\pm\sqrt{\frac{(c+a-b)(a+b-c)}{2(b+c-a)}},$$

and similarly $$y=\pm\sqrt{\frac{(a+b-c)(b+c-a)}{2(c+a-b)}},$$

and $$z=\pm\sqrt{\frac{(b+c-a)(c+a-b)}{2(a+b-c)}}.$$

Ex. 3. Solve the equations:

$$x^2+2yz=a \qquad \text{(i)},$$
$$y^2+2zx=a \qquad \text{(ii)},$$
$$z^2+2xy=b \qquad \text{(iii)}.$$

By addition $(x+y+z)^2=2a+b$;

$$\therefore\ x+y+z=\pm\sqrt{2a+b} \qquad \text{(iv)}.$$

From (i) and (ii) by subtraction

$$(x-y)(x+y-2z)=0.$$

Hence $$x=y \qquad \text{(v)},$$

or else $$x+y-2z=0 \qquad \text{(vi)}.$$

I. If $x=y$, we have from (ii) and (iii) by subtraction

$$z^2+x^2-2xz=b-a;$$

$$\therefore\ z-x=\pm\sqrt{b-a} \qquad \text{(vii)}.$$

Hence, from (iv), (v) and (vii),

$$x=y=\frac{1}{3}\{\pm\sqrt{2a+b}\mp\sqrt{b-a}\},$$

$$z=\frac{1}{3}\{\pm\sqrt{2a+b}\pm 2\sqrt{b-a}\}.$$

II. When $x+y-2z=0$, we have from (iv)

$$z=\pm\frac{1}{3}\sqrt{2a+b},$$

and $$x+y=\pm\frac{2}{3}\sqrt{2a+b}.$$

Also, from (ii), $$y^2+x(x+y)=a,$$
which with the previous equation gives

$$x=\pm\sqrt{\frac{a-b}{3}}\pm\frac{1}{3}\sqrt{2a+b},$$

and $$y=\mp\sqrt{\frac{a-b}{3}}\pm\frac{1}{3}\sqrt{2a+b}.$$

Ex. 4. Solve the equations:

$$b^2z+c^2y=c^2x+a^2z=a^2y+b^2x=xyz.$$

We have $$b^2z+c^2y=xyz \quad \text{(i)},$$
$$c^2x+a^2z=xyz \quad \text{(ii)},$$
and $$a^2y+b^2x=xyz \quad \text{(iii)}.$$

Multiply (i) by $-a^2$, (ii) by b^2, and (iii) by c^2, and add;

then $$2b^2c^2x=(-a^2+b^2+c^2)xyz.$$

Hence $$x=0,$$

or else $$yz=\frac{2b^2c^2}{-a^2+b^2+c^2}.$$

If $x=0$, y and z must also be zero.

Hence $$x=y=z=0;$$

or else $$yz=\frac{2b^2c^2}{b^2+c^2-a^2},$$

and similarly $$zx=\frac{2c^2a^2}{c^2+a^2-b^2},$$

and $$xy=\frac{2a^2b^2}{a^2+b^2-c^2}.$$

The solution then proceeds as in Ex. 2.

Ex. 5. Solve the equations:

$$x^2-yz=a,$$
$$y^2-zx=b,$$
$$z^2-xy=c.$$

We have $$(x^2-yz)^2-(y^2-zx)(z^2-xy)=a^2-bc,$$
that is $$x(x^3+y^3+z^3-3xyz)=a^2-bc.$$
Hence, from the last equation and the two similar ones,

$$\frac{x}{a^2-bc}=\frac{y}{b^2-ca}=\frac{z}{c^2-ab}.$$

Hence each fraction is equal to

$$\sqrt{\frac{x^2-yz}{(a^2-bc)^2-(b^2-ca)(c^2-ab)}}=\pm\sqrt{\frac{1}{(a^3+b^3+c^3-3abc)}}.$$

Ex. 6. Solve the equations:

$$x+y+z=a+b+c \quad\text{.........(i)},$$
$$x^2+y^2+z^2=a^2+b^2+c^2 \quad\text{.........(ii)},$$
$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3 \quad\text{.........(iii)}.$$

It is obvious that $x=a$, $y=b$, $z=c$ will satisfy the equations: put then $x=a+\lambda$, $y=b+\mu$, $z=c+\nu$, and we have after reduction

$$\lambda+\mu+\nu=0 \quad\text{.........(iv)},$$
$$\frac{\lambda}{a}+\frac{\mu}{b}+\frac{\nu}{c}=0 \quad\text{.........(v)},$$
$$2(a\lambda+b\mu+c\nu)+\lambda^2+\mu^2+\nu^2=0 \quad\text{.........(vi)}$$

From (iv) and (v)

$$\frac{\lambda}{a(b-c)}=\frac{\mu}{b(c-a)}=\frac{\nu}{c(a-b)},$$

whence from (vi)

$$\lambda=\mu=\nu=0,$$

or $$\frac{\lambda}{a(b-c)}=\frac{2(b-c)(c-a)(a-b)}{a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2}.$$

Hence $$x=a, \quad y=b, \quad z=c;$$
or else

$$\frac{x-a}{a(b-c)}=\frac{y-b}{b(c-a)}=\frac{z-c}{c(a-b)}=\frac{2(b-c)(c-a)(a-b)}{a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2}.$$

Ex. 7. Solve the equations:

$$x+y+z=6,$$
$$yz+zx+xy=11,$$
$$xyz=6.$$

This is an example of a system of three symmetrical equations. Such equations can generally be easily solved by making use of the relations of Art. 129. Thus in the present instance it is clear that x, y, z are the three roots of the cubic equation

$$\lambda^3-6\lambda^2+11\lambda-6=0.$$

The roots of the cubic are 1, 2, 3.

Hence $x=1$, $y=2$, $z=3$; or $x=1$, $y=3$, $z=2$; or $x=2$, $y=3$, $z=1$; &c.

Ex. 8. Solve the equations:

$$x+y+z=a \quad \text{..............................(i)},$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{a} \quad \text{..............................(ii)},$$

$$yz+zx+xy=-c^2 \quad \text{..........................(iii)}.$$

This again is a system of symmetrical equations, and two of the relations of Art. 129 are already given; we have therefore only to find the third.

We have from (ii),

$$\frac{yz+zx+xy}{xyz}=\frac{1}{a};$$

$$\therefore\ xyz=-ac^2 \quad \text{................................(iv)}.$$

Then, from (i), (iii) and (iv), we see that x, y, z are the roots of the cubic $\qquad \lambda^3-a\lambda^2-c^2\lambda+ac^2=0$,

that is $\qquad \lambda^2(\lambda-a)-c^2(\lambda-a)=0;$

$$\therefore\ \lambda=a, \text{ or } \lambda=\pm c.$$

Thus $x=a$, $y=c$, $z=-c$; &c.

Ex. 9. Solve the equations:

$$x^2(y-z)=a^2(b-c),$$
$$y^2(z-x)=b^2(c-a),$$
$$z^2(x-y)=c^2(a-b).$$

By addition

$$x^2(y-z)+y^2(z-x)+z^2(x-y)=a^2(b-c)+b^2(c-a)+c^2(a-b),$$

that is $\qquad (y-z)(z-x)(x-y)=(b-c)(c-a)(a-b).$

By multiplication

$$x^2y^2z^2(y-z)(z-x)(x-y)=a^2b^2c^2(b-c)(c-a)(a-b);$$

$$\therefore\ x^2y^2z^2=a^2b^2c^2.$$

Hence $\qquad xyz=abc$ ....(i),

or $\qquad xyz=-abc$ ....(ii).

Again $a^2(b-c)y+b^2(c-a)x=x^2y(y-z)+xy^2(z-x)$

$$=xyz(y-x) \quad \text{......................(iii)}.$$

Hence, if $xyz=abc$, we have from (iii)

$$\{b^2(c-a)+abc\}x+\{a^2(b-c)-abc\}y=0,$$

that is $\qquad bx(bc+ca-ab)-ay(bc+ca-ab)=0;$

$$\therefore\ \frac{x}{a}=\frac{y}{b}, \text{ and therefore each } =\frac{z}{c}.$$

Thus, when $xyz = abc$, we have $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

Hence each is equal to $\sqrt[3]{\frac{xyz}{abc}} = \sqrt[3]{1}$.

Thus $\quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = 1, \quad$ or $\quad \frac{x}{a\omega} = \frac{y}{b\omega} = \frac{z}{c\omega} = 1,$

or $\quad \frac{x}{a\omega^2} = \frac{y}{b\omega^2} = \frac{z}{c\omega^2} = 1.$

If $xyz = -abc$, we have from (iii)

$$\frac{x}{a}(bc - ca - ab) = \frac{y}{b}(ca - ab - bc).$$

Hence also each $\qquad = \frac{z}{c}(ab - bc - ca)$

$$= \sqrt[3]{\{-(bc - ca - ab)(ca - ab - bc)(ab - bc - ca)\}}.$$

EXAMPLES XIV.

Solve the following equations:

1. $yz = a^2,$
$zx = b^2,$
$xy = c^2.$

2. $x(x + y + z) = a^2,$
$y(x + y + z) = b^2,$
$z(x + y + z) = c^2.$

3. $-yz + zx + xy = a,$
$yz - zx + xy = b,$
$yz + zx - xy = c.$

4. $yz = a(y + z),$
$zx = b(z + x),$
$xy = c(x + y).$

5. $yz = by + cz,$
$zx = cz + ax,$
$xy = ax + by.$

6. $x^2 + 2yz = 12,$
$y^2 + 2zx = 12,$
$z^2 + 2xy = 12.$

7. $(y + z)(x + y + z) = a,$
$(z + x)(x + y + z) = b,$
$(x + y)(x + y + z) = c.$

8. $(y + b)(z + c) = a^2,$
$(z + c)(x + a) = b^2,$
$(x + a)(y + b) = c^2.$

9. $x^2-(y-z)^2=a^2,$
$y^2-(z-x)^2=b^2,$
$z^2-(x-y)^2=c^2.$

10. $x(y+z-x)=a,$
$y(z+x-y)=b,$
$z(x+y-z)=c.$

11. $\dfrac{y+z}{a}=\dfrac{z+x}{b}=\dfrac{x+y}{c}=2xyz.$

12. $\dfrac{y+z}{a}=\dfrac{z+x}{b}=\dfrac{x+y}{c}=\dfrac{x^2+y^2+z^2}{a^2+b^2+c^2}.$

13. $yz=a+y+z,$
$zx=b+z+x,$
$xy=c+x+y.$

14. $yz=a(y+z)+\alpha,$
$zx=a(z+x)+\beta,$
$xy=a(x+y)+\gamma.$

15. $yz-f^2=cy+bz,$
$zx-g^2=az+cx,$
$xy-h^2=bx+ay.$

16. $x+y^{-1}=\dfrac{3}{2},$
$y+z^{-1}=\dfrac{7}{3},$
$z+x^{-1}=4.$

17. $x+y+z=6,$
$x^2+y^2+z^2=14,$
$xyz=6.$

18. $x+y+z=15,$
$x^3+y^3+z^3=495,$
$xyz=105.$

19. $x+y+z=9,$
$x^2+y^2+z^2=41,$
$x^3+y^3+z^3=189.$

20. $x+y+z=10,$
$yz+zx+xy=33,$
$(y+z)(z+x)(x+y)=294.$

21. $\dfrac{yz}{bz+cy}=\dfrac{zx}{cx+az}=\dfrac{xy}{ay+bx}=\dfrac{x^2+y^2+z^2}{a^2+b^2+c^2}.$

22. $\dfrac{a}{x}+\dfrac{y}{b}+\dfrac{z}{c}=1,$
$\dfrac{x}{a}+\dfrac{b}{y}+\dfrac{z}{c}=1,$
$\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{c}{z}=1.$

23. $ax=\dfrac{y}{z}+\dfrac{z}{y},$
$by=\dfrac{z}{x}+\dfrac{x}{z},$
$cz=\dfrac{x}{y}+\dfrac{y}{x}.$

24. $y^2 + z^2 - x(y+z) = a^2,$
$z^2 + x^2 - y(z+x) = b^2,$
$x^2 + y^2 - z(x+y) = c^2.$

25. $x^2 + yz - a^2 = y^2 + zx - b^2 = z^2 + xy - c^2 = \frac{1}{2}(x^2 + y^2 + z^2).$

26. $x(x+y+z) - (y^2 + z^2 + yz) = a,$
$y(x+y+z) - (z^2 + x^2 + zx) = b,$
$z(x+y+z) - (x^2 + y^2 + xy) = c.$

27. $x + y + z = a + b + c,$
$x^2 + y^2 + z^2 = a^2 + b^2 + c^2,$
$(b-c)x + (c-a)y + (a-b)z = 0.$

28. $(x+y)(x+z) = ax,$
$(y+z)(y+x) = by,$
$(z+x)(z+y) = cz.$

29. $x^2 - yz = ax,$
$y^2 - zx = by,$
$z^2 - xy = cz.$

30. $x^2 + a(2x+y+z) = y^2 + b(2y+z+x) = z^2 + c(2z+x+y)$
$= (x+y+z)^2.$

31. $y^2 + yz + z^2 = a^2,$
$z^2 + zx + x^2 = b^2,$
$x^2 + xy + y^2 = c^2.$

32. $a^2x + b^2y + c^2z = 0,$
$$\frac{(b-c)^2}{ax} + \frac{(c-a)^2}{by} + \frac{(a-b)^2}{cz} = 0,$$
$$\frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

CHAPTER XI.

PROBLEMS.

152. We shall in the present chapter consider a class of questions called *problems*. In a problem the magnitudes of certain quantities, some of which are known and others unknown, are connected by given relations; and the values of the unknown quantities have to be found by means of these relations.

In order to solve a problem, the relations between the magnitudes of the known and unknown quantities must be expressed by means of algebraical symbols: we thus obtain equations the solution of which gives the required values of the unknown quantities.

It often happens that by solving the equations which are the algebraical statements of the relations between the magnitudes of the known and unknown quantities, we obtain results which do not all satisfy the conditions of the problem. The reason of this is that in a problem there may be restrictions, expressed or implied, on the numbers concerned, which restrictions cannot be retained in the equations. For example, in a problem which refers to a number of men, it is clear that this number must be *integral*, but this condition cannot be expressed in the equations.

Thus there are three steps in the solution of a problem. We first find the equations which are the algebraical expressions of the relations between the magnitudes of the

known and unknown quantities; we then find the values of the unknown quantities which satisfy these equations; and finally we examine whether any or all of the values we have found violate any conditions which are expressed or implied in the problem, but which are not contained in the equations. The necessity of this final examination will be seen from some of the following examples of problems.

Ex. 1. A has £5 and B has ten shillings. How much must A give to B in order that he may have just four times as much as B?

Let x be the *number of shillings* that A gives to B.

Then A will have $100 - x$ shillings, and B will have $10 + x$ shillings. But, by the question, A now has four times as much as B.

Hence we have the equation

$$100 - x = 4(10 + x);$$

$$\therefore x = 12.$$

Thus A must give 12 shillings to B.

It should be remembered that x must always stand for a *number*. It is also of importance to notice that all concrete quantities of the same kind must be expressed in terms of the same unit.

Ex. 2. One man and two boys can do in 12 days a piece of work which would be done in 6 days by 3 men and 1 boy. How long would it take one man to do it?

Let $x =$ the number of days in which one man would do the whole, and let $y =$ the number of days in which one boy would do the whole.

Then a man does $\frac{1}{x}$th of the whole in a day; and a boy does $\frac{1}{y}$th of the whole in a day.

By the question one man and two boys do $\frac{1}{12}$th of the whole in a day.

Hence we have

$$\frac{1}{x} + \frac{2}{y} = \frac{1}{12}.$$

We have also, since 3 men and 1 boy do $\frac{1}{6}$th of the whole in a day,

$$\frac{3}{x} + \frac{1}{y} = \frac{1}{6}.$$

Whence $x = 20$.

Thus one man would do the whole work in 20 days.

Ex. 3. In a certain family eleven times the number of the children is greater by 12 than twice the square of the number. How many children are there?

Let x be the number of children; then we have the equation

$$11x = 2x^2 + 12,$$

or $$2x^2 - 11x + 12 = 0,$$

that is $$(2x-3)(x-4) = 0.$$

Hence $$x = 4, \text{ or } x = \tfrac{3}{2}.$$

The value $x = \frac{3}{2}$ satisfies the *equation*, but it must be rejected, since it does not satisfy all the conditions of the problem, for the number of children must be a whole number.

Thus there are 4 children.

Ex. 4. Eleven times the number of yards in the length of a rod is greater by 12 than twice the square of the number. How long is the rod?

This leads to the same equation as Ex. 3; but in this case we cannot reject the fractional result. Thus the length of the rod may be 4 yards, or it may be a yard and a half.

Ex. 5. A number of two digits is equal to three times the product of the digits, and the digit in the ten's place is less by 2 than the digit in the unit's place. Find the number.

Let x be the digit in the ten's place; then $x+2$ will be the digit in the unit's place. The number is therefore equal to

$$10x + (x+2).$$

Hence, by the question,

$$10x + (x+2) = 3x(x+2);$$

$$\therefore\ 3x^2 - 5x - 2 = 0,$$

or $$(x-2)(3x+1) = 0.$$

Hence $$x = 2, \text{ or } x = -\tfrac{1}{3}.$$

Now the digits of a number must be positive integers not greater than nine; hence the value $x = -\frac{1}{3}$ must be rejected. The digit in the ten's place must therefore be 2, and the digit in the unit's place must be 4. Hence 24 is the required number.

Ex. 6. A number of two digits is equal to three times the sum of the digits. Find the number.

Let x be the digit in the ten's place, and y the digit in the unit's place; then the number will be equal to $10x + y$.

Hence, by the question,

$$10x + y = 3(x+y);$$

$$\therefore\ 7x = 2y.$$

Since x and y must both be positive integers not greater than 9, it follows that x must be 2 and y must be 7. Thus the required number is 27.

Ex. 7. The sum of a certain number and its square root is 90. What is the number?

Let x be the number; then we have the equation

$$x+\sqrt{x}=90;$$

$$\therefore\ (x-90)^2=x,$$

or $$x^2-181x+8100=0,$$

that is $$(x-81)(x-100)=0.$$

Hence $$x=81,\text{ or } x=100.$$

If, in the question, the square root means only the *arithmetical* square root, 81 is the only number which satisfies the conditions. If, however, 'its square root' is taken to mean '*one* of its square roots,' both 81 and 100 are admissible.

Ex. 8. The sum of the ages of a father and his son is 100 years; also one-tenth of the product of their ages, in years, exceeds the father's age by 180. How old are they?

Let the father be x years old; then the son will be $100-x$ years old. Hence, by the question,

$$\tfrac{1}{10}x\,(100-x)=x+180;$$

$$\therefore\ x^2-90x+1800=0,$$

that is $$(x-60)(x-30)=0.$$

Hence $$x=60,\text{ or } x=30.$$

If the father is 60, the son will be $100-60=40$. If the father is 30, the son will be $100-30=70$, which is impossible, since the son cannot be older than the father.

Hence the father must be 60 and the son 40 years old.

Ex. 9. A man buys pigs, geese and ducks. If each of the geese had cost a shilling less, one pig would have been worth as many geese as each goose is actually worth shillings. A goose is worth as much as two ducks, and fourteen ducks are worth seven shillings more than a pig. Find the price of a pig, a goose, and a duck respectively.

Let $x=$ the price in shillings of a pig,

$y=$,, ,, ,, ,, goose,

and $z=$,, ,, ,, ,, duck.

Then, by the question, a pig is worth y times $(y-1)$ shillings;

$$\therefore\ x=y\,(y-1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Since a goose is worth 2 ducks,

$$\therefore y = 2z \quad \text{...(ii).}$$

And, since 14 ducks are worth 7 shillings more than a pig,

$$14z = 7 + x \quad \text{...(iii).}$$

From (i) and (ii) we have the values of x and z in terms of y; and, substituting these values in (iii), we have

$$7y = 7 + y(y-1),$$

or

$$y^2 - 8y + 7 = 0;$$

$$\therefore y = 7, \text{ or } y = 1.$$

If $y=7$, $x=42$ from (i), and $z=\frac{7}{2}$ from (ii).

If $y=1$, $x=0$ from (i), and $z=\frac{1}{2}$ from (ii). These values are however inadmissible, since pigs cannot be bought for nothing.

Hence a pig cost 42*s*., a goose 7*s*., and a duck 3*s*. 6*d*.

EXAMPLES XV.

1. Divide 50 into two parts, such that twice one part is equal to three times the other.

2. *A* has £5 less than *B*, *C* has as much as *A* and *B* together, and *A*, *B*, *C* have £50 between them. How much has each?

3. One man is 70 and another is 45 years of age; when was the first twice as old as the second?

4. How much are eggs a score, if a rise of 25 per cent. in the price would make a difference of 40 in the number which could be bought for a sovereign?

5. A bag contains 50 coins which are worth £14 altogether. A certain number of the coins are sovereigns, there are three times as many half-sovereigns, and the rest are shillings. Find the number of each.

6. *A* can do a piece of work in 20 days, which *B* can do in 12 days. *A* begins the work, but after a time *B* takes his place, and the whole work is finished in 14 days from the beginning. How long did *A* work?

7. A man buys a certain number of eggs at two a penny, four times as many at 5*d*. a dozen, five times as many at 8*d*. a score, and sells them at 3*s*. 8*d*. a hundred, gaining by the transaction 3*s*. 6*d*. How many eggs did he buy?

8. A bill of £63. 5*s*. was paid in sovereigns and half-crowns, and the number of coins used was 100; how many sovereigns were paid?

9. A man walking from a town A to another B at the rate of 4 miles an hour, starts one hour before a coach which goes 12 miles an hour, and is picked up by the coach. On arriving at B he observes that his coach journey lasted two hours. Find the distance from A to B.

10. Two passengers have altogether 600 lbs. of luggage and are charged for the excess above the weight allowed 3*s*. 4*d*. and 11*s*. 8*d*. respectively. If all the luggage had belonged to one person he would have been charged £1. How much luggage is each passenger allowed free of charge?

11. A piece of work can be done by A and B in 4 days, by A and C in 6 days, and by B and C in 12 days: find in what time it would be done by A, B and C working together.

12. A father's age is equal to those of his three children together. In 9 years it will amount to those of the two eldest, in 3 years after that to those of the eldest and youngest, and in 3 years after that to those of the two youngest. Find their present ages.

13. A and B start simultaneously from two towns to meet one another: A travels 2 miles per hour faster than B and they meet in 3 hours: if B had travelled one mile per hour slower, and A at two-thirds his previous pace they would have met in 4 hours. Find the distance between the towns.

14. A traveller walks a certain distance: if he had gone half a mile an hour faster, he would have walked it in $\frac{4}{5}$ of the time: if he had gone half a mile an hour slower he would have been $2\frac{1}{2}$ hours longer on the road. Find the distance.

15. Divide 243 into three parts such that one-half of the first, one-third of the second, and one-fourth of the third part, shall all be equal to one another.

16. A sum of money consisting of pounds and shillings would be reduced to one-eighteenth of its original value if the pounds were shillings, and the shillings pence. Shew that its value would be increased in the ratio of 15 to 2 if the pounds were five-pound notes, and the shillings pounds.

17. £1000 is divided between A, B, C and D. B gets half as much as A, the excess of C's share over D's share is equal to one-third of A's share, and if B's share were increased by £100 he would have as much as C and D have between them; find how much each gets.

18. Find two numbers, one of which is three-fifths of the other, so that the difference of their squares may be equal to 16.

19. Find two numbers expressed by the same two digits in different orders whose sum is equal to the square of the sum of the two digits, and whose difference is equal to five times the square of the smaller digit.

20. A man rode one-third of a journey at 10 miles per hour, one-third more at 9 miles per hour, and the rest at 8 miles per hour. If he had ridden half the journey at 10 miles per hour and the other half at 8 miles per hour, he would have been half a minute longer on the journey. What distance did he ride?

21. Two bicyclists start at 12 o'clock, one from Cambridge to Stortford and back, and the other from Stortford to Cambridge and back. They meet at 3 o'clock for the second time, and they are then 9 miles from Cambridge. The distance from Cambridge to Stortford is 27 miles. When and where did they meet for the first time?

22. Divide £1015 among A, B, C so that B may receive £5 less than A, and C as many times B's share as there are shillings in A's share.

23. On a certain road the telegraph posts are at equal distances, and the number per mile is such that if there were one less in each mile the interval between the posts would be increased by $2\frac{14}{15}$ yards. Find the number of posts in a mile.

24. The sum of two numbers multiplied by the greater is 144, and their difference multiplied by the less is 14: find them.

25. *A* and *B* start simultaneously from two towns and meet after five hours; if *A* had travelled one mile per hour faster and *B* had started one hour sooner, or if *B* had travelled one mile per hour slower and *A* had started one hour later, they would in either case have met at the same spot they actually met at. What was the distance between the towns?

26. A battalion of soldiers, when formed into a solid square, present sixteen men fewer in the front than they do when formed in a hollow square four deep. Required the number of men.

27. A number of two digits is equal to seven times the sum of the digits; shew that if the digits be reversed, the number thus formed will be equal to four times the sum of the digits.

28. *A* sets out to walk to a town 7 miles off, and *B* starts 20 minutes afterwards to follow him. When *B* has overtaken *A* he immediately turns back, and reaches the place from which he started at the same instant that *A* reaches his destination. Supposing *B* to have walked at the rate of 4 miles an hour: find *A*'s rate.

29. *A* starts to bicycle from Cambridge to London, and *B* at the same time from London to Cambridge, and they travel uniformly: *A* reaches London 4 hours, and *B* reaches Cambridge 1 hour, after they have met on the road. How long did *B* take to perform the journey?

30. A number consists of 3 digits whose sum is 10. The middle digit is equal to the sum of the other two; and the number will be increased by 99 if its digits be reversed. Find the number.

31. Two vessels contain each a mixture of wine and water. In the first vessel the quantity of wine is to the quantity of water as 1 : 3, and in the second as 3 : 5. What quantity must be taken from each in order to form a third mixture, which shall contain 5 gallons of wine and 9 gallons of water?

32. Supposing that it is now between 10 and 11 o'clock, and that 6 minutes hence the minute hand of a watch will be exactly opposite to the place where the hour hand was 3 minutes ago: find the time.

33. A, B and C start from Cambridge, at 3, 4 and 5 o'clock respectively to walk, drive and ride respectively to London. C overtakes B at 7 o'clock, and C overtakes A $4\frac{1}{2}$ miles further on at half-past seven. When and where will B overtake A?

34. A train 60 yards long passed another train 72 yards long, which was travelling in the same direction on a parallel line of rails, in 12 seconds. Had the slower train been travelling half as fast again, it would have been passed in 24 seconds. Find the rates at which the trains were travelling.

35. A distributes £180 in equal sums amongst a certain number of people. B distributes the same sum but gives to each person £6 more than A, and gives to 40 persons less than A does. How much does A give to each person?

36. Three vessels ply between the same two ports. The first sails half a mile per hour faster than the second, and makes the passage in an hour and a half less. The second sails three-quarters of a mile per hour faster than the third and makes the passage in $2\frac{1}{2}$ hours less. What is the distance between the ports?

37. Two persons A, B walk from P to Q and back. A starts 1 hour after B, overtakes him 2 miles from Q, meets him 32 minutes afterwards, and arrives at P when B is 4 miles off. Find the distance from P to Q.

CHAPTER XII.

MISCELLANEOUS THEOREMS AND EXAMPLES.

153. **Elimination.** When more equations are given than are necessary to determine the values of the unknown quantities, the constants in the equations must be connected by one or more relations, and it is often of importance to determine these relations.

Since the relations required are not to contain any of the unknown quantities, what we have to do is to *eliminate* all the unknown quantities from the given system.

The following are some examples of Elimination:

Ex. 1. Eliminate x from the equations $ax+b=0$, $a'x+b'=0$.

From the first equation we have $x=-\frac{b}{a}$, and from the second equation we have $x=-\frac{b'}{a'}$.

Hence we must have $\frac{b}{a}=\frac{b'}{a'}$, or $ba'-b'a=0$; which is the required result.

Ex. 2. Eliminate x and y from the equations

$$ax+by+c=0,$$
$$a'x+b'y+c'=0,$$
$$a''x+b''y+c''=0.$$

From the first two equations we have [Art. 143]

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{1}{ab'-a'b}.$$

These values of x and y must satisfy the third equation; hence

$$a''\frac{bc'-b'c}{ab'-a'b}+b''\frac{ca'-c'a}{ab'-a'b}+c''=0,$$

or $$a''(bc'-b'c)+b''(ca'-c'a)+c''(ab'-a'b)=0,$$

the required result.

The general case of the elimination of $n-1$ unknown quantities from n equations of the first degree will be considered in the Chapter on Determinants.

Ex. 3. Eliminate x from the equations

$$ax^2+bx+c=0,$$
$$a'x^2+b'x+c'=0.$$

As in Art. 143, we have

$$\frac{x^2}{bc'-b'c}=\frac{x}{ca'-c'a}=\frac{1}{ab'-a'b}.$$

Hence $$(bc'-b'c)(ab'-a'b)=(ca'-c'a)^2,$$

the required result.

It should be remarked that the above condition is also the condition that the two expressions ax^2+bx+c and $a'x^2+b'x+c'$ may have a common factor of the form $x-a$; for if the expressions have a common factor of the form $x-a$ they must both vanish for the same value of x.

Ex. 4. Eliminate x from the equations

$$ax^3+bx+c=0,$$
$$a'x^3+b'x+c'=0.$$

As in Ex. 3, we have

$$\frac{x^3}{bc'-b'c}=\frac{x}{ca'-c'a}=\frac{1}{ab'-a'b};$$

$$\therefore\ \frac{bc'-b'c}{ab'-a'b}=\left(\frac{ca'-c'a}{ab'-a'b}\right)^3;$$

$$\therefore\ (bc'-b'c)(ab'-a'b)^2=(ca'-c'a)^3,$$

the required relation.

Ex. 5. Eliminate x from the equations

$$ax^2+bx+c=0 \quad\text{.............................(i)},$$

$$a'x^3+b'x^2+c'x+d'=0 \quad\text{.............................(ii)}.$$

Multiply (i) by $a'x$, (ii) by a, and subtract; then,

$$(ab'-ba')x^2+(ac'-ca')x+ad'=0 \quad\text{....................(iii)}.$$

We can now eliminate x from (i) and (iii) as in Ex. 3.

Ex. 6. Eliminate x, y, z from the equations

$$x+y+z=a \quad\text{.............................(i)},$$

$$x^2+y^2+z^2=b^2 \quad\text{.............................(ii)},$$

$$x^3+y^3+z^3=c^3 \quad\text{.............................(iii)},$$

$$xyz=d^3 \quad\text{.............................(iv)}.$$

From (i) and (ii) we have

$$2yz+2zx+2xy=a^2-b^2.$$

From (iii) and (iv) we have

$$x^3+y^3+z^3-3xyz=c^3-3d^3,$$

i.e. $$(x+y+z)(x^2+y^2+z^2-yz-zx-xy)=c^3-3d^3.$$

Hence $$a\{b^2-\tfrac{1}{2}(a^2-b^2)\}=c^3-3d^3;$$

$$\therefore\ a^3+2c^3-6d^3-3ab^2=0,$$

the required result.

Ex. 7. Eliminate x, y, z from the equations

$$x^2(y+z)=a^2 \quad\text{.............................(i)},$$

$$y^2(z+x)=b^2 \quad\text{.............................(ii)},$$

$$z^2(x+y)=c^2 \quad\text{.............................(iii)},$$

$$xyz=abc \quad\text{.............................(iv)}.$$

From (i), (ii), (iii) by multiplication

$$x^2y^2z^2(y+z)(z+x)(x+y)=a^2b^2c^2.$$

Hence, from (iv),

$$(y+z)(z+x)(x+y)=1,$$

that is,

$$2xyz+x^2(y+z)+y^2(z+x)+z^2(x+y)=1;$$

$$\therefore\ 2abc+a^2+b^2+c^2=1,$$

the required result.

Ex. 8. Eliminate l, m, n, l', m', n' from the equations

$$ll'=a,\quad mm'=b,\quad nn'=c,$$

$$mn'+m'n=2f,\quad nl'+n'l=2g,\quad lm'+l'm=2h.$$

By continued multiplication of the last three equations, we have

$$8fgh = 2lmnl'm'n' + ll'(m^2n'^2 + m'^2n^2)$$
$$+ mm'(n^2l'^2 + n'^2l^2) + nn'(l^2m'^2 + l'^2m^2)$$
$$= ll'(mn' + m'n)^2 + mm'(nl' + n'l)^2$$
$$+ nn'(lm' + l'm)^2 - 4ll'mm'nn'$$
$$= 4af^2 + 4bg^2 + 4ch^2 - 4abc.$$

Hence $$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

154. *To find the condition that the most general quadratic expression in x and y may be expressed as the product of two factors of the first degree in x and y.*

The most general quadratic expression in x and y may be written in the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c \text{ ....(i).}$$

What is required is the condition that the above expression may be identically equal to

$$(lx + my + n)(l'x + m'y + n') \text{ ....(ii),}$$

where l, m, n, l', m', n' do not contain x or y.

Now if (i) and (ii) are identically equal we may equate the coefficients of the different powers of x and also of y [Art. 91]. Hence we have

$$ll' = a, \quad mm' = b, \quad nn' = c,$$
$$mn' + m'n = 2f, \quad nl' + n'l = 2g, \quad lm' + l'm = 2h.$$

Eliminating l, m, n, l', m', n' [Art. 153, Ex. 8], we have

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0,$$

the condition required.

Ex. 1. For what value of λ is

$$12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda$$

the product of two factors of the first degree in x and y?

Ans. $\lambda = 2$.

Ex. 2. For what value of λ is

$$12x^2 + 36xy + \lambda y^2 + 6x + 6y + 3$$

the product of two factors of the first degree in x and y?

Ans. $\lambda = 28$.

155. Equations in which there is some restriction on the values of the letters. A single equation which contains two or more unknown quantities can be satisfied by an indefinite number of values of the unknown quantities, provided that these values are not in any way restricted. If however the values of the unknown quantities are subject to any restriction, a single equation may be sufficient to determine more than one unknown quantity.

For example, if we have the single equation $2x+5y=7$, and restrict both x and y to positive integral values, the equation can only be satisfied by one set of values, namely by the values $x=1$, $y=1$.

Again, from the single equation

$$3(x-a)^2+4(y-b)^2=0,$$

with the restriction that all the quantities must be *real*, we can conclude *both* that $x-a=0$, and that $y-b=0$; for the squares of real quantities must be positive, and the sum of two or more positive quantities cannot be zero unless they are all zero.

Ex. 1. If $(a+b+c)^2=3(bc+ca+ab)$, then $a=b=c$.

We have $$a^2+b^2+c^2-bc-ca-ab=0,$$

that is $$\tfrac{1}{2}\{(b-c)^2+(c-a)^2+(a-b)^2\}=0.$$

Whence $b-c$, $c-a$ and $a-b$ must all be zero.

Ex. 2. If x, x', y, y' be all real, and

$$2(x^2+x'^2-xx')(y^2+y'^2-yy')=x^2y^2+x'^2y'^2;$$

then will $x=x'$ and $y=y'$.

We have

$$x^2(y^2+2y'^2-2yy')-2xx'(y^2+y'^2-yy')+x'^2(2y^2+y'^2-2yy')=0;$$

$$\therefore\ (x^2-2xx'+x'^2)(y^2-2yy'+y'^2)+x^2y'^2-2xx'yy'+x'^2y^2=0,$$

that is $$(x-x')^2(y-y')^2+(xy'-x'y)^2=0.$$

Hence $xy'-x'y=0$ and $(x-x')(y-y')=0$.

From the second relation $x=x'$ or $y=y'$; and either of these combined with the first relation shews that both $x=x'$ and $y=y'$.

Ex. 3. If
$$a_1^2+a_2^2+a_3^2+\ldots\ldots=p^2,$$
$$b_1^2+b_2^2+b_3^2+\ldots\ldots=q^2,$$
and
$$a_1b_1+a_2b_2+a_3b_3+\ldots\ldots=pq,$$
the quantities being all real; then will
$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\text{ &c. }=\frac{p}{q}.$$

Multiply the equations in order by q^2, p^2 and $-2pq$ respectively, and add; we then have
$$(qa_1-pb_1)^2+(qa_2-pb_2)^2+(qa_3-pb_3)^2+\ldots\ldots=0.$$
Hence
$$qa_1-pb_1=0=qa_2-pb_2=qa_3-pb_3=\ldots\ldots\text{&c.}$$
Therefore
$$\frac{a_1}{b_1}=\frac{p}{q}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\ldots\ldots\text{&c.}$$

156. We have already proved that
$$\begin{aligned}a^3+b^3+c^3-3abc&=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)\\&=\tfrac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}\\&=(a+b+c)(a+\omega b+\omega^2c)(a+\omega^2b+\omega c),\end{aligned}$$
where ω is either of the cube roots of unity. [See Art. 139.]

From the above many other identities can be found.

Ex. 1. $(b+c)^3+(c+a)^3+(a+b)^3-3(b+c)(c+a)(a+b)$
$$=2(a^3+b^3+c^3-3abc).$$

$$\begin{aligned}\text{Left side}&=\tfrac{1}{2}\{b+c+c+a+a+b\}\{(\overline{c+a}-\overline{a+b})^2+\text{two similar terms}\}\\&=(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}\\&=2(a^3+b^3+c^3-3abc).\end{aligned}$$

Ex. 2. $(b+c-a)^3+(c+a-b)^3+(a+b-c)^3$
$$-3(b+c-a)(c+a-b)(a+b-c)=4(a^3+b^3+c^3-3abc).$$

$$\begin{aligned}\text{Left side}&=\tfrac{1}{2}(a+b+c)\{(2b-2c)^2+\text{two similar terms}\}\\&=4(a^3+b^3+c^3-3abc).\end{aligned}$$

Ex. 3. $(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3-3(x^2-yz)(y^2-zx)(z^2-xy)$
$$=(x^3+y^3+z^3-3xyz)^2.$$

$$\begin{aligned}\text{Left side}&=\tfrac{1}{2}(x^2+y^2+z^2-yz-zx-xy\,[(\overline{y^2-zx}-\overline{z^2-xy})^2+\text{two similar terms}]\\&=\tfrac{1}{2}(x^2+y^2+z^2-yz-zx-xy)(x+y+z)^2[(y-z)^2+\text{two similar terms}]\\&=(x+y+z)^2(x^2+y^2+z^2-yz-zx-xy)^2\\&=(x^3+y^3+z^3-3xyz)^2.\end{aligned}$$

Ex. 4. Shew that $(x^3+y^3+z^3-3xyz)(a^3+b^3+c^3-3abc)$ can be expressed in the form $X^3+Y^3+Z^3-3XYZ$.

We have

$$(x+y+z)(a+b+c)=(ax+by+cz)+(bx+cy+az)+(cx+ay+bz),$$

$$(x+\omega y+\omega^2 z)(a+\omega^2 b+\omega c)=(ax+by+cz)+\omega^2(bx+cy+az)+\omega(cx+ay+bz),$$

and

$$(x+\omega^2 y+\omega z)(a+\omega b+\omega^2 c)=(ax+by+cz)+\omega(bx+cy+az)+\omega^2(cx+ay+bz).$$

The continued product of the left members of the above equations is

$$(x^3+y^3+z^3-3xyz)(a^3+b^3+c^3-3abc);$$

and the continued product of the expressions on the right is

$$(ax+by+cz)^3+(bx+cy+az)^3+(cx+ay+bz)^3-3(ax+by+cz)(bx+cy+az)(cx+ay+bz),$$

which is of the required form.

157. **Definitions.** The symbol $\equiv$ is often used to denote that the two expressions between which it is placed are *identically* equal. Thus $a^2-b^2\equiv(a+b)(a-b)$.

The sum of any number of quantities of the same type is often expressed by writing only one of the terms preceded by the symbol Σ. Thus Σbc means the sum of all such terms as bc; so that if there are three letters a, b, c, $\Sigma bc\equiv bc+ca+ab$. So also the identity

$$(a+b+c+\ldots)^2\equiv a^2+b^2+c^2+\ldots+2(ab+bc+\ldots),$$

may be written $(\Sigma a)^2\equiv\Sigma a^2+2\Sigma ab$.

The product of any number of quantities of the same type is often expressed by writing only one of the factors preceded by the symbol Π. Thus $\Pi(b+c)$ means the product of all such factors as $(b+c)$; so that if there are three letters a, b, c, $\Pi(b+c)\equiv(b+c)(c+a)(a+b)$.

158. The following examples illustrate cases of frequent occurrence.

Ex. 1. If $a^3+b^3+c^3=(a+b+c)^3$, then will

$$a^{2n+1}+b^{2n+1}+c^{2n+1}=(a+b+c)^{2n+1},$$

where n is any positive integer.

Since $(a+b+c)^3=a^3+b^3+c^3+3(b+c)(c+a)(a+b)$, the given relation shews that $(b+c)(c+a)(a+b)=0$.

Hence either $b+c=0$, or $c+a=0$ or $a+b=0$.

If $b+c=0$; then $b^{2n+1}=(-c)^{2n+1}=-c^{2n+1}$, and therefore $b^{2n+1}+c^{2n+1}=0$.

Thus if $b+c=0$, $a^{2n+1}+b^{2n+1}+c^{2n+1}-(a+b+c)^{2n+1}$ becomes $a^{2n+1}+b^{2n+1}+c^{2n+1}-a^{2n+1}=b^{2n+1}+c^{2n+1}=0$.

Hence $a^{2n+1}+b^{2n+1}+c^{2n+1}=(a+b+c)^{2n+1}$ if $b+c=0$; and so also if $c+a=0$, or if $a+b=0$. This proves the proposition, since $(b+c)(c+a)(a+b)=0$.

Ex. 2. If x, y, z be unequal, and if

$$y^3+z^3+m(y^2+z^2)=z^3+x^3+m(z^2+x^2)=x^3+y^3+m(x^2+y^2),$$

prove that each equals $2xyz$, and that $x+y+z+m=0$.

Since $y^3+z^3+m(y^2+z^2)=z^3+x^3+m(z^2+x^2)$, we have

$$y^3-x^3+m(y^2-x^2)=0,$$

that is $$(y-x)\{y^2+xy+x^2+m(x+y)\}=0.$$

Therefore, $y-x$ not being equal to zero, we have

$$y^2+xy+x^2+m(x+y)=0 \qquad\text{(i)}.$$

So also, since $y \neq z$,

$$z^2+yz+y^2+m(z+y)=0 \qquad\text{(ii)}.$$

From (i) and (ii) we have by subtraction

$$x^2-z^2+y(x-z)+m(x-z)=0.$$

Hence, as $x \neq z$, we have

$$x+y+z+m=0 \qquad\text{(iii)}.$$

Substitute $-(x+y+z)$ for m in (i); and we have

$$x^2+xy+y^2-(x+y)(x+y+z)=0;$$

$$\therefore\ yz+zx+xy=0 \qquad\text{(iv)}.$$

Then $y^3+z^3+m(y^2+z^2)=y^3+z^3-(y^2+z^2)(x+y+z)$ from (iii)

$$=-(y^2x+z^2y+y^2z+z^2x)$$

$$=-y(xy+yz)-z(yz+zx)$$

$$=2xyz \qquad \text{from (iv)}.$$

Ex. 3. Shew that, if $a+b+c+d=0$, then will

$$a^4+b^4+c^4+d^4=2(ab-cd)^2+2(ac-bd)^2+2(ad-bc)^2+4abcd.$$

We have to prove that

$$\Sigma a^4=2\Sigma a^2b^2-8abcd.$$

Since $a+b+c+d=0$; we have, by squaring and transposing,

$$a^2+b^2+c^2+d^2=-2(bc+ca+ab+ad+bd+cd).$$

Hence by squaring

$$\Sigma a^4+2\Sigma a^2b^2=4(\Sigma bc)^2.$$

Now $(\Sigma bc)^2=\Sigma b^2c^2+6abcd+2bcd(b+c+d)+2cda(c+d+a)$
$+2dab(d+a+b)+2abc(a+b+c)=\Sigma b^2c^2+6abcd-8abcd.$

Hence $\qquad \Sigma a^4+2\Sigma a^2b^2=4\Sigma a^2b^2-8abcd;$

$$\therefore\ \Sigma a^4=2\Sigma a^2b^2-8abcd.$$

Ex. 4. Prove that, if $ax+by+cz=0$, and $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$, then will

$$ax^3+by^3+cz^3=-(a+b+c)(y+z)(z+x)(x+y).$$

From the given relations we have, as in Art. 143,

$$\frac{a}{\frac{y}{z}-\frac{z}{y}}=\frac{b}{\frac{z}{x}-\frac{x}{z}}=\frac{c}{\frac{x}{y}-\frac{y}{x}}.$$

Hence [Art. 113] each fraction is equal to

$$\frac{ax^3+by^3+cz^3}{x^3\left(\frac{y}{z}-\frac{z}{y}\right)+y^3\left(\frac{z}{x}-\frac{x}{z}\right)+z^3\left(\frac{x}{y}-\frac{y}{x}\right)}=\frac{a+b+c}{\frac{y}{z}-\frac{z}{y}+\frac{z}{x}-\frac{x}{z}+\frac{x}{y}-\frac{y}{x}}.$$

Hence

$$\frac{ax^3+by^3+cz^3}{a+b+c}=\frac{x^4(y^2-z^2)+y^4(z^2-x^2)+z^4(x^2-y^2)}{x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)}$$
$$=\frac{-(y^2-z^2)(z^2-x^2)(x^2-y^2)}{(y-z)(z-x)(x-y)}$$
$$=-(y+z)(z+x)(x+y).$$

EXAMPLES XVI.

1. Shew that, if $\frac{x}{y+z}=a$, $\frac{y}{z+x}=b$ and $\frac{z}{x+y}=c$; then will $\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1.$

2. Shew that, if $ax+by=0$ and $cx^2+dxy+ey^2=0$, then will $a^2e+b^2c=abd$.

3. Eliminate x, y, z from the equations

$$\frac{y-z}{y+z}=a,\ \frac{z-x}{z+x}=b,\ \frac{x-y}{x+y}=c.$$

4. Eliminate x, y, z from the equations

$$\frac{y}{x}+\frac{x}{z}=a,\ \frac{z}{y}+\frac{y}{x}=b,\ \frac{x}{z}+\frac{z}{y}=c.$$

5. If $x+\frac{1}{y}=1$ and $y+\frac{1}{z}=1$; prove that $z+\frac{1}{x}=1$.

6. Eliminate x from the equations

$$a+c=\frac{b}{x}-dx,$$

$$a-c=\frac{d}{x}-bx.$$

7. Eliminate x, y, z from the equations

$$x^2-yz=a,\ y^2-zx=b,\ z^2-xy=c,\ ax+by+cz=d.$$

8. Prove that the equations

$$x+y+z=a,$$

$$x^2+y^2+z^2=b^2,$$

$$x^3+y^3+z^3-3xyz=c^3,$$

do not give any roots, but simply a relation between a, b and c.

9. Shew that, if

$bz+cy=cx+az=ay+bx$, and $x^2+y^2+z^2-2yz-2zx-2xy=0$;

then will $$a\pm b\pm c=0.$$

10. Shew that, if $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$; then will $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

11. If $x+\frac{1}{y}=y+\frac{1}{z}=z+\frac{1}{x}$; then $x^2y^2z^2=1$, or $x=y=z$.

12. Shew that, if $x = cy + bz$, $y = az + cx$ and $z = bx + ay$; then

$$\frac{x^2}{1-a^2} = \frac{y^2}{1-b^2} = \frac{z^2}{1-c^2}.$$

13. Shew that, if $x^2 = y^2 + z^2 + 2ayz$, $y^2 = z^2 + x^2 + 2bzx$ and $z^2 = x^2 + y^2 + 2cxy$; then

$$\frac{x^2}{1-a^2} = \frac{y^2}{1-b^2} = \frac{z^2}{1-c^2}.$$

14. Shew that, if x, y, z be unequal, and

$$y = \frac{a+bz}{c+dz},\quad z = \frac{a+bx}{c+dx} \text{ and } x = \frac{a+by}{c+dy},$$

then will $ad + bc + b^2 + c^2 = 0$.

15. Eliminate x, y, z from the equations

$$\frac{x^2}{yz} + \frac{yz}{x^2} = l,\quad \frac{y^2}{zx} + \frac{zx}{y^2} = m,\quad \frac{z^2}{xy} + \frac{xy}{z^2} = n.$$

16. Eliminate x, y, z from the equations $bx^2 + lx + c = 0$, $cy^2 + my + a = 0$, $az^2 + nz + b = 0$, $xyz = 1$.

17. Eliminate x, y, z from the equations

$$y^2 + z^2 = ayz,\quad z^2 + x^2 = bzx,\quad x^2 + y^2 = cxy,$$

xyz not being zero.

18. Eliminate (i) x, y, z and (ii) a, b, c from the equations

$$b\frac{y}{z} + c\frac{z}{y} = a,\quad c\frac{z}{x} + a\frac{x}{z} = b, \text{ and } a\frac{x}{y} + b\frac{y}{x} = c.$$

19. Eliminate x, y, z from the equations

$$ax + yz = bc,\quad by + zx = ca,\quad cz + xy = ab, \text{ and } xyz = abc.$$

20. Eliminate x, y, z from the equations

$$\frac{x^2 - xy - xz}{a} = \frac{y^2 - yz - yx}{b} = \frac{z^2 - zx - zy}{c},$$

and $ax + by + cz = 0$.

21. From the equations $a^2yz = \alpha^2(y+z)^2$, $b^2zx = \beta^2(z+x)^2$, $c^2xy = \gamma^2(x+y)^2$, deduce the relation

$$\pm\frac{abc}{\alpha\beta\gamma} = \frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2} - 4.$$

22. Prove that, if

$$y^2 + z^2 + yz = a^2, \quad z^2 + zx + x^2 = b^2, \quad x^2 + xy + y^2 = c^2,$$

and $yz + zx + xy = 0$; then will $a \pm b \pm c = 0$.

23. Prove that, if $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = \dfrac{1}{(a+b+c)}$, then will

$$\frac{1}{a^{2n+1}} + \frac{1}{b^{2n+1}} + \frac{1}{c^{2n+1}} = \frac{1}{(a+b+c)^{2n+1}},$$

where n is any positive integer.

24. Shew that, if

$$\frac{b^2+c^2-a^2}{2bc} + \frac{c^2+a^2-b^2}{2ca} + \frac{a^2+b^2-c^2}{2ab} = 1,$$

then $$(b+c-a)(c+a-b)(a+b-c) = 0,$$

and

$$\left(\frac{b^2+c^2-a^2}{2bc}\right)^{2n+1} + \left(\frac{c^2+a^2-b^2}{2ca}\right)^{2n+1} + \left(\frac{a^2+b^2-c^2}{2ab}\right)^{2n+1} = 1.$$

25. If $$a^2x^2 + b^2y^2 + c^2z^2 = 0,$$
$$a^2x^3 + b^2y^3 + c^2z^3 = 0,$$

and $$\frac{1}{x} - a^2 = \frac{1}{y} - b^2 = \frac{1}{z} - c^2;$$

prove that $$a^4x^3 + b^4y^3 + c^4z^3 = 0,$$

and $$a^6x^3 + b^6y^3 + c^6z^3 = a^4x^2 + b^4y^2 + c^4z^2.$$

26. If $x - \dfrac{ayz}{x^2} = y - \dfrac{azx}{y^2} = z - \dfrac{axy}{z^2}$, and x, y, z be unequal; then each member of the equations is equal to $x + y + z - a$.

27. If x, y, z be unequal, and if $2a - 3y = \dfrac{(z-x)^2}{y}$ and $2a - 3z = \dfrac{(x-y)^2}{z}$, then will $2a - 3x = \dfrac{(y-z)^2}{x}$, and

$$x + y + z = a.$$

28. If $x + \dfrac{yz - x^2}{x^2 + y^2 + z^2}$ be not altered in value by interchanging x and y, it will not be altered by interchanging x and z, and it will vanish if $x + y + z = 1$, the letters being all unequal.

29. If x, y, z be unequal, and
$y^3 + z^3 + m(y+z) = z^3 + x^3 + m(z+x) = x^3 + y^3 + m(x+y)$,
then each will equal $2xyz$.

30. If x, y, z be unequal, and

$$y^2 + z^2 + myz = z^2 + x^2 + mzx = x^2 + y^2 + mxy,$$

then each will equal $\frac{1}{2}(x^2 + y^2 + z^2)$.

31. If x, y be unequal, and if $\dfrac{(2x - y - z)^3}{x} = \dfrac{(2y - z - x)^3}{y}$, then will each equal $\dfrac{(2z - x - y)^3}{z}$.

32. Shew that, if a, b, c, d be all real quantities not zero, and $(a^2 + b^2)(c^2 + d^2) = 4abcd$: then will $a = \pm b$ and $c = \pm d$.

33. If a, b, c, x be all real quantities, and

$$(a^2 + b^2)x^2 - 2b(a+c)x + b^2 + c^2 = 0;$$

then

$$\frac{c}{b} = \frac{b}{a} = x.$$

34. Shew that, if

$$(x^2 + y^2 + z^2)(a^2 + b^2 + c^2) = (ax + by + cz)^2,$$

then

$$x/a = y/b = z/c.$$

35. Prove the following:

(i) If $2(a^2 + b^2) = (a+b)^2$, then $a = b$.

(ii) If $3(a^2 + b^2 + c^2) = (a+b+c)^2$, then $a = b = c$.

(iii) If $4(a^2 + b^2 + c^2 + d^2) = (a+b+c+d)^2$, then

$$a = b = c = d.$$

and

(iv) If $n(a^2 + b^2 + c^2 + \ldots\ldots) = (a + b + c + \ldots\ldots)^2$, then $a = b = c = \ldots\ldots$, n being the number of the letters.

36. Prove that, if a, b, c, d be all real and positive, and

$$a^4 + b^4 + c^4 + d^4 = 4abcd;$$

then will

$$a = b = c = d.$$

37. If

$$(n-1)x^2 + 2x(a_1 - a_n) + a_1^2 + 2a_2^2 + 2a_3^2 + \ldots + 2a_{n-1}^2 + a_n^2$$
$$= 2(a_1a_2 + a_2a_3 + \ldots\ldots + a_{n-1}a_n)$$

for real values of x, a_1, a_2, ..., a_n; then will

$$a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = x.$$

Verify the following identities:

38. $a^3(b+c) + b^3(c+a) + c^3(a+b) + abc(a+b+c)$
$$= (a^2+b^2+c^2)(bc+ca+ab).$$

39. $(b+c-a-d)^4(b-c)(a-d) + (c+a-b-d)^4(c-a)(b-d)$
$+ (a+b-c-d)^4(a-b)(c-d)$
$$\equiv 16(b-c)(c-a)(a-b)(d-a)(d-b)(d-c).$$

40. $8(a+b+c)^3 - (b+c)^3 - (c+a)^3 - (a+b)^3$
$$\equiv 3(2a+b+c)(a+2b+c)(a+b+2c).$$

41. $(a+b+c+d)^5 - (b+c+d)^5 - (c+d+a)^5 - (d+a+b)^5$
$- (a+b+c)^5 + (b+c)^5 + (c+a)^5 + (a+b)^5 + (a+d)^5$
$+ (b+d)^5 + (c+d)^5 - a^5 - b^5 - c^5 - d^5 \equiv 60abcd(a+b+c+d).$

42. $(a+b+c)^3abc - (bc+ca+ab)^3 \equiv abc(a^3+b^3+c^3)$
$$- (b^3c^3 + c^3a^3 + a^3b^3).$$

43. $(a^2+b^2+c^2)^3 + 2(bc+ca+ab)^3$
$$- 3(a^2+b^2+c^2)(bc+ca+ab)^2 \equiv (a^3+b^3+c^3-3abc)^2.$$

44. $(ca-b^2)(ab-c^2) + (ab-c^2)(bc-a^2) + (bc-a^2)(ca-b^2)$
$$\equiv (bc+ca+ab)(bc+ca+ab-a^2-b^2-c^2).$$

45. $2(c^2+ca+a^2)(a^2+ab+b^2)-(b^2+bc+c^2)^2$
$+2(a^2+ab+b^2)(b^2+bc+c^2)-(c^2+ca+a^2)^2$
$+2(b^2+bc+c^2)(c^2+ca+a^2)-(a^2+ab+b^2)^2$
$\equiv 3(bc+ca+ab)^2.$

46. Shew that

$$(3a-b-c)^3+(3b-c-a)^3+(3c-a-b)^3$$
$$-3(3a-b-c)(3b-c-a)(3c-a-b)\equiv 16(a^3+b^3+c^3-3abc).$$

47. Shew that

$$(na-b-c)^3+(nb-c-a)^3+(nc-a-b)^3$$
$$-3(na-b-c)(nb-c-a)(nc-a-b)$$
$$\equiv (n+1)^2(n-2)(a^3+b^3+c^3-3abc).$$

48. Shew that

$$(x^2+2yz)^3+(y^2+2zx)^3+(z^2+2xy)^3$$
$$-3(x^2+2yz)(y^2+2zx)(z^2+2xy)\equiv (x^3+y^3+z^3-3xyz)^2.$$

49. Shew that

$$(by+az)^3+(bz+ax)^3+(bx+ay)^3-3(by+az)(bz+ax)(bx+ay)$$
$$\equiv (a^3+b^3)(x^3+y^3+z^3-3xyz).$$

50. Shew that, if $1+\omega+\omega^2=0$, then

$$[(b-c)(x-a)+\omega(c-a)(x-b)+\omega^2(a-b)(x-c)]^3$$
$$+[(b-c)(x-a)+\omega^2(c-a)(x-b)+\omega(a-b)(x-c)]^3$$
$$\equiv 27(b-c)(c-a)(a-b)(x-a)(x-b)(x-c).$$

51. Shew that the product of any number of factors, each of which is the sum of two squares, can be expressed as the sum of two squares.

52. Verify the identity

$$(a^2+b^2+c^2+d^2)(p^2+q^2+r^2+s^2)\equiv (ap+bq+cr+ds)^2$$
$$+(aq-bp+cs-dr)^2+(ar-bs-cp+dq)^2$$
$$+(as+br-cq-dp)^2.$$

Hence shew that the product of any number of factors, each of which is the sum of four squares, can be expressed as the sum of four squares.

53. Shew that $(x^2+xy+y^2)(a^2+ab+b^2)$ can be expressed in the form X^2+XY+Y^2.

54. Shew that $(x^2+pxy+qy^2)(a^2+pab+qb^2)$ can be expressed in the form $X^2+pXY+qY^2$.

55. Shew that, if $2s \equiv a+b+c$,

(i) $$a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b) + 2(s-a)(s-b)(s-c) = abc.$$

(ii) $$(s-a)^3+(s-b)^3+(s-c)^3+3abc = s^3.$$

(iii) $$\begin{aligned}&(b+c)s(s-a)+a(s-b)(s-c)-2sbc\\ &=(c+a)s(s-b)+b(s-c)(s-a)-2sca\\ &=(a+b)s(s-c)+c(s-a)(s-b)-2sab.\end{aligned}$$

(iv) $$a(b-c)(s-a)^2+b(c-a)(s-b)^2 + c(a-b)(s-c)^2 = 0.$$

(v) $$s(s-b)(s-c)+s(s-c)(s-a)+s(s-a)(s-b) - (s-a)(s-b)(s-c) = abc.$$

(vi) $$\begin{aligned}&(s-a)^2(s-b)^2(s-c)^2+s^2(s-b)^2(s-c)^2\\ &+s^2(s-c)^2(s-a)^2+s^2(s-a)^2(s-b)^2\\ &+s(s-a)(s-b)(s-c)(a^2+b^2+c^2) = a^2b^2c^2.\end{aligned}$$

56. Shew that, if $2s = a+b+c+d$,

$$4(bc+ad)^2-(b^2+c^2-a^2-d^2)^2 = 16(s-a)(s-b)(s-c)(s-d).$$

Shew also that

$$\begin{aligned}&a(s-b)(s-c)(s-d)+b(s-c)(s-d)(s-a)+c(s-d)(s-a)(s-b)\\ &+d(s-a)(s-b)(s-c)+2(s-a)(s-b)(s-c)(s-d)\\ &-s(bcd+cda+dab+abc) = -2abcd.\end{aligned}$$

57. Shew that, if $a+b+c+d=0$, then

$$\begin{aligned}&ad(a+d)^2+bc(a-d)^2+ab(a+b)^2+cd(a-b)^2\\ &+ac(a+c)^2+bd(a-c)^2+4abcd = 0.\end{aligned}$$

58. Shew that, if

$$(a+b)(b+c)(c+d)(d+a) = (a+b+c+d)(bcd+cda+dab+abc);$$

then $ac = bd$.

59. Shew that, if $a+b+c=0$ and $x+y+z=0$, then

$$4(ax+by+cz)^3 - 3(ax+by+cz)(a^2+b^2+c^2)(x^2+y^2+z^2) - 2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y) = 54abcxyz.$$

60. Shew that, if $a+b+c=0$; then

(i) $2(a^7+b^7+c^7) = 7abc(a^4+b^4+c^4)$.

(ii) $6(a^7+b^7+c^7) = 7(a^3+b^3+c^3)(a^4+b^4+c^4)$.

(iii) $a^6+b^6+c^6 = 3a^2b^2c^2 + \frac{1}{4}(a^2+b^2+c^2)^3$.

(iv) $25(a^7+b^7+c^7)(a^3+b^3+c^3) = 21(a^5+b^5+c^5)^2$.

61. If $a+b+c+d=0$, prove that

$$(a^3+b^3+c^3+d^3)^2 = 9(bcd+cda+dab+abc)^2 = 9(bc-ad)(ca-bd)(ab-cd).$$

62. Shew that, if $a+b+c=0$, then

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right) = 9.$$

63. Prove that, if

$$\frac{1}{1+l+ln}+\frac{m}{1+m+ml}+\frac{nm}{1+n+nm}=1,$$

and
$$\frac{l}{1+l+ln}+\frac{ml}{1+m+ml}+\frac{1}{1+n+nm}=1,$$

and none of the denominators be zero, then will $l=m=n$.

64. Shew that

$$a+(1-a)b+(1-a)(1-b)c+(1-a)(1-b)(1-c)d + \ldots \equiv 1-(1-a)(1-b)(1-c)(1-d) \ldots$$

65. Shew that

$$\frac{1}{a} = 1 + 2(1-a) + 3(1-a)(1-2a) + \ldots$$
$$+ \{n(1-a)(1-2a)\ldots(1-\overline{n-1}a)\}$$
$$+ \frac{1}{a}\{(1-a)(1-2a)\ldots(1-na)\}.$$

66. Shew that

$$a^n + a^{n-1}(1-a^n) + a^{n-2}(1-a^n)(1-a^{n-1}) + \ldots$$
$$+ \{a(1-a^n)(1-a^{n-1})\ldots(1-a^2)\} + \{(1-a^n)(1-a^{n-1})\ldots(1-a)\}$$
$$= 1.$$

67. Shew that, if n be any positive integer,

$$\frac{1-a^n}{1-a} + \frac{(1-a^n)(1-a^{n-1})}{1-a^2} + \frac{(1-a^n)(1-a^{n-1})(1-a^{n-2})}{1-a^3} + \ldots$$
$$+ \frac{(1-a^n)(1-a^{n-1})\ldots(1-a)}{1-a^n} = n.$$

68. Prove that, if

$$a + b + c + d = 0,$$
$$x + y + z + u = 0,$$

and $$ax + by + cz + du = 0;$$

then $$2(a^4x + b^4y + c^4z + d^4u)$$
$$= (a^2x + b^2y + c^2z + d^2u)(a^2 + b^2 + c^2 + d^2).$$

69. Prove that, if n be any positive integer,

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots\ldots - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{2n}.$$

70. Prove that, if

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{u+a} + \frac{1}{v-b} = \frac{1}{u+a'} + \frac{1}{v-b'} = \frac{1}{f};$$

then $$f^2(ab' - a'b)^2 = aa'bb'(a-a')(b-b').$$

CHAPTER XIII.

POWERS AND ROOTS. FRACTIONAL AND NEGATIVE INDICES.

159. THE process by which the powers of quantities are obtained is often called *involution*; and the inverse process, namely that by which the roots of quantities are obtained, is called *evolution*.

We proceed to consider some cases of involution and of evolution.

160. **Index Laws.** We have proved in Art. 31, that when m and n are any positive integers,

$$a^m \times a^n = a^{m+n} \dots\dots\dots\dots\dots\dots\dots\dots(\text{i}).$$

This result is called the Index Law.

From the Index Law, we have

$$a^m \times a^n \times a^p = a^{m+n} \times a^p = a^{m+n+p},$$

and so on, however many factors there may be.

Hence $\qquad a^m \times a^n \times a^p \times \dots = a^{m+n+p+\dots} \quad \dots\dots\dots\dots(\text{ii}).$

Thus *the index of the product of any number of powers of the same quantity is the sum of the indices of the factors.*

Also, $\qquad a^m \times a^m \times a^m \times \dots$ to n factors

$$= a^{m+m+m+\dots\dots \text{to } n \text{ terms}}$$

$$= a^{mn}.$$

Hence $$(a^m)^n = a^{mn} \qquad \text{......................(iii)}.$$

Thus, *to raise any power of a quantity to any other power, its original index must be multiplied by the index of the power to which it is to be raised.*

Again, to find $(ab)^m$.

$(ab)^m = ab \times ab \times ab \times \text{......}$ to m factors, by definition,

$= (a \times a \times a \text{......}$ to m factors$) \times (b \times b \times b \text{......}$ to m factors$)$, by the Commutative Law,

$= a^m \times b^m$, by definition.

Hence $$(ab)^m = a^m \times b^m.$$

Similarly $$(abc\ldots)^m = a^m \times b^m \times c^m \times \ldots \qquad \ldots\text{(iv)}.$$

Thus, *the mth power of a product is the product of the mth powers of its factors.*

The most general case of a monomial expression is $a^x b^y c^z\text{......}$

Now $$(a^x b^y c^z\text{......})^m = (a^x)^m (b^y)^m (c^z)^m \text{......from (iv)}$$
$$= a^{xm} b^{ym} c^{zm} \text{ ....from (iii)}.$$

Hence $$(a^x b^y c^z\text{......})^m = a^{xm} b^{ym} c^{zm}\ldots \qquad \ldots\text{(v)}.$$

Thus *any power of an expression is obtained by taking each of its factors to a power whose index is the product of its original index and the index of the power to which the whole expression is to be raised.*

As a particular case

$$\left(\frac{a}{b}\right)^m = \left(a \times \frac{1}{b}\right)^m = a^m \times \left(\frac{1}{b}\right)^m = a^m \times \frac{1}{b^m} = \frac{a^m}{b^m}.$$

161. It follows from the Law of Signs that all powers of a positive quantity are positive, but that successive powers of a negative quantity are alternately positive and negative. For we have

$$(-a)^2 = (-a)\ (-a) = +a^2,$$
$$(-a)^3 = (-a)^2\ (-a) = (+a^2)\ (-a) = -a^3,$$
$$(-a)^4 = (-a)^3\ (-a) = (-a^3)\ (-a) = +a^4,$$

and so on.

Thus $(-a)^{2n} = +a^{2n}$, and $(-a)^{2n+1} = -a^{2n+1}$.

Hence *all even powers, whether of positive or of negative quantities, are positive; and all odd powers of any quantity have the same sign as the original quantity.*

162. **Roots of Arithmetical numbers.** The approximate value of the square or of any other root of an arithmetical number can always be found: this we proceed to prove. It will be seen that the process described would be an extremely laborious one; we are not however here concerned with the actual calculation of surds.

Consider, for example, $\sqrt[2]{62}$. First write down the squares of the numbers 1, 2, 3, &c. until one is found which is greater than 62: it will then be seen that 7^2 is less and 8^2 is greater than 62. Now write down the squares of the numbers 7·1, 7·2, 7·3, ..., 7·9: it will then be seen that $(7{\cdot}8)^2$ is less, and $(7{\cdot}9)^2$ greater than 62. Now write down the squares of 7·81, 7·82, ..., 7·89: it will then be seen that $(7{\cdot}83)^2$ is less, and $(7{\cdot}84)^2$ greater than 62.

By continuing this process, we get at every stage two numbers such that 62 is intermediate between their squares, and such that their difference becomes smaller and smaller at every successive stage; moreover, this difference can, by sufficiently continuing the process, be made *less than any assigned quantity however small.*

Thus, although we can never find any number whose square is *exactly* equal to 62, we can find two numbers whose squares are the one greater and the other less than 62, and whose difference is less than any assigned quantity however small. The limiting value of these two numbers,

when the process is continued indefinitely, is called *the square root* of 62.

The process above described for finding a *square* root can clearly be applied to find any other root.

Thus an nth root of any integral or fractional number can always be found.

163. Surds obey the Fundamental Laws of Algebra. The fundamental laws of Algebra were proved for integral or fractional values of the letters; and it can be proved that they are also true for surds.

Consider, for example, the Commutative Law.

We have to prove that

$$\sqrt[n]{a} \times \sqrt[m]{b} = \sqrt[m]{b} \times \sqrt[n]{a}.$$

We can find whole numbers or fractions x, y and p, q such that

$$x > \sqrt[n]{a} > y,$$

and $$p > \sqrt[m]{b} > q;$$

and the difference between x and y, and also the difference between p and q, can be made less than any assigned quantity however small.

Hence $$x \times p > \sqrt[n]{a} \times \sqrt[m]{b} > y \times q,$$

and $$p \times x > \sqrt[m]{b} \times \sqrt[n]{a} > q \times y.$$

But, since x, y, p, q are integral or fractional numbers, we know that $x \times p = p \times x$, and $y \times q = q \times y$; also the difference between px and qy can be made less than any assigned quantity however small.

It therefore follows that $\sqrt[n]{a} \times \sqrt[m]{b}$ and $\sqrt[m]{b} \times \sqrt[n]{a}$, which are both always intermediate to xp and yq, must be equal.

Thus the Commutative Law holds for Surds, and the other laws can be proved in a similar manner.

164. We already know that there are *two* square roots, and *three* cube roots of every quantity; and we may remark that there are always n nth roots. Thus there is

an important difference between powers and roots; for there is only one nth power, but there is more than one nth root.

165. We have proved in Art. 160 that the mth power of a product is the product of the mth powers of its factors; and, since surds obey the fundamental laws of Algebra, the proposition holds good when all or any of the factors are irrational. Hence

$$(\sqrt{a} \times \sqrt{b} \ldots)^2 = (\sqrt{a})^2 \times (\sqrt{b})^2 \ldots = ab \ldots.$$

Also $\quad (\sqrt{ab\ldots})^2 = ab\ldots$, by definition.

$$\therefore (\sqrt{a} \times \sqrt{b} \ldots)^2 = (\sqrt{ab\ldots})^2.$$

Hence $\sqrt{a} \times \sqrt{b} \ldots$ must be equal to *one* of the square roots of $ab \ldots$.

We can write this

$$\sqrt{a}\sqrt{b} \ldots = \sqrt{ab\ldots},$$

meaning thereby that the continued product of either of the square roots of a, either of the square roots of b, &c. is equal to *one or other* of the square roots of $ab \ldots$

Similarly we have, *with a corresponding limitation,*

$$\sqrt[n]{a}\,\sqrt[n]{b} \ldots = \sqrt[n]{ab\ldots}, \text{ and } \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

Also $\sqrt[n]{a^m} = \sqrt[np]{a^{mp}}$, for their npth powers are both equal to a^{mp}.

Again, since the nth *power* of a monomial expression is obtained by *multiplying* the index of each of its factors by n, it follows conversely that *an* nth *root* of a monomial expression is obtained by *dividing* the index of each of its factors by n, provided the division can be performed.

Thus one value of $\sqrt[2]{a^4}$ is a^2, one value of $\sqrt[3]{a^6 b^9 c^3}$ is $a^2 b^3 c$, and one value of $\sqrt[n]{a^{n\alpha} b^{n\beta} c^{n\gamma}}$ is $a^\alpha b^\beta c^\gamma$.

Fractional and Negative Indices.

166. We have hitherto supposed that an *index* was always a positive integer; and this is necessarily the case so long as we retain the definition of Art. 9; for, with that definition, such expressions as $a^{\frac{1}{2}}$ and a^{-2} have no meaning whatever.

We might extend the meaning of an index by assigning meanings to a^n when n is fractional and negative. It is, however, essential that algebraical symbols should always obey the same laws whatever their values may be; we therefore do not begin by assigning any meaning to a^n when n is not a positive integer, but we first impose the restriction that *the meaning of a^n must in all cases be such that the fundamental index law,* namely

$$a^m \times a^n = a^{m+n},$$

shall always be true; and it will be found that the above restriction is of itself sufficient to define the meaning of a^n in all cases, so that there is no further freedom of choice.

For example, to find the meaning of $a^{\frac{1}{2}}$.

Since the meaning is to be consistent with the Index Law, we must have

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$$

Thus $a^{\frac{1}{2}}$ must be such that its square is a, that is $a^{\frac{1}{2}}$ must be $\sqrt{a}$.

Again, to find the meaning of a^{-1}.

By the index law

$$a^{-1} \times a^3 = a^{-1+3} = a^2; \text{ therefore } a^{-1} = \frac{a^2}{a^3} = \frac{1}{a}.$$

Thus a^{-1} must be $\dfrac{1}{a}$.

167. We now proceed to consider the most general cases.

I. To find the meaning of $a^{\frac{1}{n}}$, where n is any positive integer.

By the index law,

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots\ldots \text{ to } n \text{ factors}$$

$$= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\ldots\ldots \text{ to } n \text{ terms}} = a^{\frac{n}{n}} = a^1 = a.$$

Hence $a^{\frac{1}{n}}$ must be such that its nth power is a, that is $a^{\frac{1}{n}} = \sqrt[n]{a}$.

II. To find the meaning of $a^{\frac{m}{n}}$, where m and n are any positive integers.

By the index law,

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \ldots\ldots \text{ to } n \text{ factors} = a^{\frac{m}{n}+\frac{m}{n}+\ldots\ldots \text{ to } n \text{ terms}} = a^{\frac{m}{n} \times n} = a^m.$$

Hence $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

We have also

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots\ldots \text{ to } m \text{ factors} = a^{\frac{1}{n}+\frac{1}{n}+\ldots\ldots \text{ to } m \text{ terms}} = a^{\frac{m}{n}}.$$

$$\text{Hence } a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m.$$

Thus we may consider that $a^{\frac{m}{n}}$ is an nth root of the mth power of a, or that it is the mth power of an nth root of a; which we express by

$$a^{\frac{m}{n}} = \sqrt[n]{(a^m)} = (\sqrt[n]{a})^m.$$

With the above meaning of $a^{\frac{m}{n}}$ it follows from Art. 165 that $a^{\frac{m}{n}} = a^{\frac{mp}{np}}$.

NOTE. It should be remarked that it is not strictly true that $\sqrt[n]{(a^m)} = (\sqrt[n]{a})^m$ except with a limitation corresponding to that of Art. 165, or unless by the nth root of a quantity is meant only the *arithmetical* root. For example, $\sqrt[2]{(a^4)}$ has *two* values, namely $\pm a^2$, whereas $(\sqrt[2]{a})^4$ has only the value $+a^2$.

III. To find the meaning of a^0.

By the index law

$$a^0 \times a^m = a^{0+m} = a^m; \quad \therefore \ a^0 = a^m \div a^m = 1.$$

Thus $a^0 = 1$, whatever a may be.

IV. To find the meaning of a^{-m}, where m has any positive value.

By the index law,

$$a^{-m} \times a^m = a^{-m+m} = a^0; \text{ and } a^0 = 1, \text{ by III.}$$

Hence $a^{-m} = \dfrac{1}{a^m}$, and $a^m = \dfrac{1}{a^{-m}}$.

168. We have in the preceding Article found that in order that the fundamental index law, $a^m \times a^n = a^{m+n}$, may always be obeyed, a^n must have a definite meaning when n has any given positive or negative value. We have now to shew that, *with the meanings thus obtained,*

$$a^m \times a^n = a^{m+n}, \quad (a^m)^n = a^{mn}, \quad \text{and} \quad (ab)^n = a^n b^n,$$

are true for *all values* of m and n. When these have been proved, the final result of Art. 160 is easily seen to be true in all cases.

I. To prove that $a^m \times a^n = a^{m+n}$, for all values of m and n.

We already know that this is true when m and n are positive integers. Let m and n be any positive fractions $\frac{p}{q}$ and $\frac{r}{s}$ respectively. Then

$$a^m \times a^n = a^{\frac{p}{q}} \times a^{\frac{r}{s}} = \sqrt[q]{a^p} \times \sqrt[s]{a^r}. \qquad \text{by definition}$$

$$= \sqrt[qs]{a^{ps}} \times \sqrt[qs]{a^{rq}} = \sqrt[qs]{a^{ps+rq}} \qquad \text{[Art. 165]}$$

$$= a^{\frac{ps+rq}{qs}}, \qquad \text{by definition}$$

$$= a^{\frac{p}{q}+\frac{r}{s}} = a^{m+n}.$$

Thus the proposition is true for all *positive* values of m and n. To shew that it is true also for negative values, it is necessary and sufficient to prove that

$$a^{-m} \times a^{-n} = a^{-m-n}, \text{ and } a^m \times a^{-n} = a^{m-n}$$

where m and n are positive.

Now $a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n} = \frac{1}{a^{m+n}} = a^{-m-n}$.

And, if $m - n$ be positive,

$$a^{m-n} \times a^n = a^m, \text{ and } a^m \times a^{-n} \times a^n = a^m;$$

therefore $\qquad a^{m-n} = a^m \times a^{-n}$.

Hence, if $m - n$ be negative, $\frac{1}{a^{-m}} \times \frac{1}{a^n} = \frac{1}{a^{n-m}}$,

that is, $\qquad a^m \times a^{-n} = a^{m-n}$.

Hence $a^m \times a^n = a^{m+n}$, for *all values* of m and n.

COR. Since $a^{m-n} \times a^n = a^m$ for all values of m and n, it follows that $a^m \div a^n = a^{m-n}$.

II. To prove that $(a^m)^n = a^{mn}$, for all values of m and n.

First, let n be a positive integer, m having any value whatever.

Then $(a^m)^n = a^m \times a^m \times a^m \times \ldots\ldots$ to n factors,

$$= a^{m+m+m+\ldots\ldots \text{ to } n \text{ terms}}, \qquad \text{by I.}$$

$$= a^{mn}.$$

Next, let n be a positive fraction $\dfrac{p}{q}$, where p and q are positive integers.

Then $(a^m)^n = (a^m)^{\frac{p}{q}} = \sqrt[q]{\{(a^m)^p\}}$, $= \sqrt[q]{(a^{mp})}$, since p is an integer,

$$= a^{\frac{mp}{q}} = a^{mn}.$$

Finally, let n be negative, and equal to $-p$.

Then $(a^m)^n = (a^m)^{-p} = \dfrac{1}{(a^m)^p} = \dfrac{1}{a^{mp}} = a^{-mp} = a^{mn}.$

Hence for all values of m and n we have

$$(a^m)^n = a^{mn}.$$

III. To prove that $(ab)^n = a^n b^n$, for all values of n.

We have proved in Art. 160 that $(ab)^n = a^n b^n$, where n is a positive integer.

And, whatever m may be, provided that q is a positive integer, we have

$$(a^m b^m)^q = a^m b^m \times a^m b^m \times \ldots \text{ to } q \text{ factors}$$

$$= a^{m+m+\ldots \text{ to } q \text{ terms}} \times b^{m+m+\ldots \text{ to } q \text{ terms}}$$

$$= a^{mq} b^{mq}.$$

Let n be a positive fraction $\dfrac{p}{q}$, where p and q are positive integers. Then

$(ab)^n=(ab)^{\frac{p}{q}}=\sqrt[q]{(ab)^p}=\sqrt[q]{(a^pb^p)}$, since p is a positive integer.

Also $(a^nb^n)^q=a^{nq}b^{nq}$, since q is a positive integer.

Hence $\qquad a^nb^n=\sqrt[q]{(a^pb^p)}=(ab)^n$.

Thus $(ab)^n=a^nb^n$, for all positive values of n.

Finally, if n be negative, and equal to $-m$, we have

$$(ab)^n=(ab)^{-m}=\frac{1}{(ab)^m}=\frac{1}{a^mb^m}=a^{-m}b^{-m}=a^nb^n.$$

Ex. (i) Simplify $a^{\frac{2}{3}}\times a^{-\frac{1}{2}}$.

$$a^{\frac{2}{3}}\cdot a^{-\frac{1}{2}}=a^{\frac{2}{3}-\frac{1}{2}}=a^{\frac{1}{6}}.$$

Ex. (ii). Simplify $a^{\frac{2}{3}}b^{\frac{4}{5}}\times a^{\frac{4}{3}}b^{\frac{6}{5}}$.

$$a^{\frac{2}{3}}b^{\frac{4}{5}}\times a^{\frac{4}{3}}b^{\frac{6}{5}}=a^{\frac{2}{3}+\frac{4}{3}}b^{\frac{4}{5}+\frac{6}{5}}=a^{\frac{6}{3}}b^{\frac{10}{5}}=a^2b^2.$$

Ex. (iii). Simplify $\left(a^{-2}b^{\frac{4}{3}}\right)^{-\frac{3}{2}}$.

$$\left(a^{-2}b^{\frac{4}{3}}\right)^{-\frac{3}{2}}=a^{(-2)\left(-\frac{3}{2}\right)}b^{\frac{4}{3}\left(-\frac{3}{2}\right)}=a^3b^{-2}=\frac{a^3}{b^2}.$$

Ex. (iv). Simplify $\sqrt{\left(a^{-\frac{5}{3}}b^3c^{-\frac{2}{3}}\right)}\div\sqrt[3]{\left(a^{\frac{1}{2}}b^4c^{-1}\right)}$.

$$\sqrt{\left(a^{-\frac{5}{3}}b^3c^{-\frac{2}{3}}\right)}\div\sqrt[3]{\left(a^{\frac{1}{2}}b^4c^{-1}\right)}=a^{-\frac{5}{6}}b^{\frac{3}{2}}c^{-\frac{1}{3}}\div a^{\frac{1}{6}}b^{\frac{4}{3}}c^{-\frac{1}{3}}$$
$$=a^{-\frac{5}{6}}b^{\frac{3}{2}}c^{-\frac{1}{3}}\times a^{-\frac{1}{6}}b^{-\frac{4}{3}}c^{\frac{1}{3}}=a^{-1}b^{\frac{1}{6}}.$$

169. **Rationalizing Factors.** It is sometimes required to find an expression which when multiplied by a given irrational expression will give a rational product. The following are examples of rationalizing factors.

Since $(a+\sqrt{b})(a-\sqrt{b})=a^2-b$, it follows that $a\pm\sqrt{b}$ is made rational by multiplying by $a\mp\sqrt{b}$.

So also $a\sqrt{b}\pm c\sqrt{d}$ is made rational by multiplying by $a\sqrt{b}\mp c\sqrt{d}$.

Again from the known identity

$$2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$$
$$=(a+b+c)(-a+b+c)(a-b+c)(a+b-c),$$

it follows that the rationalizing factor of

$\sqrt{p}+\sqrt{q}+\sqrt{r}$ is $(-\sqrt{p}+\sqrt{q}+\sqrt{r})(\sqrt{p}-\sqrt{q}+\sqrt{r})(\sqrt{p}+\sqrt{q}-\sqrt{r})$.

The rationalizing factor of $\sqrt{p}+\sqrt{q}+\sqrt{r}$ may also be found as follows,

$$(\sqrt{p}+\sqrt{q}+\sqrt{r})(\sqrt{p}+\sqrt{q}-\sqrt{r})=p+q-r+2\sqrt{pq},$$

and

$$(p+q-r+2\sqrt{pq})(p+q-r-2\sqrt{pq})=(p+q-r)^2-4pq.$$

Thus the required rationalizing factor is

$$(\sqrt{p}+\sqrt{q}-\sqrt{r})(p+q-r-2\sqrt{pq}),$$

which is the same as before.

Again, from the identity

$$(a+b)(a^2-ab+b^2)=a^3+b^3,$$

the rationalizing factor of $a+b^{\frac{1}{3}}$ is seen to be $a^2-ab^{\frac{1}{3}}+b^{\frac{2}{3}}$.

170. *To find the rationalizing factor of any binomial.*

Let the expression to be rationalized be $ax^{\frac{p}{q}} \pm by^{\frac{r}{s}}$.

Put $X=ax^{\frac{p}{q}}$, and $Y=by^{\frac{r}{s}}$, and let n be the L.C.M. of q and s.

Then it is easily seen that X^n and Y^n are both rational.

Hence, from the identities

$$(X+Y)\{X^{n-1}-X^{n-2}Y+\ldots+(-1)^{n-1}Y^{n-1}\}=X^n+(-1)^{n-1}Y^n$$

and $(X-Y)(X^{n-1}+X^{n-2}Y+\ldots\ldots+Y^{n-1})=X^n-Y^n$,

the rationalizing factors of $X+Y$ and $X-Y$ are seen to be respectively

$$X^{n-1}-X^{n-2}Y+\ldots\ldots+(-1)^{n-1}Y^{n-1},$$

$$\text{and } X^{n-1}+X^{n-2}Y+\ldots\ldots+Y^{n-1}.$$

Ex. To find a factor which will rationalize

$$x^{\frac{2}{3}} - ay^{\frac{5}{6}}.$$

Here $$X = x^{\frac{2}{3}},\ Y = ay^{\frac{5}{6}},\ n = 6.$$

The factor required is therefore

$$x^{\frac{10}{3}} + ax^{\frac{8}{3}}y^{\frac{5}{6}} + a^2x^2y^{\frac{5}{3}} + a^3x^{\frac{4}{3}}y^{\frac{5}{2}} + a^4x^{\frac{2}{3}}y^{\frac{10}{3}} + a^5y^{\frac{25}{6}}.$$

EXAMPLES XVII.

1. Simplify $a^{\frac{2}{3}}b^{\frac{5}{6}} \times a^{-\frac{1}{2}}b^{-\frac{2}{3}}$.

2. Simplify $a^{\frac{3}{2}} \times a^{-\frac{3}{4}} \times (a^2)^{-\frac{1}{6}} \times \dfrac{1}{(a^{-\frac{1}{12}})^5}$.

3. Simplify $(ab^{-2}c^3)^{\frac{1}{2}} \times (a^3b^2c^{-3})^{\frac{1}{3}}$.

4. Simplify $\left(x^{\frac{b+c}{c-a}}\right)^{\frac{1}{a-b}} \times \left(x^{\frac{c+a}{a-b}}\right)^{\frac{1}{b-c}} \times \left(x^{\frac{a+b}{b-c}}\right)^{\frac{1}{c-a}}$.

5. Multiply $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

6. Multiply $x^2 + 1 + x^{-2}$ by $x^2 - 1 + x^{-2}$.

7. Multiply

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} - z^{\frac{1}{3}}x^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} \text{ by } x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}}.$$

8. Divide $x^{\frac{4}{3}} - 2 + x^{-\frac{4}{3}}$ by $x^{\frac{2}{3}} - x^{-\frac{2}{3}}$.

9. Divide $a^{\frac{1}{2}} - x$ by $a^{\frac{1}{10}} - x^{\frac{1}{5}}$.

10. Divide $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

11. Shew that

$$x^{\frac{5}{3}} - 4x^{\frac{4}{3}} + 2x^{\frac{7}{6}} + 4x - 4x^{\frac{5}{6}} + x^{\frac{2}{3}} = (x^{\frac{5}{6}} - 2x^{\frac{1}{2}} + x^{\frac{1}{3}})^2.$$

12. Multiply $4x^3 - 5x - 4 - 7x^{-1} + 6x^{-2}$ by $3x - 4 + 2x^{-1}$ and divide the product by $3x - 10 + 10x^{-1} - 4x^{-2}$.

13. Divide

$$x - x^{-1} - 2\left(x^{\frac{1}{6}} - x^{-\frac{1}{6}}\right) + 2\left(x^{\frac{5}{6}} - x^{-\frac{5}{6}}\right) \text{ by } x^{\frac{1}{3}} - x^{-\frac{1}{3}}.$$

14. Simplify $\dfrac{ax^{-1} + a^{-1}x + 2}{a^{\frac{1}{3}}x^{-\frac{1}{3}} + a^{-\frac{1}{3}}x^{\frac{1}{3}} - 1}$.

15. Divide $\dfrac{x^{\frac{7}{3}}}{y^{\frac{14}{5}}} + \dfrac{y^{\frac{14}{5}}}{x^{\frac{7}{3}}}$ by $\dfrac{x^{\frac{1}{3}}}{y^{\frac{2}{5}}} + \dfrac{y^{\frac{2}{5}}}{x^{\frac{1}{3}}}$.

16. Shew that

$$\frac{x}{x^{\frac{1}{3}} - 1} - \frac{x^{\frac{2}{3}}}{x^{\frac{1}{3}} + 1} - \frac{1}{x^{\frac{1}{3}} - 1} + \frac{1}{x^{\frac{1}{3}} + 1} = x^{\frac{2}{3}} + 2.$$

17. Shew that

$$(2x + y^{-1})(2y + x^{-1}) = \left(2x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{-\frac{1}{2}}y^{-\frac{1}{2}}\right)^2.$$

18. Shew that

$$\frac{a^2 + b^2 - a^{-2} - b^{-2}}{a^2b^2 - a^{-2}b^{-2}} + \frac{(a - a^{-1})(b - b^{-1})}{ab + a^{-1}b^{-1}} = 1.$$

19. Shew that, if

$$x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0, \quad \text{then} \quad (x + y + z)^3 = 27xyz.$$

20. Find factors which will rationalize the following expressions:

(i) $a^{\frac{1}{2}} + b^{\frac{4}{3}}$, (ii) $a^{\frac{2}{3}}x^{\frac{5}{6}} + y^{\frac{1}{2}}$,

(iii) $a + bx^{\frac{1}{3}} + cx^{\frac{2}{3}}$, and (iv) $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}}$.

21. Shew that, if

$$(1 - x^3)^{\frac{1}{3}}(y - z) + (1 - y^3)^{\frac{1}{3}}(z - x) + (1 - z^3)^{\frac{1}{3}}(x - y) = 0,$$

and x, y, z are all unequal, then

$$(1 - x^3)(1 - y^3)(1 - z^3) = (1 - xyz)^3.$$

CHAPTER XIV.

SURDS. IMAGINARY AND COMPLEX QUANTITIES.

171. **Definitions.** A surd is a root of an arithmetical number which can only be found approximately.

An algebraical expression such as $\sqrt{a}$ is also often called a surd, although a *may* have such a value that $\sqrt{a}$ is not in reality a surd.

Surds are said to be of the same *order* when the same root is required to be taken. Thus $\sqrt{2}$ and $\sqrt{6}$ are called surds of the *second order*, or *quadratic* surds; also $\sqrt[3]{4}$ is a surd of the *third order*, or a *cubic* surd; and $\sqrt[n]{a}$ is a surd of the nth order.

Two surds are said to be *similar* when they can be reduced so as to have the same irrational factors. Thus $\sqrt{8}$ and $\sqrt{18}$ are similar surds, for they are equivalent to $2\sqrt{2}$ and $3\sqrt{2}$ respectively.

The rules for operations with surds follow at once from the principles established in the previous chapter.

Note. It should be remarked that when a root symbol is placed before an *arithmetical* number it denotes only the *arithmetical root,* but when the root symbol is placed before an algebraical expression it denotes *one* of the roots. Thus $\sqrt{a}$ has *two* values but $\sqrt{2}$ is only supposed to denote the arithmetical root, unless it is written $\pm\sqrt{2}$.

172. Any rational quantity can be written in the form of a surd. For example,

$$2=\sqrt[2]{4}=\sqrt[3]{8}=\sqrt[n]{2^n},$$

and
$$a=\sqrt[2]{a^2}=\sqrt[3]{a^3}=\sqrt[n]{a^n}.$$

Also, since $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$ [Art. 165],

we have
$$2\sqrt{2}=\sqrt{4}\times\sqrt{2}=\sqrt{(4\times 2)}=\sqrt{8},$$

$$5\sqrt[3]{3}=\sqrt[3]{5^3}\times\sqrt[3]{3}=\sqrt[3]{(5^3\times 3)}=\sqrt[3]{375},$$

and
$$a\sqrt[n]{ab}=\sqrt[n]{a^n}\times\sqrt[n]{ab}=\sqrt[n]{(a^n\times ab)}=\sqrt[n]{a^{n+1}b}.$$

Conversely, we have $\sqrt{18}=\sqrt{(9\times 2)}=\sqrt{9}\times\sqrt{2}=3\sqrt{2}$, and

$$\sqrt[3]{135}+\sqrt[3]{40}=\sqrt[3]{(3^3\times 5)}+\sqrt[3]{(2^3\times 5)}=3\sqrt[3]{5}+2\sqrt[3]{5}=5\sqrt[3]{5}.$$

173. Any two surds can be reduced to surds of the same order. For if the surds be $\sqrt[n]{a}$ and $\sqrt[m]{b}$, we have $\sqrt[n]{a}=\sqrt[nm]{a^m}$, and $\sqrt[m]{b}=\sqrt[nm]{b^n}$ [Art. 165].

Ex. Which is the greater, $\sqrt[3]{14}$ or $\sqrt[2]{6}$?

The surds must be reduced to equivalent surds of the same order. Now $\sqrt[3]{14}=\sqrt[6]{14^2}=\sqrt[6]{196}$, and $\sqrt[2]{6}=\sqrt[6]{6^3}=\sqrt[6]{216}$. Hence, as $\sqrt[6]{216}$ is greater than $\sqrt[6]{196}$, $\sqrt[2]{6}$ must be greater than $\sqrt[3]{14}$.

Thus we can determine which is the greater of two surds without finding either of them.

174. The product of two surds of the same order can be written down at once, for we have $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$. Hence, in order to find the product of any number of surds, the surds are first reduced to surds of the same order: their product is then given by the formula

$$\sqrt[n]{a}\times\sqrt[n]{b}\times\sqrt[n]{c}\ldots=\sqrt[n]{abc\ldots}$$

Ex. 1. Multiply $\sqrt{5}$ by $\sqrt[3]{2}$.

$$\sqrt{5}\times\sqrt[3]{2}=\sqrt[6]{5^3}\times\sqrt[6]{2^2}=\sqrt[6]{(5^3\times 2^2)}=\sqrt[6]{500}.$$

Ex. 2. Multiply $3\sqrt{5}$ by $2\sqrt[3]{2}$.

$$3\sqrt{5}\times 2\sqrt[3]{2}=3\times 2\times\sqrt{5}\times\sqrt[3]{2}=6\times\sqrt[6]{5^3}\times\sqrt[6]{2^2}=6\sqrt[6]{500}.$$

Ex. 3. Multiply $\sqrt{2}$ by $\sqrt[3]{2}$.

$$\sqrt{2}\times\sqrt[3]{2}=\sqrt[6]{2^3}\times\sqrt[6]{2^2}=\sqrt[6]{2^3\times 2^2}=\sqrt[6]{32}.$$

Or thus: $\sqrt{2}\times\sqrt[3]{2}=2^{\frac{1}{2}}\times 2^{\frac{1}{3}}=2^{\frac{1}{2}+\frac{1}{3}}=2^{\frac{5}{6}}=\sqrt[6]{2^5}.$

Ex. 4. Multiply $\sqrt{2}+\sqrt{3}$ by $\sqrt{3}+\sqrt{5}$.

$$(\sqrt{3}+\sqrt{2})(\sqrt{3}+\sqrt{5})=\sqrt{3}\times\sqrt{3}+\sqrt{2}\times\sqrt{3}+\sqrt{3}\times\sqrt{5}+\sqrt{2}\times 5$$
$$=3+\sqrt{6}+\sqrt{15}+\sqrt{10}.$$

Ex. 5. Divide $\sqrt[3]{4}$ by $\sqrt[3]{8}$.

$$\sqrt[3]{4}\div\sqrt[3]{8}=\sqrt[6]{4^2}\div\sqrt[6]{8^3}=\sqrt[6]{\frac{4^2}{8^3}}=\sqrt[6]{\frac{1}{32}}.$$

175. The determination of the approximate value of an expression containing surds is an arithmetical rather than an algebraical problem; but an expression containing surds must always be reduced to the form most suitable for arithmetical calculation. For this reason when surds occur in the denominators of fractions, the denominators must be rationalized. [See Art. 169.]

The following examples will illustrate the process:

$$\frac{2}{\sqrt{5}}=\frac{2\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}}=\frac{2}{5}\sqrt{5}.$$

$$\frac{3}{\sqrt{5}-1}=\frac{3(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)}=\frac{3}{4}(\sqrt{5}+1).$$

$$\frac{1}{1+\sqrt{3}+\sqrt{5}+\sqrt{15}}=\frac{1}{(1+\sqrt{3})(1+\sqrt{5})}=\frac{1}{8}(\sqrt{3}-1)(\sqrt{5}-1).$$

176. The product and the quotient of two similar quadratic surds are both rational.

This is obvious; for any two similar quadratic surds can be reduced to the forms $a\sqrt{b}$ and $c\sqrt{b}$.

Conversely, if the product of the quadratic surds $\sqrt{a}$ and $\sqrt{b}$ is rational and equal to x, we have $x=\sqrt{a}\times\sqrt{b}$; therefore $x\sqrt{b}=\sqrt{a}\times\sqrt{b}\times\sqrt{b}=b\sqrt{a}$, which shews that the surds are similar. So also, if $\sqrt{a}\div\sqrt{b}$ is rational, the surds must be similar.

177. The following theorem is important.

Theorem. *If $a+\sqrt{b}=x+\sqrt{y}$, where a and x are rational, and $\sqrt{b}$ and $\sqrt{y}$ are irrational; then will $a=x$, and $b=y$.*

For we have $$a-x+\sqrt{b}=\sqrt{y}.$$

Square both sides; then, after transformation, we have

$$2(a-x)\sqrt{b}=y-b-(a-x)^2.$$

Hence, *unless the coefficient of $\sqrt{b}$ is zero,* we must have an irrational quantity equal to a rational one, which is impossible.

The coefficient of $\sqrt{b}$ in the last equation must therefore be zero, so that $a=x$. And when $a=x$, the given relation shews that $\sqrt{b}=\sqrt{y}$, and therefore $b=y$.

As a particular case of the above,

$$\sqrt{a}\neq b+\sqrt{c}, \text{ unless } b=0 \text{ and } a=c.$$

Hence $\sqrt{a}\mp\sqrt{c}$ can only be rational when it is zero.

Ex. 1. Shew that $\sqrt{a}+\sqrt{b}+\sqrt{c}\neq 0$, unless the surds are all similar.

For we should have $\sqrt{a}+\sqrt{b}=-\sqrt{c}$; and therefore $a+b+2\sqrt{a}\sqrt{b}=c$. Hence $\sqrt{a}\sqrt{b}$ is rational, which shews [Art. 176], that $\sqrt{a}$ and $\sqrt{b}$ are similar surds.

178. The expressions $a+\sqrt{b}$ and $a-\sqrt{b}$ are said to be *conjugate* quadratic surd expressions.

It is clear that the sum and the product of two conjugate quadratic surd expressions are both rational.

Conversely, if the sum and the product of the expressions $a+\sqrt{b}$ and $c+\sqrt{d}$ are both rational, then $a=c$ and $\sqrt{b}+\sqrt{d}=0$, so that the two expressions are conjugate.

For $a+c+\sqrt{b}+\sqrt{d}$ can only be rational when $\sqrt{b}+\sqrt{d}$ is zero. [Art. 177.]

And, when $\sqrt{d}=-\sqrt{b}$, the product $(a+\sqrt{b})(c+\sqrt{d})$ $=ac+(c-a)\sqrt{b}-b$, which cannot be rational unless $c=a$.

179. In the expression

$$ax^n+bx^{n-1}+cx^{n-2}+\ldots\ldots+k,$$

where $a, b, c, \ldots\ldots k$ are all rational, let $\alpha+\sqrt{\beta}$ be substi-

tuted for x; and let P be the sum of all the rational terms in the result and $Q\sqrt{\beta}$ the sum of all the irrational terms. Then the given expression becomes $P+Q\sqrt{\beta}$.

Since P and Q are rational, they contain only squares and higher *even* powers of $\sqrt{\beta}$, and hence P and Q will not be changed by changing the sign of $\sqrt{\beta}$. Therefore when $\alpha-\sqrt{\beta}$ is substituted for x in the given expression the result will be $P-Q\sqrt{\beta}$.

If now the given expression vanish when $\alpha+\sqrt{\beta}$ is substituted for x, we have

$$P+Q\sqrt{\beta}=0.$$

Hence, as P and Q are rational and $\sqrt{\beta}$ is irrational, we must have *both* $P=0$ and $Q=0$; and therefore $P-Q\sqrt{\beta}=0$.

Therefore if the given expression vanish when $\alpha+\sqrt{\beta}$ is substituted for x it will also vanish when $\alpha-\sqrt{\beta}$ is substituted for x.

Hence [Art. 88], if $x-\alpha-\sqrt{\beta}$ be a factor of the given expression, $x-\alpha+\sqrt{\beta}$ will also be a factor.

Thus, *if a rational and integral expression be divisible by either of two conjugate quadratic surd expressions it will also be divisible by the other.*

180. The square root of a binomial expression which is the sum of a rational quantity and a quadratic surd can sometimes be found in a simple form. The process is as follows.

To find $\sqrt{(a+\sqrt{b})}$, where $\sqrt{b}$ is a surd.

Let $$\sqrt{(a+\sqrt{b})}=\sqrt{x}+\sqrt{y}.$$

Square both sides; then

$$a+\sqrt{b}=x+y+2\sqrt{xy}.$$

Now, since $\sqrt{b}$ is a surd, we can [Art. 177] equate the rational and irrational terms on the different sides of the last equation; hence $x+y=a$, and $4xy=b$.

Hence x and y are the roots of the equation

$$x^2 - ax + \frac{b}{4} = 0,$$

and these roots are

$$\tfrac{1}{2}\{a + \sqrt{(a^2 - b)}\} \text{ and } \tfrac{1}{2}\{a - \sqrt{(a^2 - b)}\}.$$

Thus $$\sqrt{(a+\sqrt{b})} = \sqrt{\frac{a+\sqrt{(a^2-b)}}{2}} + \sqrt{\frac{a-\sqrt{(a^2-b)}}{2}}.$$

It is clear that, unless $\sqrt{(a^2 - b)}$ is rational, the right side of the last equation is less suitable for calculation than the left. Thus the above process fails entirely unless $a^2 - b$ is a square number; and as this condition will not often be satisfied, the process has not much practical utility.

It should be remarked that if x and y are really rational, they can generally be written down by inspection.

Ex. 1. Find $\sqrt{(6+2\sqrt{5})}$.

Let $\sqrt{(6+2\sqrt{5})} = \sqrt{x} + \sqrt{y}$. Then, by squaring, we have $6+2\sqrt{5} = x+y+2\sqrt{xy}$. Hence, equating the rational and irrational parts, $x+y=6$ and $xy=5$. Whence obviously $x=1$ and $y=5$. Thus $\sqrt{(6+2\sqrt{5})} = 1+\sqrt{5}$.

Ex. 2. Find $\sqrt{(28-5\sqrt{12})}$.

Let $\sqrt{(28-5\sqrt{12})} = \sqrt{x} - \sqrt{y}$. Then, as before, $4xy = 25\times 12$, or $xy = 75$ and $x+y=28$; whence $x=25$ and $y=3$. Thus $\sqrt{(28-5\sqrt{12})} = 5-\sqrt{3}$. [If we had taken $x=3$ and $y=25$ we should have had the *negative* root, namely $\sqrt{3}-5$.]

Ex. 3. Find $\sqrt{(18+12\sqrt{3})}$.

In this case $\sqrt{(a^2-b)}$ is irrational and therefore the required root cannot be expressed in the form $\sqrt{x}+\sqrt{y}$ where x and y are rational. The root can however be expressed in the form $\sqrt[4]{x}+\sqrt[4]{y}$; for $\sqrt{(18+12\sqrt{3})} = \sqrt{\{\sqrt{3}\,(12+6\sqrt{3})\}} = \sqrt[4]{3}\times\sqrt{(12+6\sqrt{3})} = \sqrt[4]{3}\times(3+\sqrt{3}) = \sqrt[4]{243}+\sqrt[4]{27}$.

Ex. 4. Find $\sqrt{(10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15})}$.

Assume $\sqrt{(10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15})} = \sqrt{x}+\sqrt{y}+\sqrt{z}$; then $10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15} = x+y+z+2\sqrt{xy}+2\sqrt{xz}+2\sqrt{yz}$. We have now to find, *if possible*, rational values of x, y, z such that $xy=6$, $xz=10$, $yz=15$ and $x+y+z=10$. The first three equations are satisfied by the values $x=2$, $y=3$, $z=5$, and these values satisfy $x+y+z=10$. Hence $\sqrt{(10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15})} = \sqrt{2}+\sqrt{3}+\sqrt{5}$.

Ex. 5. Prove that, if $\sqrt[3]{(a+\sqrt{b})}=x+\sqrt{y}$; then will $\sqrt[3]{(a-\sqrt{b})}=x-\sqrt{y}$.

We have $a+\sqrt{b}=(x+\sqrt{y})^3=x^3+3xy+\sqrt{y}\,(3x^2+y)$.

Hence, equating the rational and irrational parts, we have

$$a=x^3+3xy, \text{ and } \sqrt{b}=\sqrt{y}\,(3x^2+y).$$

Hence $$a-\sqrt{b}=x^3+3xy-\sqrt{y}\,(3xz+y);$$

$$\therefore \sqrt[3]{(a-\sqrt{b})}=x-\sqrt{y}.$$

EXAMPLES XVIII.

Simplify the following:

1. $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$.

2. $\dfrac{2\sqrt{5}}{\sqrt{5}+\sqrt{3}}$.

3. $\dfrac{1}{\sqrt{8}+\sqrt{3}}+\dfrac{1}{\sqrt{8}-\sqrt{3}}$.

4. $(2-\sqrt{3})^{-3}+(2+\sqrt{3})^{-3}$.

5. $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$.

6. $\dfrac{(7-2\sqrt{5})(5+\sqrt{7})(31+13\sqrt{5})}{(6-2\sqrt{7})(3+\sqrt{5})(11+4\sqrt{7})}$.

7. $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

8. $\dfrac{1}{\sqrt{3}+\sqrt{5}-\sqrt{2}}$.

9. $\dfrac{1}{\sqrt{10}+\sqrt{14}+\sqrt{15}+\sqrt{21}}$.

10. $\dfrac{1}{\sqrt{6}+\sqrt{21}-\sqrt{10}-\sqrt{35}}$.

11. $\dfrac{1}{\sqrt[3]{2}-1}+\dfrac{1}{\sqrt[3]{2}+1}$.

12. $\dfrac{4}{\sqrt[3]{9}-1}+\dfrac{5}{\sqrt[3]{9}+1}$.

13. $\dfrac{1}{1+\sqrt[3]{2}+\sqrt[3]{4}}$.

14. $\dfrac{1}{\sqrt[3]{2}+\sqrt[3]{6}+\sqrt[3]{18}}$.

15. $\sqrt{(101-28\sqrt{13})}$. 16. $\sqrt{(28-5\sqrt{12})}$.

17. $\sqrt{\{11+2(1+\sqrt{5})(1+\sqrt{7})\}}$.

18. $\sqrt{\{6-4\sqrt{3}+\sqrt{(16-8\sqrt{3})}\}}$.

19. $\sqrt[4]{(97-56\sqrt{3})}$. 20. $\dfrac{\sqrt{(3+2\sqrt{2})}-\sqrt{2}}{\sqrt{2}-\sqrt{(3-2\sqrt{2})}}$.

21. $\dfrac{\sqrt{2}+\sqrt{45}}{\sqrt{2}+\sqrt{(7-2\sqrt{10})}}$.

22. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+\sqrt{(2+\sqrt{3})}}-\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-\sqrt{(2+\sqrt{3})}}$.

23. $\dfrac{\sqrt{(5+2\sqrt{6})}-\sqrt{(5-2\sqrt{6})}}{\sqrt{(5+2\sqrt{6})}+\sqrt{(5-2\sqrt{6})}}$.

24. $\sqrt{\{6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}\}}$.

25. $\sqrt{\{11+6\sqrt{2}+4\sqrt{3}+2\sqrt{6}\}}$.

26. $\sqrt{\{17+4\sqrt{2}-4\sqrt{3}-4\sqrt{6}-4\sqrt{5}-2\sqrt{10}+2\sqrt{30}\}}$.

27. Shew that

$$\frac{1}{\sqrt{(12-\sqrt{140})}}-\frac{1}{\sqrt{(8-\sqrt{60})}}-\frac{2}{\sqrt{(10+\sqrt{84})}}=0.$$

28. Shew that

$$\frac{1}{\sqrt{(11-2\sqrt{30})}}-\frac{4}{\sqrt{(7-2\sqrt{10})}}-\frac{4}{\sqrt{(8+4\sqrt{3})}}=0.$$

IMAGINARY AND COMPLEX QUANTITIES.

181. We have already seen that in order that the formula obtained in Art. 81 for the factors of a quadratic expression may be applicable to all cases, it is necessary to consider expressions of the form $\sqrt{-a}$, where a is

positive, and to assume that such expressions obey all the fundamental laws of algebra.

Since all squares, whether of positive or of negative quantities, are positive, it follows that $\sqrt{-a}$ cannot represent any positive or negative quantity; it is on this account called an *imaginary* quantity. Also expressions of the form $a + b\sqrt{-1}$ where a and b are real, are called *complex quantities.*

182. The question now arises whether the meanings of the symbols of algebra can be so extended as to include these imaginary quantities. It is clear that nothing would be gained, and that very much would be lost, by extending the meanings of the symbols, except it be possible to do this consistently with all the fundamental laws remaining true.

Now we have not to determine all the possible systems of meanings which might be assigned to algebraical symbols, both to the symbols which have hitherto been regarded as symbols of quantity and to the symbols of operation, subject only to the restriction that the fundamental laws should be satisfied *in appearance* whatever the symbols may mean: our problem is the much simpler and more definite one of finding a meaning for the imaginary expression $\sqrt{-a}$ which is consistent with the truth of all the fundamental laws.

183. We already know that -1 is an operation which performed upon any quantity changes it into a magnitude of a diametrically opposite kind. And, if we suppose that $\sqrt{-1}$ obeys the law expressed by $1 \times \sqrt{-1} \times \sqrt{-1} = -1$, it follows that $\sqrt{-1}$ must be an operation which when repeated is equivalent to a reversal.

Now any species of magnitude whatever can be represented by lengths set off along a straight line; and, when a magnitude is so represented, we may consider the

operation $\sqrt{-1}$ to be a revolution through a right angle, for a repetition of the process will turn the line in the same direction through a second right angle, and the line will then be directly opposite to its original direction.

Hence, when magnitudes are represented by lengths measured along a straight line, we see that $\sqrt{-1}$, regarded as a symbol of *operation*, has a perfectly definite meaning.

The symbol $\sqrt{-1}$ is generally for shortness denoted by i, and the operation denoted by i is considered to be a revolution through a right angle counter-clockwise, $-i$ denoting revolution through a right angle in the opposite direction.

184. It is clear that to take a units of length and then rotate through a right angle counter-clockwise gives the same result as to rotate the unit through a right angle counter-clockwise and then multiply by a. Thus $ai = ia$.

Again, to multiply ai by bi is to do to ai what is done to the unit to obtain bi, that is to say we must multiply by b and then rotate through a right angle; we thus obtain ab units rotated through two right angles, so that $ai \times bi = -ab = abii$.

From the above we see that the symbol i is commutative with other symbols in a product.

Since $(ai) \times (ai) = aaii = a^2(-1) = -a^2$, it follows that $\sqrt{-a^2} = ai$; it is therefore only necessary to use one imaginary expression, namely $\sqrt{-1}$.

185. With the above definition of $\sqrt{-1}$ or i, namely that it represents the *operation* of turning through a right angle counter-clockwise, magnitudes being represented by lengths measured along a straight line, the truth of the fundamental laws of algebra for imaginary and complex expressions can be proved. Some simple cases have been considered in the previous Article: for a full discussion see De Morgan's *Double Algebra*; see also Clifford's *Common Sense of the Exact Sciences*, Chapter IV. §§ 12 and 13.

186. If $a+bi=0$, where a and b are real, we have $a=-bi$. But a real quantity cannot be equal to an imaginary one, unless they are both zero.

Hence, if $a+bi=0$, we have both $a=0$ and $b=0$.

Note. In future, when an expression is written in the form $a+bi$, it will always be understood that a and b are both real.

187. If $a+bi=c+di$, we have $a-c+(b-d)i=0$; and hence, from Art. 186, $a-c=0$ and $b-d=0$.

Thus, *two complex expressions cannot be equal to one another, unless the real and imaginary parts are separately equal.*

188. The expressions $a+bi$ and $a-bi$ are said to be *conjugate* complex expressions.

The sum of the two conjugate complex expressions $a+bi$ and $a-bi$ is $a+a+(b-b)i=2a$; also their product is $aa+abi-abi-b^2i^2=a^2+b^2$.

Hence *the sum and the product of two conjugate complex expressions are both real.*

Conversely, if the sum and the product of two complex expressions are both real, the expressions must be conjugate.

For let the expressions be $a+bi$ and $c+di$. The sum is $a+bi+c+di=a+c+(b+d)i$, which cannot be real unless $b+d=0$. Again,

$$(a+bi)(c+di)=ac+bci+adi+bdi^2=ac-bd+(bc+ad)i,$$

which cannot be real unless $bc+ad=0$. Now, if $b+d=0$ and also $bc+ad=0$, we have $b(c-a)=0$; whence $a=c$ or $b=0$. If $b=0$, d is also zero, and both expressions are real; and, if $b \neq 0$, we have $a=c$, which with $b=-d$, shews that the expressions are conjugate.

189. **Definition.** The positive value of the square root of a^2+b^2 is called the *modulus* of the complex

quantity $a+bi$, and is written $\text{mod}\,(a+bi)$. Thus $\text{mod}\,(a+bi) = +\sqrt{a^2+b^2}$.

It is clear that two conjugate complex expressions have the same modulus; also, since $(a+bi)(a-bi) = a^2+b^2$ [Art. 188], the modulus of either of two conjugate complex expressions is equal to the positive square root of their product.

Since a and b are both real, a^2+b^2 will be zero if, and cannot be zero unless, a and b are both zero. Thus the modulus of a complex expression vanishes if the expression vanishes, and conversely the expression will vanish if the modulus vanishes.

If in $\text{mod}\,(a+bi) = +\sqrt{a^2+b^2}$ we put $b=0$, we have $\text{mod}\,a = +\sqrt{a^2}$, so that the modulus of a real quantity is its absolute value.

190. The product of $a+bi$ and $c+di$ is

$$ac+bci+adi+bdi^2 = ac-bd+(bc+ad)\,i.$$

Hence the modulus of the product of $a+bi$ and $c+di$ is

$$\sqrt{\{(ac-bd)^2+(bc+ad)^2\}} = \sqrt{\{(a^2+b^2)(c^2+d^2)\}}$$
$$= \sqrt{(a^2+b^2)} \times \sqrt{(c^2+d^2)}.$$

Thus the modulus of the product of two complex expressions is equal to the product of their moduli.

The proposition can easily be extended to the case of the product of more than two complex expressions; and, since the modulus of a real quantity is its absolute value, we have the following

Theorem. *The modulus of the product of any number of quantities whether real or complex, is equal to the product of their moduli.*

191. Since the modulus of the product of two complex expressions is equal to the product of their moduli, it follows conversely that the modulus of the quotient of two expressions is the quotient of their moduli. This may also be proved directly as follows :

$$(a+bi)\div(c+di)=\frac{a+bi}{c+di}\times\frac{c-di}{c-di}$$
$$=\frac{ac+bd+(bc-ad)\,i}{c^2+d^2}.$$

Hence
$$\text{mod}\left\{\frac{a+bi}{c+di}\right\}=\frac{\sqrt{\{(ac+bd)^2+(bc-ad)^2\}}}{c^2+d^2}$$
$$=\frac{\sqrt{\{a^2+b^2\}}}{\sqrt{\{c^2+d^2\}}}=\frac{\text{mod}\,(a+bi)}{\text{mod}\,(c+di)}.$$

192. It is obvious that in order that the product of any number of *real* factors may vanish, it is necessary and sufficient that one of the factors should be zero, and, by means of the theorem of Art. 190, the proposition can be proved to be true when all or any of the factors are complex quantities.

For, since the modulus of a product of any number of factors is equal to the product of their moduli, and since the moduli are all real, it follows that the modulus of a product cannot vanish unless the modulus of one of its factors vanishes.

Now if the product of any number of factors vanishes its modulus must vanish [Art. 189]; therefore the modulus of one of the factors must vanish, and therefore that factor must itself vanish. Conversely, if one of the factors vanishes, its modulus will vanish; and therefore the modulus of the product and hence the product itself must vanish.

193. In the expression

$$ax^n+bx^{n-1}+cx^{n-2}+\ldots+k,$$

where $a, b, c, \ldots k$ are all real, let $\alpha+\beta i$ be substituted for x, and let P be the sum of all real terms in the result, and Qi the sum of all the imaginary terms. Then the given expression becomes $P+Qi$.

Since P and Q are both real, they can contain only

squares and higher even powers of i, and hence P and Q will not be changed by changing the sign of i. Therefore when $\alpha - \beta i$ is substituted for x in the given expression the result will be $P - Qi$.

If now the given expression vanishes when $\alpha + \beta i$ is substituted for x, we have $P + Qi = 0$.

Hence, as P and Q are real, we must have *both* $P = 0$ and $Q = 0$; and therefore $P - Qi = 0$.

Hence if the given expression vanishes when $\alpha + \beta i$ is substituted for x, it will also vanish when $\alpha - \beta i$ is substituted for x.

Therefore [Art. 88] if $x - \alpha - \beta i$ is a factor of the given expression, $x - \alpha + \beta i$ will also be a factor.

Thus, *if any expression rational and integral in x, and with all its coefficients real, be divisible by either of two conjugate complex expressions it will also be divisible by the other.*

CHAPTER XV.

SQUARE AND CUBE ROOTS.

194. We have already shewn how to find the square of a given algebraical expression; and we have now to shew how to perform the inverse operation, namely that of finding an expression whose square will be identically equal to a given algebraical expression. It will be seen that our knowledge of the mode of formation of squares will enable us in many cases to write down by *inspection* the square root of a given expression.

195. From the identity

$$a^2 \pm 2ab + b^2 \equiv (a \pm b)^2,$$

we see that when a trinomial expression consists of the sum of the squares of any two quantities plus (or minus) twice their product, it is equal to the square of their sum (or difference).

Hence, to write down the square root of a trinomial expression which is a perfect square, arrange the expression according to descending powers of some letter; the square root of the whole expression will then be found by taking the square roots of the extreme terms with the same or with different signs according as the sign of the middle term is positive or negative.

Thus, to find the square root of

$$4a^8 - 12a^4b^3 + 9b^6.$$

The square roots of the extreme terms are $\pm 2a^4$ and $\pm 3b^3$. Hence, the middle term being negative, the required square root is $\pm(2a^4 - 3b^3)$.

Note. In future only one of the two square roots of an expression will be given, namely that one for which the sign of the first term is positive: to find the other root all the signs must be changed.

196. When an expression which contains only two different powers of a particular letter is arranged according to ascending or descending powers of that letter, it will only consist of *three* terms. For example, the expression $a^2 + b^2 + c^2 + 2bc + 2ca + 2ab$ when arranged according to powers of a is the trinomial

$$a^2 + 2a(b + c) + (b^2 + c^2 + 2bc).$$

It follows therefore from the preceding article that however many terms there may be in an expression which is a perfect square, the square root can be written down *by inspection*, provided that the expression contains only *two different powers of some particular letter.*

Ex. 1. To find the square root of

$$a^2 + b^2 + c^2 + 2bc + 2ca + 2ab.$$

Arranged according to powers of a, we have

$$a^2 + 2a(b + c) + (b + c)^2, \text{ that is } \{a + (b + c)\}^2.$$

Hence the required square root is $a + b + c$.

Ex. 2. To find the square root of

$$4x^4 + 9y^4 + 16z^4 + 12x^2y^2 - 16x^2z^2 - 24y^2z^2.$$

The given expression is

$$4x^4 + 4x^2(3y^2 - 4z^2) + 9y^4 - 24y^2z^2 + 16z^4,$$

that is, $$(2x^2)^2 + 2(2x^2)(3y^2 - 4z^2) + (3y^2 - 4z^2)^2,$$

which is $$\{2x^2 + (3y^2 - 4z^2)\}^2.$$

Hence the required square root is $2x^2 + 3y^2 - 4z^2$.

Ex. 3. To find the square root of

$$a^2+2abx+(b^2+2ac)x^2+2bcx^3+c^2x^4.$$

Arrange according to powers of a; we then have

$$a^2+2a(bx+cx^2)+b^2x^2+2bcx^3+c^2x^4,$$

that is, $$a^2+2a(bx+cx^2)+(bx+cx^2)^2.$$

Hence the required square root is $a+bx+cx^2$.

Ex. 4. To find the square root of

$$x^6-2x^5+3x^4+2x^3(y-1)+x^2(1-2y)+2xy+y^2.$$

The expression only contains y^2 and y; we therefore arrange it according to powers of y, and have

$$y^2+2y(x^3-x^2+x)+x^6-2x^5+3x^4-2x^3+x^2.$$

Now, if the expression is a complete square at all, the last of the three terms must be the square of half the coefficient of y; and it is easy to verify that

$$(x^3-x^2+x)^2=x^6-2x^5+3x^4-2x^3+x^2.$$

Hence the required square root is $y+x^3-x^2+x$.

197. *To find the square root of any algebraical expression.*

Suppose that we have to find the square root of $(A+B)^2$, where A stands for any number of terms of the root, and B for the rest; the terms in A and B being *arranged according to descending* (*or ascending*) *powers of some letter,* so that *every term in A is of higher* (*or lower*) *degree in that letter than any term of B.*

Also suppose that the terms in A are known, and that we have to find the terms in B.

Subtracting A^2 from $(A+B)^2$, we have the remainder $(2A+B)B$.

Now from the mode of arrangement it follows that the term of the highest (or lowest) degree in the remainder is twice the product of the *first* term in A and the *first* term in B.

Hence, to obtain the *next term* of the required root, that

is, to obtain the highest (or lowest) term of B, we *subtract from the whole expression the square of that part of the root which is already found, and divide the highest (or lowest) term of the remainder by twice the first term of the root.*

The first term of the root is clearly the square root of the first term of the given expression; and, when we have found the first term of the root, the second and other terms of the root can be obtained in succession by the above process.

For example, to find the square root of

$$x^6-4x^5+6x^4-8x^3+9x^2-4x+4.$$

The process is written as follows:

$$\begin{array}{rl}
 & x^6-4x^5+6x^4-8x^3+9x^2-4x+4\ \big(\,x^3-2x^2+x-2 \\
(x^3)^2= & \underline{x^6} \\
(x^3-2x^2)^2= & \underline{x^6-4x^5+4x^4} \\
(x^3-2x^2+x)^2= & \underline{x^6-4x^5+6x^4-4x^3+x^2} \\
(x^3-2x^2+x-2)^2= & \underline{x^6-4x^5+4x^4-8x^3+9x^2-4x+4}
\end{array}$$

We first take the square root of the first term of the given expression, *which must be arranged according to ascending or descending powers of some letter:* we thus obtain x^3, the *first term* of the required root.

Now subtract the square of x^3 from the given expression, and divide the first term of the remainder, namely $-4x^5$, by $2x^3$: we thus obtain $-2x^2$, the *second term* of the root.

Now subtract the square of x^3-2x^2 from the given expression, and divide the first term of the remainder, namely $2x^4$, by $2x^3$: we thus obtain x, the *third term* of the root.

Now subtract the square of x^3-2x^2+x from the given expression, and divide the first term of the remainder, namely $-4x^3$, by $2x^3$: we thus obtain -2, the *fourth term* of the root.

Subtract the square of x^3-2x^2+x-2 from the given expression and there is no remainder.

Hence x^3-2x^2+x-2 is the required square root.

The squares of x^3, x^3-2x^2, &c. are placed under the given expression, like terms being placed in the same column, so that in every case the first term of the remainder is obvious.

198. The square root of an algebraical expression may also be obtained by means of the theorem of Art. 91.

Take for example the case just considered.

The required root will be ax^3+bx^2+cx+d, provided that the given expression is equal to $(ax^3+bx^2+cx+d)^2$, that is equal to

$$a^2x^6+2abx^5+(2ac+b^2)x^4+2(ad+bc)x^3$$
$$+(2bd+c^2)x^2+2cdx+d^2.$$

Hence, equating the coefficients of corresponding powers of x in the last expression and in the expression whose root is required, we have

$$a^2=1;\ 2ab=-4;\ 2ac+b^2=6;\ 2ad+2bc=-8;$$
$$2bd+c^2=9;\ 2cd=-4;\ d^2=4.$$

The first four of these equations are sufficient to determine the values of a, b, c, d; these values are (taking only the positive value of a), $a=1$, $b=-2$, $c=1$, $d=-2$.

The last three equations will be satisfied by the values of a, b, c, d found from the first four, provided the given expression is a perfect square, which is really the case.

Thus the required square root is x^3-2x^2+x-2.

199. **Extended Definition of Square Root.** The definition of the Square Root of an algebraical expression may be extended so as to include the case of an expression which is not a perfect square. For, although an expression may not be a perfect square, we can find, by the methods of Art. 197 or Art. 198, a second expression whose square is equal to the given expression so far as certain terms are concerned.

Thus the square root of x^2+2x may be said to be $x+1$, $(x+1)^2$ being equal to x^2+2x so far as the terms which contain x are concerned.

Again, the square root of $1+x$ may be said to be $1+\frac{x}{2}$ or $1+\frac{x}{2}-\frac{x^2}{8}$, the square of the former differing from $1+x$ by $\frac{x^2}{4}$, and the square of the latter differing by

$-\frac{1}{8}x^3+\frac{1}{64}x^4$. Thus, *provided x is small*, $1+\frac{x}{2}$ is an approximation to the square root of $1+x$, and $1+\frac{x}{2}-\frac{x^2}{8}$ is a closer approximation, and by continuing the process we can approximate as closely as we please to the square root of $1+x$; this however is by no means the case when x is not a small quantity.

200. *When any number of terms of a square root have been obtained as many more can be found by ordinary division.*

For suppose the expression whose square root is to be found is the square of

$$(a_1x^n+a_2x^{n-1}+\dots+a_rx^{n-r+1})+(a_{r+1}x^{n-r}+\dots+a_{2r}x^{n-2r+1})+R.$$

The coefficients $a_1, a_2, \dots a_{2r}$ can be found by equating the coefficients of the first $2r$ powers of x in the square of the above to the coefficients of the corresponding powers of x in the given expression.

The square of the above expression is

$$(a_1x^n+a_2x^{n-1}+\dots+a_rx^{n-r+1})^2+2(a_1x^n+\dots+a_rx^{n-r+1})$$
$$(a_{r+1}x^{n-r}+\dots a_{2r}x^{n-2r+1})$$
$$+[(a_{r+1}x^{n-r}+\dots+a_{2r}x^{n-2r+1})^2+2R(a_1x^n+\dots+a_rx^{n-r+1})$$
$$+2R(a_{r+1}x^{n-r}+\dots+a_{2r}x^{n-2r+1})+R^2].$$

Now, since the highest power of x in R is x^{n-2r}, the highest power of x in the expression within square brackets is x^{2n-2r}.

Hence the expression within square brackets will not affect any of the terms from which $a_1, a_2, \dots a_{2r}$ are determined, for the first $2r$ terms of the given expression extend from x^{2n} to $x^{2n-2r+1}$.

It therefore follows that if the square of the sum of the first r terms of the root be subtracted from the given

expression, and the remainder be divided by twice the sum of the first r terms, the quotient will give the next r terms of the root.

201. *When n figures of a square root of a number have been found by the ordinary method, $n-1$ more figures can be found by division, provided that the number is a perfect square of $2n-1$ figures; if however this be not the case, there may be an error in the last figure.*

Let N be the given number, which is the perfect square of a number containing $2n-1$ figures, and let p be the number formed by the first n figures followed by $n-1$ zeros, and let q be the number formed by the remaining $n-1$ figures.

Then $$\sqrt{N} = p + q;$$

$$\therefore \ (N - p^2)/2p = q + q^2/2p.$$

Now $2p \nless 2.10^{2n-2}$ and $q \ngtr 10^{n-1}$. Hence $q^2/2p$ must be a fraction; whence it follows that if p^2 be subtracted from N and the remainder be divided by $2p$, the integral part of the quotient will be q.

Next, let $\sqrt{N}$ contain m figures, where m is greater than $2n-1$.

Let p be the number formed by the first n figures of the root followed by $m-n$ zeros, let q be the number formed by the next $n-1$ figures followed by $m-2n+1$ zeros, and let r be the number formed by the $m-2n+1$ remaining figures. Then

$$N = (p + q + r)^2;$$

$$\therefore \ (N - p^2)/2p - q = (q^2 + r^2 + 2qr)/2p + r.$$

Now $$10^m > p \nless 10^{m-1},$$

$$10^{m-n} > q \nless 10^{m-n-1},$$

and $$10^{m-2n+1} > r \nless 10^{m-2n},$$

whence it follows that $(q^2+r^2+2qr)/2p$ is less than 10^{m-2n+1}.

Hence $(q^2+r^2+2qr)/2p+r$ is less than $2\times 10^{m-2n+1}$, but it is not necessarily less than 10^{m-2n+1}. Hence $(N-p^2)/2p$ may differ from q by more than 10^{m-2n+1}; it must however differ by less than $2\times 10^{m-2n+1}$; so that the $n-1$ first figures of the quotient $(N-p^2)/2p$ are either the $n-1$ figures of q or differ only in the last figure, and in that case by 1 in excess.

Cube Root.

202. From the identity

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3,$$

we see that the cube of a binomial expression has *four* terms, and that when the cube is arranged according to ascending or descending powers of some letter, the cube roots of its extreme terms are the terms of the original binomial.

Hence the cube root of any perfect cube which has only four terms can be written down by inspection, for we have only to arrange the expression according to powers of some letter and then take the cube roots of its extreme terms.

For example, if $27a^6-54a^5b+36a^4b^2-8a^3b^3$ is a perfect cube its cube root must be $3a^2-2ab$; and by forming the cube of $3a^2-2ab$ it is seen that the given expression is really a perfect cube.

When an expression which contains only three different powers of a particular letter is arranged according to powers of that letter, there will be only *four* terms.

It therefore follows that however many terms there may be in an expression which is a perfect cube, the cube root can be written down *by inspection*, provided that the

expression contains only three different powers of some particular letter.

For example, to find the cube root of

$$a^3+b^3+c^3+3a^2b+3a^2c+3ab^2+3ac^2+6abc+3b^2c+3bc^2.$$

Arranged according to powers of a, we have

$$a^3+3a^2(b+c)+3a(b^2+c^2+2bc)+b^3+c^3+3b^2c+3bc^2,$$

that is, $$a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3.$$

Hence the required root is $a+b+c$.

203. *To find the cube root of any algebraical expression.*

Suppose we have to find the cube root of $(A+B)^3$, where A stands for any number of terms of the root, and B for the rest; the terms in A and B being arranged according to descending (or ascending) powers of some letter, so that every term of A is of higher (or lower) degree in that letter than any term of B.

Also suppose the terms in A are known, and that we have to find the terms in B.

Subtracting A^3 from $(A+B)^3$, we have the remainder $(3A^2+3AB+B^2)B$.

Now from the mode of arrangement it follows that the term of the highest (or lowest) degree in the remainder is $3 \times$ square of the first term of $A \times$ first term of B.

Hence to obtain the *next term* of the required root, that is, to obtain the highest (or lowest) term of B we *subtract from the whole expression the cube of that part of the root which is already found and divide the highest (or lowest) term of the remainder by three times the square of the first term of the root.*

This gives a method of finding the successive terms of the root after the first; and the first term of the root is clearly the cube root of the first term of the given expression.

For example, to find the cube root of

$$x^6-6x^5y+21x^4y^2-44x^3y^3+63x^2y^4-54xy^5+27y^6.$$

The process is written as follows:

$$\begin{aligned}
& x^6-6x^5y+21x^4y^2-44x^3y^3+63x^2y^4-54xy^5+27y^6\\
(x^2)^3 &= x^6\\
(x^2-2xy)^3 &= \overline{x^6-6x^5y+12x^4y^2-8x^3y^3}\\
(x^2-2xy+3y^2)^3 &= \overline{x^6-6x^5y+21x^4y^2-44x^3y^3}+63x^2y^4-54xy^5+27y^6.
\end{aligned}$$

Having arranged the given expression according to descending powers of x, we take the cube root of the first term: we thus obtain x^2, the *first term* of the required root.

We then subtract the cube of x^2 from the given expression, and divide the first term of the remainder, namely $-6x^5y$, by $3\times(x^2)^2$: we thus obtain $-2xy$, the *second term* of the root.

We then subtract the cube of x^2-2xy from the given expression, and divide the first term of the remainder by $3\times(x^2)^2$: this will give the *third term* of the root.

Note. The above rule for finding the cube root of an algebraical expression is rarely, if ever, necessary.

In actual practice cube roots are found as follows.

Take the case just considered; the first and last terms of the root are x^2 and $3y^2$, the cube roots of the first and last terms of the given expression; also the second term of the root will be found by dividing the second term of the given expression by $3\times(x^2)^2$, so that the second term of the root is $-2xy$.

Hence, *if the given expression is really a perfect cube*, it must be $(x^2-2xy+3y^2)^3$, and it is easy to verify that $(x^2-2xy+3y^2)^3$ is equal to the given expression.

Again, to find the cube root of

$$x^9-6x^8y+15x^7y^2-29x^6y^3+51x^5y^4-60x^4y^5+64x^3y^6\\-63x^2y^7+27xy^8-27y^9.$$

If the given expression is really a perfect cube the first and last terms of the root must be $\sqrt[3]{x^9}$ and $\sqrt[3]{-27y^9}$ respectively, that is x^3 and $-3y^3$.

The second term of the root must be $-6x^8y \div 3(x^3)^2 = -2x^2y$; and the term next to the last must be $27xy^8 \div 3(-3y^3)^2 = +xy^2$.

Hence the given expression, *if a cube at all*, must be $(x^3 - 2x^2y + xy^2 - 3y^3)^3$; and by expanding $(x^3 - 2x^2y + xy^2 - 3y^3)^3$ it will be found that the given expression is really a perfect cube.

204. From the identity [see Art. 253]

$$(a+b)^n = a^n + na^{n-1}b + \text{terms of lower degree in } a,$$

it is easy to shew, as in Articles 197 and 203, that the n^{th} root of any algebraical expression can be found by the following

Rule. *Arrange the expression according to descending or ascending powers of some letter, and take the n^{th} root of the first term: this gives the first term of the root.*

Also, having found any number of terms of the root, subtract from the given expression the n^{th} power of that part of the root which is already found, and divide the first term of the remainder by n times the $(n-1)^{\text{th}}$ power of the first term of the root: this gives the next term of the root.

EXAMPLES XIX.

Write down the square roots of the following expressions:

1. $4x^{10} - 12x^5y^3 + 9y^6$.

2. $x^8 + 9x^4y^{12} - 6x^6y^6$.

3. $a^2 + 4b^2 + 9c^2 + 12bc - 6ca - 4ab$.

4. $25a^4 + 9b^4 + 4c^4 + 12b^2c^2 - 20c^2a^2 - 30a^2b^2$.

Find the square roots of

5. $x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1.$

6. $4x^4 - 8x^3y^2 + 4xy^6 + y^8.$

7. $49 + 112x^2 + 70x^3 + 64x^4 + 80x^5 + 25x^6.$

8. $x^4 - 2x^3 + 5x^2 - 6x + 8 - 6x^{-1} + 5x^{-2} - 2x^{-3} + x^{-4}.$

9. $\dfrac{25x^2}{y^2} + \dfrac{y^2}{25x^2} - 20\dfrac{x}{y} + \dfrac{4y}{5x} + 2.$

10. $x^{\frac{5}{3}} - 4x^{\frac{4}{3}} + 2x + 4x^{\frac{2}{3}} + x^{\frac{1}{3}}.$

11. $x^{\frac{2}{3}} - 4x^{\frac{5}{6}} + 4x + 2x^{\frac{7}{6}} - 4x^{\frac{4}{3}} + x^{\frac{5}{3}}.$

12. $x^{\frac{8}{5}} - 2x^{-\frac{3}{5}}x^{\frac{11}{5}} + 2a^{\frac{4}{5}}x^{\frac{4}{5}} + a^{-\frac{6}{5}}x^{\frac{14}{5}} - 2a^{\frac{1}{5}}x^{\frac{7}{5}} + a^{\frac{8}{5}}.$

Find the cube roots of

13. $x^3 - 24x^2 + 192x - 512.$

14. $x^6 - 3x^5y + 6x^4y^2 - 7x^3y^3 + 6x^2y^4 - 3xy^5 + y^6.$

15. $1 - 9x^2 + 33x^4 - 63x^6 + 66x^8 - 36x^{10} + 8x^{12}.$

16. Find the square root of

$$2a^2(b+c)^2 + 2b^2(c+a)^2 + 2c^2(a+b)^2 + 4abc(a+b+c).$$

17. Find the square root of

$$x^2(x^2 + y^2 + z^2) + y^2z^2 + 2x(y+z)(yz - x^2).$$

18. Find the square root of

$$(a-b)^4 - 2(a^2+b^2)(a-b)^2 + 2(a^4+b^4).$$

19. Shew that $(x+a)(x+2a)(x+3a)(x+4a) + a^4$ is a perfect square.

20. Prove that $x^4 + px^3 + qx^2 + rx + s$ is a square, if $p^2s = r^2$ and $p^3 - 4pq + 8r = 0$.

21. Find the values of A, B and C in order that

$$4x^6 - 24x^5 + Ax^4 + Bx^3 + Cx^2 - 40x + 25$$

may be a perfect square.

22. Shew that, if $ax^3 + bx^2 + cx + d$ be a perfect cube, then $b^2 = 3ac$ and $c^2 = 3bd$.

23. Find the conditions that

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

may be the square of an expression which is rational in x, y and z.

24. Shew that if

$$(a - \lambda)\, x^2 + (b - \lambda)\, y^2 + (c - \lambda)\, z^2 + 2fyz + 2gzx + 2hxy$$

be the square of an expression which is rational in x, y and z, then will

$$a - \frac{gh}{f} = b - \frac{hf}{g} = c - \frac{fg}{h} = \lambda.$$

25. Shew that when the first r terms of the cube root of an algebraical expression are known, r more terms can be found by ordinary division.

26. When $n + 2$ figures of the cube root of a number have been obtained by the ordinary method, n more can be obtained by ordinary division, provided the number is a perfect cube of $2n + 2$ figures.

27. Shew that, if $n + 2$ figures whose numerical value is a have been found of a positive root of the equation $x^3 + qx - r = 0$, q being supposed positive, then the result of dividing $r - qa - a^3$ by $3a^2 + q$ will give at least $n - 1$ more figures correctly.

CHAPTER XVI.

RATIO. PROPORTION.

205. **Definitions.** The relative magnitude of two quantities, measured by the number of times the one contains the other, is called their *ratio*.

Concrete quantities of different kinds can have no ratio to one another: we cannot, for example, compare with respect to magnitude miles and tons, or shillings and weeks.

The ratio of a to b is expressed by the notation $a : b$; and a is called the first *term*, and b the second *term*, of the ratio. Sometimes the first and second terms of a ratio are called respectively the *antecedent* and the *consequent*.

It is clear that a ratio is greater, equal or less than unity according as its first term is greater, equal or less than the second. A ratio which is greater than unity is sometimes called a ratio of *greater inequality*, and a ratio which is less than unity is similarly called a ratio of *less inequality*.

The ratio of the product of the first terms of any number of ratios to the product of their second terms, is called the ratio *compounded* of the given ratios.

Thus $ac : bd$ is the ratio compounded of the two ratios $a : b$ and $c : d$.

The ratio $a^2 : b^2$ is sometimes called the *duplicate* ratio of $a : b$; so also $a^3 : b^3$, and $\sqrt{a} : \sqrt{b}$ are called respectively the *triplicate*, and the *sub-duplicate* ratio of $a : b$.

206. Magnitudes must always be expressed by means of *numbers*, and the number of times which one number contains another is found by dividing the one by the other.

Thus ratios can be expressed as *fractions*.

The principal properties of fractions and therefore of ratios have already been considered in Chapter VIII.

Thus, a ratio is unaltered in value by multiplying each of its terms by the same number. [Art. 107.]

Different ratios can be compared by reducing to a common denominator the fractions which express their values. [Art. 109.]

The theorems of Art. 113 are also true for ratios.

The following theorem is of importance:

207. **Theorem.** *Any ratio is made more nearly equal to unity by adding the same positive quantity to each of its terms.*

By adding x to each term of the ratio $a : b$, the ratio $a+x : b+x$ is obtained.

Now $\dfrac{a}{b} - 1 = \dfrac{a-b}{b}$, and $\dfrac{a+x}{b+x} - 1 = \dfrac{a-b}{b+x}$,

and it is clear that the *absolute* value of $\dfrac{a-b}{b+x}$ is less than that of $\dfrac{a-b}{b}$, for the numerators are the same and the denominator of the former is the larger: this proves the proposition.

When x is very great, the fraction $\dfrac{a-b}{b+x}$ is very small; and $\dfrac{a-b}{b+x}$, which is the difference between $\dfrac{a+x}{b+x}$ and 1, can be made *less than any assignable difference* by taking x sufficiently great.

This is expressed by saying that the *limiting value* of $\frac{a+x}{b+x}$, when x is *infinite*, is unity.

Now two quantities, whether finite or not, are *equal* to one another when *their ratio is unity*. Thus $a+x$ and $b+x$ are equal to one another when x is infinite, a being supposed not equal to b. [See Art. 118.]

208. Since any ratio is made more nearly equal to unity by the addition of the same quantity to each of its terms, it follows that a ratio is diminished or increased by such addition according as it was originally greater or less than unity. This proposition is sometimes enunciated: *A ratio of greater inequality is diminished and a ratio of less inequality is increased by the addition of the same quantity to each of its terms.*

209. **Incommensurable numbers.** The ratio of two quantities cannot always be expressed by the ratio of two whole numbers; for example, the ratio of a diagonal to a side of a square cannot be so expressed, for this ratio is $\sqrt{2} : 1$, and we cannot find any fraction which is *exactly* equal to $\sqrt{2}$.

Magnitudes whose ratio cannot be exactly expressed by the ratio of two whole numbers, are said to be *incommensurable.*

Although the ratio of two incommensurable numbers cannot be found *exactly*, the ratio can be found to any degree of approximation which may be desired; and the different theorems which have been proved with respect to ratios can, by the method of Art. 163, be proved to be true for the ratios of incommensurable numbers.

PROPORTION.

210. Four quantities are said to be *proportional* when the ratio of the first to the second is equal to the ratio of the third to the fourth.

Thus a, b, c, d are proportional, if

$$a : b = c : d.$$

This is sometimes expressed by the notation

$$a : b :: c : d,$$

which is read "a is to b as c is to d."

The first and fourth of four quantities in proportion, are sometimes called the *extremes*, and the second and third of the quantities are called the *means*.

211. If the four quantities a, b, c, d are proportional, we have by definition,

$$\frac{a}{b} = \frac{c}{d}.$$

Multiply each of these equals by bd; then

$$ad = bc.$$

Thus *the product of the extremes is equal to the product of the means.*

Conversely, *if $ad = bc$, then a, b, c, d will be proportional.*

For, if $$ad = bc, \text{ then}$$

$$\frac{ad}{bd} = \frac{bc}{bd};$$

$$\therefore \frac{a}{b} = \frac{c}{d},$$

that is $$a : b = c : d.$$

Hence also, the four relations

$$a : b = c : d,$$

$$a : c = b : d,$$

$$b : a = d : c,$$

and $$b : d = a : c,$$

are all true, provided that $ad = bc$. Hence the four proportions are all true when any one of them is true.

Ex. If $a : b = c : d$, then will $a + b : a - b = c + d : c - d$.

This has already been proved in Art. 113: it may also be proved as follows:

$$a + b : a - b = c + d : c - d,$$

if $$(a+b)(c-d) = (a-b)(c+d),$$

that is, *if* $$ac - bd + bc - ad = ac - bd - bc + ad;$$

or, *if* $$bc = ad.$$

But bc is equal to ad, since $a : b = c : d$.

212. Quantities are said to be in *continued proportion* when the ratios of the first to the second, of the second to the third, of the third to the fourth, &c., are all equal.

Thus a, b, c, d, &c. are in continued proportion if

$$a : b = b : c = c : d = \&c.,$$

that is, if $$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \&c.$$

If $a : b = b : c$, then b is called the *mean proportional* between a and c; also c is called the *third proportional* to a and b.

If a, b, c be in continued proportion, we have

$$\frac{a}{b} = \frac{b}{c};$$

$$\therefore\ b^2 = ac, \text{ or } b = \sqrt{ac}.$$

Thus *the mean proportional between two given quantities is the square root of their product.*

Also $$\frac{a}{b} \times \frac{b}{c} = \frac{b}{c} \times \frac{b}{c},$$

that is $$\frac{a}{c} = \frac{b^2}{c^2} = \frac{a^2}{b^2}.$$

Thus, *if three quantities are in continued proportion, the ratio of the first to the third is the duplicate ratio of the first to the second.*

213. The definition of proportion given in Euclid is as follows: Four quantities are proportionals, when if any equimultiples whatever be taken of the first and the third, and also any equimultiples whatever of the second and the fourth, the multiple of the third is always greater than, equal to or less than the multiple of the fourth, according as the multiple of the first is greater than, equal to or less than the multiple of the second.

If the four quantities a, b, c, d satisfy the algebraical test of proportionality, we have $\frac{a}{b} = \frac{c}{d}$; therefore for all values of m and n, $\frac{ma}{nb} = \frac{mc}{nd}$.

Hence $mc \gtreqless nd$, according as $ma \gtreqless nb$. Thus a, b, c, d satisfy also Euclid's test of proportionality.

Next, suppose that a, b, c, d satisfy Euclid's definition of proportion.

If a and b are commensurable, so that $a : b = m : n$, where m and n are whole numbers; then

$$\frac{a}{b} = \frac{m}{n}; \ \therefore\ na = mb.$$

But by definition

$$nc \gtreqless md \text{ according as } na \gtreqless mb.$$

Hence $$nc = md;$$

$$\therefore \frac{c}{d} = \frac{m}{n} = \frac{a}{b}.$$

Thus a, b, c, d satisfy the algebraical definition.

If a and b are incommensurable we cannot find two whole numbers m and n such that $a : b = m : n$. But, if we take any multiple na of a, this must lie between two consecutive multiples, say mb and $(m+1)b$ of b, so that

$$na > mb \text{ and } na < (m+1)b.$$

Hence by the definition,

$$nc > md \text{ and } nc < (m+1)d.$$

Hence both $\frac{a}{b}$ and $\frac{c}{d}$ lie between $\frac{m}{n}$ and $\frac{m+1}{n}$.

Thus the difference between $\frac{a}{b}$ and $\frac{c}{d}$ is less than $\frac{1}{n}$; and as this is the case however great n may be, $\frac{c}{d}$ must be equal to $\frac{a}{b}$, for their difference can be made less than any assignable difference by sufficiently increasing n.

Ex. 1. For what value of x will the ratio $7+x : 12+x$ be equal to the ratio $5 : 6$? *Ans.* 18.

Ex. 2. If $6x^2+6y^2=13xy$, what is the ratio of x to y? *Ans.* $2 : 3$ or $3 : 2$.

Ex. 3. What is the least integer which when added to both terms of the ratio $5 : 9$ will make a ratio greater than $7 : 10$? *Ans.* 5.

Ex. 4. Find x in order that $x+1 : x+6$ may be the duplicate ratio of $3 : 5$. *Ans.* $\frac{29}{16}$.

Ex. 5. Shew that, if $a : b :: c : d$, then

(i) $a^2+ab+b^2 : c^2+cd+d^2 :: a^2-ab+b^2 : c^2-cd+d^2$.

(ii) $a+b : c+d :: \surd(2a^2-3b^2) : \surd(2c^2-3d^2)$.

(iii) $a^2+b^2+c^2+d^2 : (a+b)^2+(c+d)^2 :: (a+c)^2+(b+d)^2 : (a+b+c+d)^2$.

[See Art. 113.]

Ex. 6. If $a : b :: c : d$, then will $ab+cd$ be a mean proportional between a^2+c^2 and b^2+d^2.

VARIATION.

214. One magnitude is said to *vary as* another when the two are so related that the ratio of any two values of the one is equal to the ratio of the corresponding values of the other.

Thus, if a_1, a_2 be any two measures of one of the quantities, and b_1, b_2 be the corresponding measures of the other, we have

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}; \text{ and therefore } \frac{a_1}{b_1}=\frac{a_2}{b_2}.$$

Hence the measures of corresponding values of the two magnitudes are in a constant ratio.

The symbol $\propto$ is used for the words *varies as*: thus $A \propto B$ is read 'A varies as B'.

If $a \propto b$, the ratio $a : b$ is constant; and if we put m for this constant ratio, we have

$$\frac{a}{b}=m; \therefore a=mb.$$

To find the constant m in any case it is only necessary to know one set of corresponding values of a and b.

For example, if $a \propto b$, and a is 15 when b is 5, we have $\frac{a}{b}=m=\frac{15}{5}$; $\therefore a=3b$.

215. **Definitions.** One quantity is said to *vary inversely as* another when the first varies as the reciprocal of the second.

Thus a varies inversely as b if the ratio $a : \frac{1}{b}$ is constant, and therefore $ab = m$.

One quantity is said to vary as two others *jointly* when the first varies as the product of the other two. Thus a varies as b and c jointly if $a \propto bc$, that is if $a = mbc$, where m is a constant.

One quantity is said to *vary directly as a second and inversely as a third* when the ratio of the first to the product of the second and the reciprocal of the third is constant.

Thus a is said to vary directly as b and inversely as c, if $a : b \times \frac{1}{c}$ is constant, that is, if $a = m\frac{b}{c}$, where m is a constant.

In all the different cases of variation defined above, the *constant* will be determined when any one set of corresponding values is given.

For example, if a varies jointly as b and c; and if a is 6 when b is 4 and c is 3, we have

$$a = mbc,$$

and

$$6 = m \times 4 \times 3.$$

Hence $m = \frac{1}{2}$, and therefore $a = \frac{1}{2}bc$.

216. **Theorem.** *If a depends only on b and c, and if a varies as b when c is constant, and varies as c when b is constant; then, when both b and c vary, a will vary as bc.*

Let a, b, c; a', b', c and a'', b', c' be three sets of corresponding values.

Then, since c is the same in the first and second, we have

$$\frac{a}{a'} = \frac{b}{b'} \quad \text{.......................(i).}$$

And, since b' is the same in the second and third, we have

$$\frac{a'}{a''} = \frac{c}{c'} \quad \text{.......................(ii).}$$

Hence from (i) and (ii), $\dfrac{a}{a''} = \dfrac{bc}{b'c'}$,

which proves the proposition.

The following are examples of the above proposition.

The cost [C] of a quantity of meat varies as the price [P] per pound if the weight [W] is constant, and the cost varies as the weight if the price per pound is constant. Hence, when both the weight and the price per pound change, the cost varies as the product of the weight and the price.

Thus, if $C \propto P$, when W is constant,

and $C \propto W$, when P is constant;

then $C \propto PW$, when both P and W change.

Again, the area of a triangle varies as the base when the height is constant; the area also varies as the height when the base is constant; hence, when both the height and the base change, the area will vary as the base and height jointly.

Again, the pressure of a gas varies as the density when the temperature is constant; the pressure also varies as the absolute temperature when the density is constant; hence when both density and temperature change, the pressure will vary as the product of the density and absolute temperature.

Ex. 1. The area of a circle varies as the square of its radius, and the area of a circle whose radius is 10 feet is 314·159 square feet. What is the area of a circle whose radius is 7 feet?

Ans. 452·38896 feet.

Ex. 2. The volume of a sphere varies as the cube of its radius, and the volume of a sphere whose radius is 1 foot is 4·188 cubic feet. What is the volume of a sphere of one yard radius? *Ans.* 113·076 feet.

Ex. 3. The distance through which a heavy body falls from rest varies as the square of the time it falls; also a body falls 64 feet in 2 seconds. How far does a body fall in 6 seconds? *Ans.* 576 feet.

Ex. 4. The volume of a gas varies as the absolute temperature and inversely as the pressure; also when the pressure is 15 and the temperature 260 the volume is 200 cubic inches. What will the volume be when the pressure becomes 18 and the temperature 390?

Ans. 250 inches.

Ex. 5. The distance of the offing at sea varies as the square root of the height of the eye above the sea level, and the distance is 3 miles when the height is 6 feet: find the distance when the height is 72 yards. *Ans.* 18 miles.

INDETERMINATE FORMS.

217. A ratio or fraction sometimes assumes an indeterminate form for some value or values of a contained letter.

Thus, when $x=0$ both the numerator and the denominator of the fraction $\frac{x^2-x}{x^3-x}$ vanish, and the fraction assumes for this value of x the indeterminate form $\frac{0}{0}$; and this is also the case when $x=1$.

Again, when $x=\infty$ both the numerator and the denominator of the above fraction become infinitely great, and the fraction assumes the indeterminate form $\frac{\infty}{\infty}$.

We proceed to shew how to find the limiting values of fractions which assume these indeterminate forms.

Consider, for example, the fraction $\frac{x^2-1}{x^3-1}$, which assumes the form $\frac{0}{0}$ when $x=1$.

Now $$\frac{x^2-1}{x^3-1}=\frac{(x-1)(x+1)}{(x-1)(x^2+x+1)};$$

and, *provided* $x-1$ *is not really zero*, we may divide the numerator and denominator by $x-1$ without altering the value of the fraction, and we can do this *however small* $x-1$ *may be.*

Hence, when $x-1$ is *very small*, $\frac{x^2-1}{x^3-1}=\frac{x+1}{x^2+x+1}$, and the limiting value of the latter fraction, as x approaches indefinitely near to 1, is at once seen to be $\frac{2}{3}$.

Hence, as x approaches indefinitely near to 1, the fraction $\dfrac{x^2-1}{x^3-1}$ approaches indefinitely near to the value $\dfrac{2}{3}$. This is expressed by the notation $L_{x=1}\dfrac{x^2-1}{x^3-1}=\dfrac{2}{3}$.

Ex. 1. Find the limiting value of $\dfrac{x^2-5x+6}{x^2-10x+16}$ when $x=2$.

It follows from Art. 88 that $x-2$ is a common factor of the numerator and denominator.

$$L_{x=2}\frac{x^2-5x+6}{x^2-10x+16}=L_{x=2}\frac{(x-2)(x-3)}{(x-2)(x-8)}=L_{x=2}\frac{x-3}{x-8}=\frac{1}{6}.$$

Ex. 2. Find the limiting value of $\dfrac{x^2+2x}{2x^2+3x}$ when $x=0$ and when $x=\infty$.

$$L_{x=0}\frac{x^2+2x}{2x^2+3x}=L_{x=0}\frac{x(x+2)}{x(2x+3)}=L_{x=0}\frac{x+2}{2x+3}=\frac{2}{3}.$$

$$L_{x=\infty}\frac{x^2+2x}{2x^2+3x}=L_{x=\infty}\frac{x^2\left(1+\frac{2}{x}\right)}{x^2\left(2+\frac{3}{x}\right)}=L_{x=\infty}\frac{1+\frac{2}{x}}{2+\frac{3}{x}}=\frac{1}{2},$$

since $\dfrac{2}{x}$ and $\dfrac{3}{x}$ are both zero when x is infinite.

Ex. 3. Find the limiting value of the ratio $1+2x : 2+3x$ when x increases without limit.

$$L_{x=\infty}\frac{1+2x}{2+3x}=L_{x=\infty}\frac{x\left(2+\frac{1}{x}\right)}{x\left(3+\frac{2}{x}\right)}=L_{x=\infty}\frac{2+\frac{1}{x}}{3+\frac{2}{x}}=\frac{2}{3}.$$

Ex. 4. Find the limiting value of $\dfrac{2x^2+100x+500}{5x^3-40}$ when x becomes indefinitely great.

$$L_{x=\infty}\frac{2x^2+100x+500}{5x^3-40}=L_{x=\infty}\frac{x^2\left(2+\frac{100}{x}+\frac{500}{x^2}\right)}{x^3\left(5-\frac{40}{x^3}\right)}$$

$$=L_{x=\infty}\frac{2x^2}{5x^3}=L_{x=\infty}\frac{2}{5x}=0.$$

EXAMPLES XX.

1. Shew that, if $a+b$, $b+c$, $c+a$ are in continued proportion, then $b+c : c+a = c-a : a-b$.

2. Shew that, if $x : a = y : b = z : c$, then

$$\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}=\frac{(x+y+z)^3}{(a+b+c)^2}.$$

3. Shew that, if $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$, then a, b, c, d are proportionals.

4. Shew that, if $b^2+c^2=a^2$, then

$$a+b+c : c+a-b = a+b-c : b+c-a.$$

5. What number must be subtracted from each of the numbers 7, 10, 19, 31 in order that the remainders may be in proportion?

6. Find $a : b : c$, having given

$$\frac{b}{a+b}=\frac{a+c-b}{b+c-a}=\frac{a+b+c}{2a+b+2c}.$$

7. If $$\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c},$$
shew that $(a+b+c)(yz+zx+xy)=(x+y+z)(ax+by+cz)$.

8. If $a(y+z)=b(z+x)=c(x+y)$, prove that

$$\frac{y-z}{a(b-c)}=\frac{z-x}{b(c-a)}=\frac{x-y}{c(a-b)}.$$

9. Shew that the ratio

$$l_1a_1+l_2a_2+l_3a_3+\ldots\ldots : l_1b_1+l_2b_2+l_3b_3+\ldots\ldots$$

is intermediate to the greatest and least of the ratios $a_1 : b_1$, $a_2 : b_2$, &c., the quantities being all positive.

10. If $a : b :: c : d$, then

$$\frac{a^{2n}+b^{2n}+c^{2n}+d^{2n}}{a^{-2n}+b^{-2n}+c^{-2n}+d^{-2n}}=(abcd)^n.$$

11. Shew that, if $(a+b)(b+c)(c+d)(d+a)$

$$=(a+b+c+d)(bcd+cda+dab+abc),$$

then $$a : b :: d : c.$$

12. If $(bcd+cda+dab+abc)^2-abcd\,(a+b+c+d)^2=0$, then it will be possible to arrange a, b, c, d so as to be proportionals.

13. Shew that, if $\dfrac{x}{a+2b+c}=\dfrac{y}{a-c}=\dfrac{z}{a-2b+c}$,

then
$$\frac{a}{x+2y+z}=\frac{b}{x-z}=\frac{c}{x-2y+z}.$$

14. Shew that, if $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ and $x+y+z=0$ are only satisfied by one set of ratios $x:y:z$, then $bc-f^2+ca-g^2+ab-h^2+2\,(gh-af)$
$$+\,2\,(hf-bg)+2\,(fg-ch)=0.$$

15. Shew that, if
$$\frac{a}{p\,(px-qy-rz)}=\frac{b}{q\,(qy-rz-px)}=\frac{c}{r\,(rz-px-qy)},$$
then $\dfrac{p}{a\,(ax-by-cz)}=\dfrac{q}{b\,(by-cz-ax)}=\dfrac{r}{c\,(cz-ax-by)}$

16. Shew that, if $ab=cd$, then either of them is equal to
$$(a+c)\,(a+d)\,(b+c)\,(b+d)/(a+b+c+d)^2.$$
Also, if $a+b=c+d$, then either of them is equal to
$$abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\Big/(ab+cd).$$

17. Find the limiting values of the following fractions when $x=2$, and when $x=\infty$.
$$\text{(i)}\ \frac{x^2-7x+10}{x^2-9x+14},\quad \text{(ii)}\ \frac{x^2-4x+4}{x^2-5x+6},\quad \text{(iii)}\ \frac{x^3+6x-16}{x^3-12x+16}.$$

18. Find the limiting values of the following when $x=a$,
$$\text{(i)}\ \frac{\sqrt{a}-\sqrt{x}}{\sqrt[3]{a}-\sqrt[3]{x}},\quad \text{(ii)}\ \frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{(x^2-a^2)}}.$$

CHAPTER XVII.

Arithmetical, Geometrical, and Harmonical Progression.

218. **Series.** A succession of quantities the members of which are formed in order according to some definite law is called a *series.*

Thus 1, 2, 3, 4, ..., in which each term exceeds the preceding by unity, is a series.

So also 3, 6, 12, 24, ..., in which each term is double the preceding, is a series.

We shall in the present Chapter consider some very simple cases of series, and shall return to the subject in a subsequent Chapter.

Arithmetical Progression.

219. **Definition.** A series of quantities is said to be in *Arithmetical Progression* when the difference between any term and the preceding one is the same throughout the series.

Thus, a, b, c, d, &c. are in Arithmetical Progression [A. P.] if $b - a = c - b = d - c =$ &c.

The difference between each term of an A. P. and the preceding term is called the *common difference.*

The following are examples of Arithmetical progressions:—

$$1, \quad 3, \quad 5, \quad 7, \quad \&c.$$
$$3, \quad -1, \quad -5, \quad -9, \quad \&c.$$
$$a, \quad a+2b, \quad a+4b, \quad \&c.$$

In the first series the common difference is 2, in the second it is -4, and in the last it is $2b$.

220. If the first term of an arithmetical progression be a, and the common difference d; then, by definition,

the 2nd term will be $a+d$,
„ 3rd „ „ $a+2d$,
„ 4th „ „ $a+3d$,

and so on, the coefficient of d being always less by unity than the number giving the position of the term in the series.

Hence the nth term will be $a+(n-1)d$.

We can therefore write down any term of an A. P. when the first term and the common difference are given.

For example, in the A. P. whose first term is 5, and whose common difference is 4, the 10th term is $5+(10-1)4=41$, and the 30th term is $5+29\times 4=121$.

221. An arithmetical progression is determined when *any two* of its terms are given.

For, suppose we know that the mth term is α, and that the nth term is β.

Let a be the first term, and d the common difference; then the mth term will be $a+(m-1)d$, and the nth term will be $a+(n-1)d$.

Hence $\qquad a+(m-1)d=\alpha,$

and $\qquad a+(n-1)d=\beta.$

Thus we have two equations of the first degree to determine a and d in terms of the known quantities m, n, α and β.

Ex. Find the 10th term of the A. P. whose 7th term is 15 and whose 21st term is 22.

If a be the first term, and d be the common difference, we have

$$a+6d=15, \text{ and } a+20d=22.$$

Hence $d=\frac{1}{2}$, $a=12$. The 10th term is therefore $12+\frac{9}{2}=16\frac{1}{2}$.

222. When three quantities are in arithmetical progression, the middle one is called the *Arithmetic Mean* of the other two.

If a, b, c are in A.P., we have, by definition,

$$b-a=c-b; \text{ and therefore } b=\tfrac{1}{2}(a+c).$$

Thus *the arithmetic mean of two given quantities is half their sum.*

When any number of quantities are in arithmetical progression all the intermediate terms may be called *arithmetic means* of the two extreme terms.

Between any two given quantities any number of arithmetic means may be inserted.

Let a and b be the two given quantities, and let n be the number of terms to be inserted.

Then b will be the $n+2$th term of the A.P. whose first term is a.

Hence, if d be the common difference, $b=a+(n+1)d$; and therefore $d=\dfrac{b-a}{n+1}$.

Then the series is

$$a,\ a+\frac{b-a}{n+1},\ a+2\frac{b-a}{n+1},\ \text{\&c.},$$

the required arithmetic means being

$$a+\frac{b-a}{n+1},\ a+2\frac{b-a}{n+1},\ \ldots\ldots\ a+n\frac{b-a}{n+1},$$

or $$\frac{na+b}{n+1},\ \frac{(n-1)a+2b}{n+1},\ \frac{(n-2)a+3b}{n+1},\ \ldots\ldots,\ \frac{a+nb}{n+1}.$$

223. *To find the sum of any number of terms of an arithmetical progression.*

Let a be the first term and d the common difference. Let n be the number of the terms whose sum is required, and let l be the last of them.

Then, since l is the nth term, we have

$$l = a + (n-1)\,d \text{.....................(i).}$$

Hence, if S be the required sum,

$$S = a + (a+d) + (a+2d) + \ldots\ldots + (l-2d) + (l-d) + l.$$

Now write the series in the reverse order; then

$$S = l + (l-d) + (l-2d) + \ldots\ldots + (a+2d) + (a+d) + a.$$

Hence, by addition of corresponding terms, we have

$$2S = (a+l) + (a+l) + (a+l) + \ldots\ldots \text{ to } n \text{ terms}$$
$$= n(a+l);$$

$$\therefore\ S = \frac{n}{2}(a+l) \text{..(ii),}$$

or, from (i),

$$S = \frac{n}{2}\{2a + (n-1)\,d\} \text{..............................(iii).}$$

From the formulæ (i), (ii), (iii) the value of all the quantities a, d, n, l, S can be found when any three are given.

Ex. 1. Find the sum of 20 terms of the arithmetical progression $3+6+9+$ &c.

Here $a=3,\ d=3,\ n=20$;

$$\therefore\ S = \frac{20}{2}\{6 + 19 \times 3\} = 630.$$

Ex. 2. Shew that the sum of any number of consecutive odd numbers, beginning with unity, is a square number.

The series of odd numbers is

$$1 + 3 + 5 + \ldots\ldots$$

Here $a=1$, $d=2$; hence the sum of n terms is given by

$$S = \frac{n}{2}\{2a + (n-1)\,d\} = \frac{n}{2}\{2 + (n-1)\,2\} = n^2.$$

Ex. 3. How many terms of the series $1+5+9+\ldots\ldots$ must be taken in order that the sum may be 190?

We have $S=\frac{n}{2}\{2a+(n-1)d\}$, where $S=190$, $a=1$, $d=4$.

Hence n is to be found from the *quadratic* equation

$$190=\frac{n}{2}\{2+4(n-1)\},$$

or $$2n^2-n-190=0,$$

that is $$(n-10)(2n+19)=0.$$

Hence $n=10$. The value $n=-\frac{19}{2}$ is to be rejected for n must necessarily be a *positive integer**.

Ex. 4. How many terms of the series $5+7+9+\ldots\ldots$ must be taken in order that the sum may be 480?

Here we have

$$480=\frac{n}{2}\{10+(n-1)2\};$$

$$\therefore n^2+4n-480=0,$$

or $$(n-20)(n+24)=0.$$

Hence n must be 20, for the value $n=-24$ must be rejected as a *negative* number of terms is altogether meaningless*.

Ex. 5. What is the 14th term of the A.P. whose 5th term is 11 and whose 9th term is 7? *Ans.* 2.

Ex. 6. What is the 2nd term of the A.P. whose 4th term is b and whose 7th term is $3a+4b$? *Ans.* $-2a-b$.

Ex. 7. Which term of the series 5, 8, 11, &c. is 320? *Ans.* The 106th.

Ex. 8. Shew that, if the same quantity be added to every term of an A.P., the sums will be in A.P.

Ex. 9. Shew that, if every term of an A.P. be multiplied by the same quantity, the products will be in A.P.

* The inadmissible value is a root of the *equation* to which the problem leads, but it is not a solution of the *problem*. [See Chapter XI.] It should be remarked that a negative value of n cannot mean a number of terms reckoned *backwards*.

Ex. 10. Shew that, if between every two consecutive terms of an A. P., a fixed number of arithmetic means be inserted, the whole will form an arithmetical progression.

Ex. 11. Find the sum of the following series:

(i) $2\frac{1}{4}+4\frac{1}{2}+6\frac{3}{4}+\ldots\ldots$ to 23 terms.

(ii) $\frac{1}{2}+\frac{1}{6}-\frac{1}{6}-\ldots\ldots$ to 12 terms.

(iii) $(a+9b)+(a+7b)+(a+5b)+\ldots\ldots$ to 10 terms.

(iv) $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+\ldots\ldots$ to n terms.

Ans. (i) 621, (ii) -16, (iii) $10a$, (iv) $\frac{1}{2}(n-1)$.

Ex. 12. The 7th term of an A. P. is 15, and the 21st term is 8; find the sum of the first 13 terms. *Ans.* 195.

Ex. 13. Find the sum of 21 terms of an A. P. whose 11th term is 20. *Ans.* 420.

Ex. 14. Shew that, if any odd number of quantities are in A. P., the first, the middle and the last are in A. P.

Ex. 15. Shew that, if unity be added to the sum of any number of terms of the series 8, 16, 24, &c., the result will be the square of an odd number.

Ex. 16. How many terms of the series $15+11+7+\ldots\ldots$ must be taken in order that the sum may be 35? *Ans.* 5.

Ex. 17. The sum of 5 terms of an A. P. is -5, and the 6th term is -13; what is the common difference? *Ans.* -4.

Ex. 18. Find the sum of all the numbers between 200 and 400 which are divisible by 7. *Ans.* 8729.

Ex. 19. If a series of terms in A. P. be collected into groups of n terms, and the terms in each group be added together, the results form an A. P. whose common difference is to the original common difference as $n^2 : 1$.

Geometrical Progression.

224. **Definition.** A series of quantities is said to be in *Geometrical Progression* when the ratio of any term to the preceding one is the same throughout the series.

Thus a, b, c, d, &c. are in Geometrical Progression (G.P.) if $\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = \&c.$

The ratio of each term of a geometrical progression to the preceding term is called the *common ratio.*

The following are examples of geometrical progressions:

$$1, \quad 3, \quad 9, \quad 27, \quad \&c.$$

$$4, \quad -2, \quad 1, \quad -\tfrac{1}{2}, \quad \&c.$$

$$a, \quad a^3, \quad a^5, \quad a^7, \quad \&c.$$

In the first series the common ratio is 3, in the second series it is $-\frac{1}{2}$, and in the third series it is a^2.

225. If the first term of a G.P. be a, and the common ratio r; then, by definition,

the 2nd term will be ar,
" 3rd " " ar^2,
" 4th " " ar^3,

and so on, the index of r being always less by unity than the number giving the position of the term in the series.

Hence the nth term will be ar^{n-1}.

We can therefore write down any term of a G.P. when the first term and the common ratio are given.

For example, in the G.P. whose first term is 2, and whose common ratio is 3, the 6th term is 2×3^5, and the 20th term is 2×3^{19}.

226. A Geometrical Progression is determined when *any two* of its terms are given.

For, suppose we know that the mth term is α, and that the nth term is β.

Let a be the first term, and r the common ratio; then the mth term will be ar^{m-1}, and the nth term will be ar^{n-1}.

Hence $ar^{m-1} = \alpha$, $ar^{n-1} = \beta$; and $\therefore\ r^{m-n} = \dfrac{\alpha}{\beta}$.

Hence $r = \alpha^{\frac{1}{m-n}} \beta^{\frac{1}{n-m}}$, and therefore $a = \alpha^{\frac{1-n}{m-n}} \beta^{\frac{1-m}{n-m}}$.

Ex. **Find the first term of the G.P. whose 3rd term is 18 and whose 5th term is $40\frac{1}{2}$.**

If a be the first term, and r the common ratio, we have

$$ar^2 = 18,\ ar^4 = \frac{81}{2};\ \therefore\ r^2 = \frac{9}{4}.$$

Hence $$a = 18 \times \frac{4}{9} = 8.$$

Thus the series is 8, 12, 18, &c.

227. When three quantities are in G.P., the middle one is called the *Geometric Mean* of the other two.

If a, b, c are in G.P., we have by definition

$$\frac{b}{a} = \frac{c}{b};\ \therefore\ b = \pm\sqrt{ac}.$$

Thus *the geometric mean of two given quantities is a square root of their product.*

When any number of quantities are in geometrical progression all the intermediate terms may be called *geometric means* of the two extreme terms.

Between two given quantities any number of geometric means may be inserted.

For let a and b be the two given quantities, and let n be the number of means to be inserted.

Then b will be the $(n+2)$th term of a G.P. of which a is the first term. Hence, if r be the common ratio, we have

$$b = ar^{n+1};\ \therefore\ r = \sqrt[n+1]{\frac{b}{a}}.$$

Hence the required means are $ar, ar^2, \ldots\ldots, ar^n$,

that is, $$a^{\frac{n}{n+1}} b^{\frac{1}{n+1}},\ a^{\frac{n-1}{n+1}} b^{\frac{2}{n+1}}, \ldots\ldots\ a^{\frac{1}{n+1}} b^{\frac{n}{n+1}}.$$

228. *To find the sum of any number of terms in geometrical progression.*

Let a be the first term, and r the common ratio. Let n be the number of the terms whose sum is required, and let l be the last of them.

Then, since l is the nth term, we have $l = ar^{n-1}$.

Hence, if S be the required sum,

$$S = a + ar + ar^2 + \dots\dots\dots\dots + ar^{n-1}.$$

Multiply by r; then

$$Sr = \quad ar + ar^2 + ar^3 + \dots\dots + ar^{n-1} + ar^n.$$

Hence, by subtraction,

$$S - Sr = a - ar^n;$$

$$\therefore S = a\frac{1-r^n}{1-r}.$$

Ex. 1. Find the sum of 10 terms of the series 3, 6, 12, &c.

Here $\qquad a=3, \; r=2, \; n=10.$

Hence $\qquad S = 3\dfrac{1-2^{10}}{1-2} = 3(2^{10}-1) = 3069.$

229. From the preceding article we have

$$S = a\frac{1-r^n}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Now when r is a *proper fraction*, whether positive or negative, the absolute value of r^n will decrease as n increases; moreover the value of r^n can be made *as small as we please* by sufficiently increasing the value of n.

Hence, *when r is numerically less than unity*, the sum of the series can be made to differ from $\dfrac{a}{1-r}$ by as small a quantity as we please by taking a sufficient number of terms.

Thus the sum of an *infinite number* of terms of the geometrical progression $a + ar + ar^2 + \ldots\ldots$, in which r is *numerically less than unity*, is $\dfrac{a}{1-r}$.

Ex. 1. Find the sum of an infinite number of terms of the series $9 - 6 + 4 - \ldots\ldots$

Here $\qquad a = 9,\ r = -\dfrac{2}{3}$.

Hence $\qquad S = \dfrac{a}{1-r} = \dfrac{9}{1 - \left(-\dfrac{2}{3}\right)} = \dfrac{27}{5}$.

Ex. 2. Find the geometrical progression whose sum to infinity is $4\frac{1}{2}$, and whose second term is -2.

Let a be the first term, and r be the common ratio.

Then we have $\qquad ar = -2$, and $\dfrac{a}{1-r} = \dfrac{9}{2}$.

Whence $\qquad 9r^2 - 9r - 4 = 0$.

Hence $\qquad r = -\dfrac{1}{3}$, or $r = \dfrac{4}{3}$.

If $\qquad r = -\dfrac{1}{3},\ a = \dfrac{-2}{r} = 6$;

and the series is $\qquad 6, \quad -2, \quad \dfrac{2}{3}$, &c.

The value $r = \dfrac{4}{3}$ is inadmissible, for r must be numerically less than unity.

Ex. 3. The 3rd term of a G.P. is 2, and the 6th term is $-\frac{1}{4}$; what is the 10th term? *Ans.* $-\frac{1}{64}$.

Ex. 4. Insert two geometric means between 8 and -1, and three means between 2 and 18. *Ans.* $-4,\ 2;\ \pm 2\sqrt{3},\ 6,\ \pm 6\sqrt{3}$.

Ex. 5. Shew that if all the terms of a G.P. be multiplied by the same quantity, the products will be in G.P.

Ex. 6. Shew that the reciprocals of the terms of a G.P. are also in G.P.

Ex. 7. Shew that, if between every two consecutive terms of a G.P., a fixed number of geometric means be inserted, the whole will form a geometrical progression.

Ex. 8. Find the sum of the following series:

(i) $12+9+6\frac{3}{4}+\ldots$ to 20 terms.

(ii) $1-\frac{2}{3}+\frac{4}{9}+\ldots$ to 6 terms.

(iii) $4+\cdot 8+\cdot 16+\ldots$ to infinity.

Ans. (i) $48\left\{1-\left(\frac{3}{4}\right)^{20}\right\}$, (ii) $\frac{133}{243}$, (iii) 5.

Ex. 9. Shew that the continued product of any number of quantities in geometrical progression is equal to $(gl)^{\frac{n}{2}}$, where n is the number of the quantities and g, l are the greatest and least of them.

Ex. 10. Shew that the product of any odd number of terms of a G.P. will be equal to the nth power of the middle term, n being the number of the terms.

Ex. 11. The sum of the first 10 terms of a certain G.P. is equal to 244 times the sum of the first 5 terms. What is the common ratio?

Ans. 3.

Ex. 12. If the common ratio of a G.P. be less than $\frac{1}{2}$, shew that each term will be greater than the sum of all that follow it.

HARMONICAL PROGRESSION.

230. **Definition.** A series of quantities is said to be in *Harmonical Progression* when the difference between the first and the second of any three consecutive terms is to the difference between the second and the third as the first is to the third.

Thus a, b, c, d &c., are in Harmonical Progression [H. P.], if

$$a-b : b-c :: a : c,$$

$$b-c : c-d :: b : d,$$

and so on.

If a, b, c be in harmonical progression, we have by definition

$$a-b : b-c :: a : c;$$

$$\therefore c(a-b)=a(b-c).$$

Hence, dividing by abc, we have

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$$

which shews that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetical progression.

Thus, *if quantities are in harmonical progression, their reciprocals are in arithmetical progression.*

231. **Harmonic Mean.** If a, b, c be in harmonical progression, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ will be in arithmetical progression.

Hence $$\frac{2}{b}=\frac{1}{a}+\frac{1}{c};$$

$$\therefore b=\frac{2ac}{a+c}.$$

Thus *the harmonic mean of two quantities is twice their product divided by their sum.*

If we put A, G, H for the arithmetic, the geometric, and the harmonic means respectively of any two quantities a and b, we have

$$A=\tfrac{1}{2}(a+b),\quad G=\sqrt{ab},\quad H=\frac{2ab}{a+b};$$

$$\therefore A \,.\, H=G^2.$$

Thus *the geometric mean of any two quantities is also the geometric mean of their arithmetic and harmonic means.*

232. **Theorem.** *The arithmetic mean of two unequal positive quantities is greater than their geometric mean.*

If a, b be the two positive quantities we have to shew that

$$\tfrac{1}{2}(a+b)>\sqrt{ab},$$

or $$\tfrac{1}{2}(\sqrt{a}-\sqrt{b})^2>0.$$

Now $(\sqrt{a}-\sqrt{b})^2$ is always positive, and therefore greater than zero, unless $a=b$.

Since the arithmetic mean of two positive quantities is greater than their geometric mean, it follows from Art. 231 that the geometric mean is greater than the harmonic.

233. To insert n harmonic means between any two quantities a and b.

Insert n arithmetic means between $\frac{1}{a}$ and $\frac{1}{b}$, and the reciprocals of these will be the required harmonic means.

The arithmetic means are

$$\frac{1}{a}+\frac{1}{n+1}\left(\frac{1}{b}-\frac{1}{a}\right),\ \frac{1}{a}+\frac{2}{n+1}\left(\frac{1}{b}-\frac{1}{a}\right),\ \&c.$$

Hence, by simplifying these terms and inverting them, the required harmonic means will be found to be

$$\frac{(n+1)\,ab}{nb+a},\ \frac{(n+1)\,ab}{(n-1)\,b+2a},\ldots\ldots,\ \frac{(n+1)\,ab}{b+na}.$$

234. It is of importance to notice that no formula can be found which will give the sum of any number of terms in harmonical progression.

EXAMPLES XXI.

1. Shew that, if a, b, c be in A.P., then will $a^2(b+c)$, $b^2(c+a)$, $c^2(a+b)$ be in A.P.

2. Find four numbers in A.P. such that the sum of their squares shall be 120, and that the product of the first and last shall be less than the product of the other two by 8.

3. If a, b, c be in A.P., and b, c, d be in H.P., then will $a:b=c:d$.

4. Find three numbers in G.P. such that their sum is 14, and the sum of their squares 84.

5. If a, b, c be in arithmetical progression, and x be the geometric mean of a and b, and y be the geometric mean of b and c; then will x^2, b^2, y^2 be in arithmetical progression.

6. Shew that, if a, b, c be in harmonical progression, then will $\frac{a}{b+c-a}$, $\frac{b}{c+a-b}$ and $\frac{c}{a+b-c}$, be also in harmonical progression.

7. Shew that, if a, b, c, d be in harmonical progression, then will

$$3(b-a)(d-c) = (c-b)(d-a).$$

8. Shew that, if a, b, c be in harmonical progression,

$$\frac{2}{b} = \frac{1}{b-a} + \frac{1}{b-c}.$$

9. Shew that, if a, b, c be in H.P., then will

$$\frac{b+a}{b-a} + \frac{b+c}{b-c} = 2.$$

10. If a, b, c be in A.P., b, c, d in G.P., and c, d, e in H.P.; then will a, c, e be in G.P.

11. If a, b, c be in H.P., then will $a-\frac{b}{2}$, $\frac{b}{2}$, $c-\frac{b}{2}$ be in G.P.

12. If a, b, c are in H.P., then a, $a-c$, $a-b$ are in H.P., and also c, $c-a$, $c-b$ are in H.P.

13. If x, a_1, a_2, y be in A.P., x, g_1, g_2, y in G.P., and x, h_1, h_2, y in H.P., then

$$\frac{g_1 g_2}{h_1 h_2} = \frac{a_1 + a_2}{h_1 + h_2}.$$

14. The sum of the first, second, and third terms of a G.P. is to the sum of the third, fourth and fifth terms as 1 : 4, and the seventh term is 384. Find the series.

15. If $a_1, a_2, a_3, \ldots\ldots, a_n$ be in harmonical progression, prove that $a_1a_2 + a_2a_3 + a_3a_4 + \ldots\ldots + a_{n-1}a_n = (n-1)\,a_1a_n$.

16. If a, x, y, b be in arithmetical progression, and a, u, v, b be in harmonical progression, then $xv = yu = ab$.

17. Three numbers are in arithmetic progression, and the product of the extremes is 5 times the mean; also the sum of the two largest is 8 times the least. Find the numbers.

18. If $\dfrac{a+b}{1-ab}$, b, $\dfrac{b+c}{1-bc}$ be in A.P.; then a, $\dfrac{1}{b}$, c will be in H.P.

19. If a, b, c be in A.P., and a^2, b^2, c^2 be in H.P., prove that $-\dfrac{a}{2}$, b, c are in G.P., or else $a = b = c$.

20. If x be any term of the arithmetical progression and y be the corresponding term of the harmonical progression whose first two terms are a, b, then will $x - a : y - a :: b : y$.

21. Shew that, if a be the arithmetic mean between b and c, and b be the geometric mean between a and c, then will c be the harmonic mean between a and b.

22. The series of natural numbers is divided into groups as follows: 1; 2, 3; 4, 5, 6; 7, 8, 9, 10; and so on. Prove that the sum of the numbers in the k^{th} group is $\frac{1}{2}k\,(k^2+1)$.

23. An A.P. and an H.P. have each the first term a, the same last term l, and the same number of terms n; prove that the product of the $(r+1)^{\text{th}}$ term of the one series and the $(n-r)^{\text{th}}$ term of the other is independent of r.

24. Terms equidistant from a given term of an A.P. are multiplied together; shew that the differences of the successive terms of the series so formed are in A.P.

25. Shew that, if S_n, S_{2n}, S_{3n} be the sum of n terms, of $2n$ terms, and of $3n$ terms respectively of any G.P., then will $S_n\,(S_{3n} - S_{2n}) = (S_{2n} - S_n)^2$.

26. If a, b, c be all positive and either in A.P., in G.P., or in H.P., and n be any positive integer, then $a^n + c^n > 2b^n$.

27. If P, Q, R be respectively the p^{th}, q^{th}, and r^{th} terms (i) of an A. P., (ii) of a G. P., and (iii) of an H. P., then will

(i) $P(q-r)+Q(r-p)+R(p-q)=0$,

(ii) $P^{q-r} . Q^{r-p} . R^{p-q}=1$,

(iii) $QR(q-r)+RP(r-p)+PQ(p-q)=0$.

28. Shew that, if $a_1, a_2, a_3, \ldots\ldots, a_n$ be in H. P., then

$$\frac{a_1}{a_2+a_3+\ldots+a_n}, \frac{a_2}{a_1+a_3+\ldots a_n}, \ldots\ldots\ldots, \frac{a_n}{a_1+a_2+\ldots+a_{n-1}}$$

will be in H. P.

29. Shew that, if $a_1, a_2, a_3 \ldots\ldots, a_n$ be all real, and if

$$(a_1^2+a_2^2+\ldots\ldots+a^2_{n-1})(a_2^2+a_3^2+\ldots\ldots+a_n^2)$$
$$=(a_1a_2+a_2a_3+\ldots\ldots+a_{n-1}a_n)^2,$$

then will $a_1, a_2, a_3, \ldots\ldots$ be in G. P.

30. Shew that any even square, $(2n)^2$, is equal to the sum of n terms of one series of integers in A. P., and that any odd square, $(2n+1)^2$, is equal to the sum of n terms of another A. P. increased by unity.

31. Prove that any positive integral power (except the first) of any positive integer, p, is the sum of p consecutive terms of the series 1, 3, 5, 7, &c.; and find the first of the p terms when the sum is p^r.

32. If an A. P. and a G. P. have the same first term and the same second term, every other term of the A. P. will be less than the corresponding term of the G. P., the terms being all positive.

CHAPTER XVIII.

SYSTEMS OF NUMERATION.

235. IN arithmetic any number whatever is represented by one or more of the ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, called figures or digits, by means of the convention that every figure placed to the left of another represents *ten* times as much as if it were in the place of that other. The cipher, 0, which stands for nothing, is necessary because one or more of the denominations, units, tens, hundreds, &c., may be wanting.

The above mode of representing numbers is called the *common scale of notation*, and 10 is said to be the *radix* or base.

236. Instead of ten any other number might be used as the base of a *System of Numeration*, that is of a system by which numbers are named according to some definite plan, and of the corresponding *Scale of Notation*, that is of a system by which numbers are represented by a few signs according to some definite plan; and to express a number, N, in the scale whose radix is r, is to write the number in the form $\ldots\ldots d_3 d_2 d_1 d_0$, where each of the digits $d_0, d_1, d_2, d_3 \ldots\ldots$ is less than r, and where d_0 stands for d_0 units, d_1 stands for $d_1 \times r$, d_2 for $d_2 \times r^2$, and so on.

Thus $$N = d_0 + d_1 r + d_2 r^2 + \ldots\ldots$$

Note. Throughout this chapter each letter stands for a positive integer, unless the contrary is stated.

237. Theorem. *Any positive integer can be expressed in any scale of notation, and this can be done in only one way.*

For divide N by r, and let Q_1 be the quotient and d_0 the remainder.

Then $$N = d_0 + r \times Q_1.$$

Now divide Q_1 by r, and let Q_2 be the quotient and d_1 the remainder.

Then $Q_1 = d_1 + rQ_2$; therefore $N = d_0 + rd_1 + r^2Q_2$.

By proceeding in this way we must sooner or later come to a quotient, $Q_n \equiv d_n$, which is less than r, when the process is completed, and we have

$$N = d_0 + rd_1 + r^2d_2 + r^3d_3 + \ldots\ldots r^n d_n,$$

so that the number would in the scale of r be written $d_n \ldots\ldots d_3 d_2 d_1 d_0$.

Each of the digits $d_0, d_1, d_2, \ldots\ldots$ is less than r, and any one or more of them, except the last, d_n, may be zero.

Since at every stage of the above process there is only *one* quotient and *one* remainder the transformation is unique.

The given number N may itself be expressed either in the common or in any other scale of notation.

Ex. 1. Express 2157 in the scale of 6.

The quotients and remainders of the successive divisions by 6 are as under:

$$\begin{array}{rll}
6 \,|\, \underline{2157} & & \\
6 \,|\, \underline{359} & \text{remainder} & 3 = d_0 \\
6 \,|\, \underline{59} & \ldots\ldots\ldots\ldots & 5 = d_1 \\
6 \,|\, \underline{9} & \ldots\ldots\ldots\ldots & 5 = d_2 \\
1 & \ldots\ldots\ldots\ldots & 3 = d_3
\end{array}$$

Thus 2157 when expressed in the scale of 6 is 13553.

Ex. 2. Change 13553 from the scale of 6 to the scale of 8.

We have the following successive divisions by 8, remembering that since 13553 is in the scale of 6 each figure is *six* times what it would be if it were moved one place to the left, so that to begin with we have to divide $1 \times 6 + 3$, and so on.

$$\begin{array}{r|l}
8 & 13558 \\
8 & 1125 \text{ remainder } 5 \\
8 & 53 \ \ldots\ldots\ldots\ldots \ 5 \\
 & 4 \ \ldots\ldots\ldots\ldots \ 1
\end{array}$$

Hence the number required is 4155.

Ex. 3. Change 4155 from the scale of 8 to the scale of 10,

Proceeding as before, we have

$$\begin{array}{r|l}
10 & 4155 \\
10 & 327 \text{ remainder } 7 \\
10 & 25 \ \ldots\ldots\ldots\ldots \ 5 \\
 & 2 \ \ldots\ldots\ldots\ldots \ 1
\end{array}$$

Thus 2157 is the number required.

Or thus:

Since $4155 = 4 \times 8^3 + 1 \times 8^2 + 5 \times 8 + 5 = \{(4 \times 8 + 1)8 + 5\}8 + 5,$

the required result may be obtained as follows :—

Multiply 4 by 8 and add 1; multiply this result by 8 and add 5; then multiply again by 8 and add 5.

Ex. 4. Express 3166 in the scale of 12. [Represent ten by t, and eleven by e.] *Ans.* $19et$.

Ex. 5. Express $\frac{17}{21}$ in the scale of 4. *Ans.* $\frac{101}{111}$.

Ex. 6. In what scale is 4950 written 20301? *Ans.* 7.

238. **Radix Fractions.** Radix fractions in any scale correspond to decimal fractions in the ordinary scale, so that $\cdot abc\ldots$ stands for $\frac{a}{r} + \frac{b}{r^2} + \frac{c}{r^3} + \ldots\ldots$

To shew that any given fraction may be expressed by a series of radix fractions in any proposed scale.

Let F be the given fraction; and suppose that, when expressed by radix fractions in the scale of r, we have

$$F = \cdot abc\ldots\ldots \equiv \frac{a}{r} + \frac{b}{r^2} + \frac{c}{r^3} + \ldots\ldots,$$

where each of $a, b, c\ldots\ldots$ is a positive integer (including zero) less than r.

Multiply by r; then

$$F \times r = a + \frac{b}{r} + \frac{c}{r^2} + \ldots\ldots$$

Hence a must be equal to the integral part, and $\frac{b}{r} + \frac{c}{r^2} + \ldots\ldots$ must be equal to the fractional part of Fr.

(If Fr be less than 1, a is zero.)

Let F_1 be the fractional part of Fr; then

$$F_1 = \frac{b}{r} + \frac{c}{r^2} + \ldots\ldots$$

Multiply by r; then

$$F_1 \times r = b + \frac{c}{r} + \ldots\ldots$$

Hence b must be equal to the integral part of $F_1 r$.

Thus a, b, c,...... can be found in succession.

Ex. 1. Express $\frac{1}{27}$ by a series of radix fractions in the scale of 6.

$$\frac{1}{27} \times 6 = 0 + \frac{6}{27}, \qquad \frac{6}{27} \times 6 = 1 + \frac{9}{27}, \qquad \frac{9}{27} \times 6 = 2.$$

Hence $\cdot 012$ is the required result.

Ex. 2. Express $\frac{1}{7}$ by a series of radix fractions in the scale of 3.

$$\frac{1}{7} \times 3 = 0 + \frac{3}{7}; \qquad \frac{3}{7} \times 3 = 1 + \frac{2}{7}; \qquad \frac{2}{7} \times 3 = 0 + \frac{6}{7};$$

$$\frac{6}{7} \times 3 = 2 + \frac{4}{7}; \qquad \frac{4}{7} \times 3 = 1 + \frac{5}{7}; \qquad \frac{5}{7} \times 3 = 2 + \frac{1}{7}.$$

Hence $\cdot\dot{0}1021\dot{2}$ is the required result.

Ex. 3. Change 324·26 from the scale 8 to the scale 6.

The integral and fractional parts must be considered separately.

$$
\begin{array}{r|l}
6 & 324 \\ \hline
6 & 43 \text{ remainder } 2 \\ \hline
 & 5 \ \ldots\ldots\ldots\ldots \ 5
\end{array}
\qquad\qquad
\begin{array}{r}
\cdot 26 \\
6 \\ \hline
2\cdot 04 \\
6 \\ \hline
0\cdot 30 \\
6 \\ \hline
2\cdot 20 \\
6 \\ \hline
1\cdot 40 \\
6 \\ \hline
3\cdot 00
\end{array}
$$

Thus the required result is $552 \cdot 20213$.

Ex. 4. In the scale of 8 express $\cdot 16\dot{3}1\dot{5}$ as a vulgar fraction.

$$N = \cdot 16\dot{3}1\dot{5};$$

$$\therefore\ 8^2 N = 16 \cdot \dot{3}1\dot{5};$$

$$\therefore\ 8^5 N = 16315 \cdot \dot{3}1\dot{5};$$

$$\therefore\ N = \frac{16315 - 16}{8^5 - 8^2} = \frac{16315 - 16}{77700} = \frac{16277}{77700}.$$

Ex. 5. In the scale of 7 express $\cdot 2\dot{3}\dot{1}$ as a vulgar fraction.

Ans. $\frac{113}{330}$.

Ex. 6. Change $314 \cdot 2\dot{3}$ from the scale of 5 to the scale of 7.

Ans. $150 \cdot \dot{3}56\dot{4}$.

239. **Theorem.** *The difference between any number and the sum of its digits is divisible by $r-1$, where r is the radix of the scale in which the number is expressed.*

Let N be the number, S the sum of the digits, and let $d_0, d_1, d_2 \ldots\ldots$ be the digits.

Then $\qquad N = d_0 + rd_1 + r^2 d_2 + \ldots\ldots + r^n d_n,$

and $\qquad S = d_0 + d_1 + d_2 + \ldots\ldots + d_n.$

$$\therefore\ N - S = (r-1) d_1 + (r^2 - 1) d_2 + \ldots\ldots + (r^n - 1) d_n.$$

Now each of the terms on the right is divisible by $r-1$ [Art. 86].

Hence $N - S$ is divisible by $r-1$.

Since $N - S$ is divisible by $r-1$, N and S must leave the same remainder when divided by $r-1$.

Ex. 1. The difference of any two numbers expressed by the same digits is divisible by $r-1$.

For the sum of the digits is the same for both; and since $N_1 - S$ and $N_2 - S$ are both divisible by $r-1$, it follows that $N_1 - N_2$ is divisible by $r-1$.

Ex. 2. Shew that in the ordinary scale a number is divisible by 9 if the sum of its digits be divisible by 9, and by 3 if the sum of its digits is divisible by 3.

$N-S$ is a multiple of 9; hence, if S be a multiple of 9, so also is N; and, if S be a multiple of 3, so also is N.

Ex. 3. Shew that any number is divisible by $r+1$ if the difference between the sum of the odd and the sum of the even digits is divisible by $r+1$.

Let $$N = d_0 + d_1 r + d_2 r^2 + d_3 r^3 + \dots\dots,$$

and $$D = d_0 - d_1 + d_2 - d_3 + \dots\dots$$

Then $$N - D = d_1(r+1) + d_2(r^2-1) + d_3(r^3+1) + \dots\dots$$

Each of the terms on the right is divisible by $r+1$ [Art. 87]; $\therefore$ $N-D$ is divisible by $r+1$. Hence if D is divisible by $r+1$ so also is N.

Ex. 4. If N_1 and N_2 be any two whole numbers, and if the remainders left after dividing the sum of the digits in N_1, N_2 and in $N_1 \times N_2$ by 9 be n_1, n_2 and p respectively; then will $n_1 n_2$ be equal to p, or differ from p by a multiple of 9.

For $N_1 = n_1 +$ a multiple of 9, and $N = n_2 +$ a multiple of 9; therefore $N_1 \times N_2 = n_1 \times n_2 +$ a multiple of 9. Hence $n_1 n_2 +$ a multiple of 9 is equal to $p +$ a multiple of 9.

If the above is applied in any case of multiplication, and it is found that $n_1 n_2$ does not equal p, or differ from it by a multiple of 9, there must be some error in the process of multiplication.

This gives a method of testing the accuracy of multiplication; the test is not however a complete one, for although it is certain that there must be an error if $n_1 \times n_2$ does not equal p, or differ from it by a multiple of 9, there *may* be errors when the condition is satisfied, provided that the errors neutralize one another so far as the sum of the digits in the product is concerned.

This is called the "**Rule for casting out the nines.**"

Ex. 5. A number of three digits in the scale of 7 has the same digits in reversed order when it is expressed in the scale of 9: find the number.

Let a, b, c be the digits; then we have

$$49a + 7b + c = 81c + 9b + a,$$

where a, b, c are positive integers less than 7.

Hence $$40c + b = 24a.$$

Now $40c$ and $24a$ are both divisible by 8; therefore b must be divisible by 8. But b is less than 7: it must therefore be zero. And since b is zero, we have $5c=3a$, which can only be satisfied when $c=3$ and $a=5$.

Thus the number required is 503.

Ex. 6. A number consisting of three digits is doubled by reversing the digits; prove that the same will hold for the number formed by the first and last digits, and also that such a number can be found in only one scale of notation out of every three.

Let the number be abc in the scale of r.

Then we have $(abc)\times 2=cba$.

Since cba is greater than abc, c must be greater than a.

Hence we must have the following equations:

$$2c=a+r \quad\text{.......................(i)},$$

$$2b+1=b+r \quad\text{.......................(ii)},$$

$$2a+1=c \quad\text{.......................(iii)}.$$

From (i) and (iii) we see that the number represented by ca is double that represented by ac.

Also $$4a+2=2c=a+r;$$

$$\therefore\ r-2=3a.$$

Hence, as a is an integer, $r-2$ must be a multiple of 3, so that the number must be in one of the scales 2, 5, 8, 11, &c., the numbers corresponding to these scales being 011, 143, 275, $3t7$, &c.

EXAMPLES XXII.

1. Find the number which has the same two digits when expressed in the scales of 7 and 9.

2. In any given scale write down the greatest and the least number which has a given number of digits.

3. A number of six digits is formed by writing down any three digits and then repeating them in the same order; shew that the number is divisible by 1001.

4. Of the weights 1, 2, 4, 8, &c. lbs., which must be taken to weigh 1027 lbs.?

5. Shew that the number represented in any scale by 144 is a square number.

6. Shew that the numbers represented in any scale by 121, 12321, and 1234321 are perfect squares.

7. Find a number of two digits, which are transposed by the addition of 18 to the number, or by converting it into the septenary scale.

8. A number is denoted by $4 \cdot 44\dot{0}$ in the quinary scale, and by $4 \cdot 5\dot{4}$ in a certain other scale. What is the radix of that other scale?

9. If S be the sum of the digits of a number N, and $2Q$ be the sum of the digits of $2N$, the number being expressed in the ordinary scale, shew that $S \sim Q$ is a multiple of 9.

10. If a whole number be expressed in a scale whose radix is odd, the sum of the digits will be even if the number be even, and odd if the number be odd.

11. Prove that, in any scale of notation, the difference of the square of any number of three digits and the square of the number formed by reversing the digits is divisible by $r^2 - 1$.

12. Prove that, in any scale of notation, the difference of the square of any number and the square of the number formed by reversing the digits is divisible by $r^2 - 1$.

13. A number of three digits in the scale of 7, when expressed in the scale of 11 has the same digits in reversed order: find the number.

14. Prove that all the numbers which are expressed in the scales of 5 and 9 by using the same digits, whether in the same order or in a different order, will leave the same remainder when divided by 4.

15. There is a certain number which is expressed by 6 digits in the scale of 3, and by the last three of those digits in the scale of 12. Find the number.

16. Find a number of four digits in the scale of 8 which when doubled will have the same digits in reverse order.

17. The digits of a number of three digits are in A. P. The number when divided by the sum of its digits gives a quotient 15; and when 396 is added to the number, the sum has the same digits in inverted order. Find the number.

18. Find the digits a, b, c in order that the number $13ab45c$ may be divisible by 792.

19. Prove that there is only one scale of notation in which the number represented by 1155 is divisible by that represented by 12, and find that scale.

20. Find a number of four digits in the ordinary scale which will have its digits reversed in order by multiplying by 9.

21. In the scale of notation whose radix is r, shew that the number $(r^2-1)(r^n-1)$ when divided by $r-1$ will give a quotient with the same digits in the reverse order.

22. Shew that, in any scale of notation,

$$\frac{1}{(r-1)^2} = \cdot\dot{0}123\ldots(r-3)\,(\dot{r-1}),$$

the circulating period consisting of all the figures in order except $r-2$ which is passed over. For example, in the ordinary scale, $\frac{1}{81} = \dot{0}1234567\dot{9}$.

23. There is a number of six digits such that when the extreme left-hand digit is transposed to the extreme right-hand, the rest being unaltered, the number is increased three-fold. Prove that the left-hand digit must be either 1 or 2, and find the number in either case.

24. Find a number of three digits, the last two of which are alike, such that when multiplied by a certain number it still consists of three digits, the first two of which are alike and the same as the former repeated ones, and the third is the same as the multiplier.

CHAPTER XIX.

PERMUTATIONS AND COMBINATIONS.

240. Definition. The different ways in which r things can be taken from n things, regard being had to the order of selection or arrangement, are called the *permutations* of the n things r at a time.

Thus two permutations will be different unless they contain the same objects arranged in the same order.

For example, suppose we have four objects, represented by the letters a, b, c, d; the permutations two at a time are $ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd$, and dc.

The number of permutations of n different things taken r at a time is denoted by the symbol ${}_nP_r$.

241. *To find the number of permutations of n different things taken r at a time.*

Let the different things be represented by the letters $a, b, c, \ldots\ldots$

It is obvious that there are n permutations of the n things when taken one at a time, so that ${}_nP_1 = n$.

Now in the permutations of the n letters r together, the number of permutations in which a particular letter occurs first in order is equal to the number of permutations of the remaining $n-1$ letters $r-1$ at a time. This is true for each one of the n letters, and therefore

$$ {}_nP_r = n \times {}_{n-1}P_{r-1}. $$

Since the above relation is true for all values of n and r, we have in succession

$$ {}_{n-1}P_{r-1} = (n-1) \times {}_{n-2}P_{r-2}, $$
$$ {}_{n-2}P_{r-2} = (n-2) \times {}_{n-3}P_{r-3}, $$
$$ \ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots $$
$$ {}_{n-r+2}P_{r-r+2} = (n-r+2) \times {}_{n-r+1}P_{r-r+1}. $$

But $\qquad {}_{n-r+1}P_1 = (n-r+1).$

Multiply all the corresponding members of the above equalities, and cancel all the common factors; we then have

$$ {}_nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1). $$

If all the n things are to be taken, r is equal to n, and we have

$$ {}_nP_n = n(n-1)(n-2)\ldots\ldots3\,.\,2\,.\,1. $$

Definitions. The product $n(n-1)(n-2)\ldots2\,.\,1$ is denoted by the symbol $\underline{|n}$ or by $n!$ The symbols $\underline{|n}$ and $n!$ are read 'factorial n.'

The continued product of the r quantities $n, n-1, n-2, \ldots\ldots(n-r+1)$, n not being necessarily an integer in this case, is denoted by n_r. Thus $n_3 \equiv n(n-1)(n-2)$.

Hence we have ${}_nP_n = \underline{|n}$, and ${}_nP_r = n_r$.

242. *To find the number of permutations of n things taken all together, when the things are not all different.*

Let the n things be represented by letters; and suppose p of them to be a's, q of them to be b's, r of them to be c's, and so on. Let P be the required number of permutations.

If in any one of the actual permutations we suppose that the a's are all changed into p letters different from each other and from all the rest; then, by changing only the arrangement of these p new letters, we should, instead of a single permutation, have $\underline{|p}$ different permutations.

Hence, if the a's were all changed into p letters different from each other and from all the rest, the b's, c's, &c. being unaltered, there would be $P \times \underline{|p}$ permutations.

Similarly, if in any one of these new permutations we suppose that the b's are all changed into q letters different from each other and from all the rest, we should obtain $\underline{|q}$ permutations by changing the order of these q new letters. Hence the whole number of permutations would now be $P \times \underline{|p} \times \underline{|q}$.

By proceeding in this way we see that if all the letters were changed so that no two were alike, the total number of permutations would be $P \times \underline{|p} \times \underline{|q} \times \underline{|r}\ldots$

But the number of permutations all together of n different things is $\underline{|n}$. Hence $P \times \underline{|p} \times \underline{|q} \times \underline{|r}\ldots = \underline{|n}$;

$$\therefore\ P = \frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}\ldots}.$$

Ex. 1. Find ${}_6P_3$, ${}_5P_4$ and ${}_7P_7$. *Ans.* 120, 120, 5040.

Ex. 2. Shew that ${}_{10}P_4 = {}_7P_7$.

Ex. 3. If ${}_nP_5 = 12 \times {}_nP_3$, find n. *Ans.* 7.

Ex. 4. If ${}_{2n}P_3 = 100 \times {}_nP_2$, find n. *Ans.* 13.

Ex. 5. If ${}_{2n}P_3 = 2 \times {}_nP_4$, find n. *Ans.* 8.

Ex. 6. Find the number of permutations of all the letters of each of the words *acacia, hannah, success* and *mississippi*. *Ans.* 60, 90, 420, 34650.

Ex. 7. In how many ways may a party of 8 take their places at a round table; and in how many ways can 8 different beads be strung on a necklace? *Ans.* $\underline{|7}$, $\frac{1}{2}\underline{|7}$.

Ex. 8. In how many ways may a party of 4 ladies and 4 gentlemen be arranged at a round table, the ladies and gentlemen being placed alternately? *Ans.* 144.

Ex. 9. The number of permutations of n things all together in which r specified things are to be in an assigned order though not necessarily consecutive is $\underline{|n}/\underline{|r}$.

Ex. 10. The number of ways in which n books can be arranged on a shelf so that two particular books shall not be together is $(n-2)\,\underline{|n-1}$.

Ex. 11. Find the number of permutations of n things r together, when each thing can be repeated any number of times.

Here any one of the n things can be put in the first place; and, however the first place is filled, any one of the n things can be put in the second place; and so on. Hence the number required

$$=n\times n\times n\times\ldots=n^r.$$

COMBINATIONS.

243. Definition. The different ways in which a selection of r things can be made from n things, without regard to the order of selection or arrangement, are called the *combinations* of the n things r at a time.

Thus the different combinations of the letters a, b, c, d three at a time are abc, abd, acd and bcd.

The number of combinations of n different things r at a time is denoted by the symbol ${}_nC_r$.

244. *To find the number of combinations of n different things taken r at a time.*

Let the different things be represented by the letters $a, b, c, \ldots$

Now in the combinations of the n letters r together the number in which a particular letter occurs is equal to the number of combinations of the remaining $n-1$ letters $r-1$ at a time. Hence in the whole number of combinations r together every letter occurs ${}_{n-1}C_{r-1}$ times, and therefore the total number of letters is $n\times{}_{n-1}C_{r-1}$; but, since there are r letters in each combination, the total number of the letters must be $r\times{}_nC_r$.

Hence $$r \times {}_nC_r = n \times {}_{n-1}C_{r-1}.$$

Since the above relation is true for all values of n and of r, we have in succession

$$(r-1) \times {}_{n-1}C_{r-1} = (n-1) \times {}_{n-2}C_{r-2},$$
$$(r-2) \times {}_{n-2}C_{r-2} = (n-2) \times {}_{n-3}C_{r-3},$$
$$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$$
$$2 \times {}_{n-r+2}C_2 = (n-r+2) \times {}_{n-r+1}C_1.$$

Also $${}_{n-r+1}C_1 = n-r+1.$$

Hence, by multiplying corresponding members of the above equations and cancelling the common factors, we have

$$\underline{|r} \times {}_nC_r = n(n-1)(n-2)\ldots\ldots(n-r+1),$$

that is $${}_nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}} = \frac{{}_nP_r}{\underline{|r}} \ldots\ldots\text{(i)}.$$

By multiplying the numerator and denominator of the fraction on the right by $\underline{|n-r}$, we have

$${}_nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1) \times \underline{|n-r}}{\underline{|r}\,\underline{|n-r}}$$
$$= \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

By comparing the above result with that obtained in Art. 242, it will be seen that ${}_nP_r = {}_nC_r \times \underline{|r}$. The relation ${}_nP_r = {}_nC_r \times \underline{|r}$ can however be at once obtained from the consideration that every combination of r different things would give rise to $\underline{|r}$ permutations, if the order of the letters were altered in every possible way.

Note. In order that the formula (ii) may be true when $r=n$, we must suppose that $\underline{|0} = 1$, since ${}_nC_n = 1$. We should also obtain the result $\underline{|0} = 1$ by putting $n=1$ in the relation $\underline{|n} = n\,\underline{|n-1}$.

245. Theorem. *The number of combinations of n different things r together is equal to the number of the combinations $n-r$ together.*

The proposition follows at once from the fact that whenever r things are taken out of n things, $n-r$ must be left, and if every set of r things differs in some particular from every other, the corresponding set of $n-r$ things will also differ in some particular from every other; and therefore the number of different ways of taking r things must be just the same as the number of different ways of leaving or taking $n-r$ things.

The result can also be obtained from the formula (ii) of the last Article.

For $\quad {}_nC_r = \dfrac{\lfloor n}{\lfloor r \, \lfloor n-r}$, and ${}_nC_{n-r} = \dfrac{\lfloor n}{\lfloor n-r \, \lfloor r}$.

It should be remarked that the first method of proof holds good when the n things are not all different, to which case the formulae of Art. 244 are not applicable.

Ex. 1. Find ${}_{10}C_4$, ${}_{12}C_9$ and ${}_{20}C_{17}$. *Ans.* 210, 220, 1140.

Ex. 2. If ${}_nC_6 = {}_nC_{12}$, find ${}_nC_{16}$. *Ans.* 153.

Ex. 3. Find n, having given that ${}_nC_5 = {}_nC_6$. *Ans.* 11.

Ex. 4. Find n, having given that $3 \times {}_nC_4 = 5 \times {}_{n-1}C_3$. *Ans.* 10.

Ex. 5. Find n, having given that ${}_nC_4 = 210$. *Ans.* 10.

Ex. 6. Find n and r having given that ${}_nP_r = 272$ and ${}_nC_r = 136$. *Ans.* $n=17$, $r=2$.

Ex. 7. Find n and r having given ${}_nC_{r-1} : {}_nC_r : {}_nC_{r+1} :: 2 : 3 : 4$. *Ans.* $n=34$, $r=14$.

Ex. 8. How many words each containing 3 consonants and 2 vowels can be formed from 6 consonants and 4 vowels?

The consonants can be chosen in ${}_6C_3 = 20$ ways; the vowels can be chosen in ${}_4C_2 = 6$ ways; hence 20×6 different sets of letters can be chosen, and each of these sets can be arranged in ${}_5P_5 = 120$ ways. Hence the required number is $20 \times 6 \times 120$.

Ex. 9. How many different sums can be formed with a sovereign, a half-sovereign, a crown, a half-crown, a shilling and a sixpence?

Number required $={}_6C_1+{}_6C_2+{}_6C_3+{}_6C_4+{}_6C_5+{}_6C_6=63$.

Ex. 10. Shew that, in the combinations of $2n$ different things n together, the number of combinations in which a particular thing occurs is equal to the number in which it does not occur.

Ex. 11. Shew that, in the combinations of $4n$ different things n together, the number of combinations in which a particular thing occurs is equal to one-third of the number in which it does not occur.

Ex. 12. Out of a party of 4 ladies and 3 gentlemen one game at lawn-tennis is to be arranged, each side consisting of one lady and one gentleman. In how many ways can this be done? *Ans.* 36.

Ex. 13. The figures 1, 2, 3, 4, 5 are written down in every possible order: how many of the numbers so formed will be greater than 23000? *Ans.* 90.

Ex. 14. At an election there are four candidates and three members to be elected, and an elector may vote for any number of candidates not greater than the number to be elected. In how many ways may an elector vote? *Ans.* 14.

246. Greatest value of ${}_nC_r$. *To find the greatest value of ${}_nC_r$ for a given value of n.*

From the formulae of Art. 244 we have

$$ {}_nC_r={}_nC_{r-1}\times\frac{n-r+1}{r}. $$

Hence ${}_nC_r \gtreqless {}_nC_{r-1}$, according as $n-r+1 \gtreqless r$; that is,

according as $r \lesseqgtr \frac{1}{2}(n+1)$.

Thus the number of combination of n things r together *increases* with r so long as r is less than $\frac{1}{2}(n+1)$.

If then n be even, ${}_nC_r$ is greatest when $r=\dfrac{n}{2}$.

If n be odd, ${}_nC_r \gtrless {}_nC_{r-1}$ as $r \gtrless \frac{1}{2}(n+1)$, and ${}_nC_r=$ ${}_nC_{r-1}$ when $r=\frac{1}{2}(n+1)$. Thus, when n is odd, ${}_nC_{\frac{1}{2}(n-1)}={}_nC_{\frac{1}{2}(n+1)}$ and these are the greatest values of ${}_nC_r$.

For example, if $n=10$, ${}_nC_r$ is greatest when $r=5$. Also if $n=11$, ${}_nC_r$ is the same for the values 5 and 6 of r, and ${}_nC_r$ is greater for these values than for any other value.

247. *To prove that* ${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$.

If the total number of combinations of $(n+1)$ things r together be divided into two groups according as they do or do not contain a certain particular letter, it is clear that the number of the combinations which do not contain the letter is the number of combinations r together of the remaining n things, and the number of the combinations which do contain the letter is the number of ways in which $r-1$ of the remaining n things can be taken. Thus

$${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}.$$

The above result can also be proved as follows:

From Art. 244 we have

$${}_nC_r + {}_nC_{r-1} = \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots\ldots r} + \frac{n(n-1)\ldots(n-r+2)}{1.2\ldots\ldots(r-1)}$$

$$= \frac{n(n-1)\ldots(n-r+2)}{1.2\ldots\ldots r}\{n-r+1+r\}$$

$$= \frac{(n+1)n(n-1)\ldots(n-r+2)}{1.2\ldots\ldots r} = {}_{n+1}C_r.$$

Ex. To prove that ${}_{n+1}P_r = {}_nP_r + r\,.\,{}_nP_{r-1}$.

A particular thing is absent in ${}_nP_r$ of the permutations of the $(n+1)$ things and occurs in ${}_nP_{r-1}$; also when it does occur it can be in either of r places. Hence

$${}_{n+1}P_r = {}_nP_r + r\,.\,{}_nP_{r-1}.$$

248. **Theorem.** *To prove that, if x and y be any two positive integers such that $x+y=m$, then will*

$${}_mC_n = {}_xC_n + {}_xC_{n-1}\,.\,{}_yC_1 + {}_xC_{n-2}\,.\,{}_yC_2 + \ldots + {}_xC_1\,.\,{}_yC_{n-1} + {}_yC_n.$$

Suppose that m letters $a, b, \ldots, p, q, \ldots$, are divided into two groups $a, b, \ldots$, and $p, q, \ldots$, there being x and y letters respectively in these groups. Then the whole number of sets of n out of the m things will clearly be

equal to the sum of the number of sets formed by taking n out of the first group and none out of the second, $n-1$ out of the first group and one out of the second, $n-2$ out of the first group and 2 out of the second, and so on.

Now n can be chosen from the first group in ${}_xC_n$ ways. Also $n-1$ can be chosen from the first group in ${}_xC_{n-1}$ ways, and any one of these sets of $n-1$ things can be taken with any one of the ${}_yC_1$ sets of 1 from the second group, so that the number of sets formed by taking $n-1$ from the first group and 1 from the second is ${}_xC_{n-1} \times {}_yC_1$.

Similarly, the number of sets formed by taking $n-2$ from the first group and 2 from the second is ${}_xC_{n-2} \times {}_yC_2$.

And, in general, the number of sets formed by taking $n-r$ from the first group and r from the second is ${}_xC_{n-r} \times {}_yC_r$.

Hence we have

$$ {}_{x+y}C_n = {}_xC_n + {}_xC_{n-1} \cdot {}_yC_1 + {}_xC_{n-2} \cdot {}_yC_2 + \dots + {}_xC_{n-r} \cdot {}_yC_r + \dots + {}_yC_n. $$

If x or y be less than n some of the terms on the right will vanish; for ${}_nC_r = 0$ if $r > n$.

249. **Vandermonde's Theorem.** From the last Article, if x, y and n be any positive integers such that $x+y$ is greater than n, we have, since ${}_nC_r = \frac{n_r}{\underline{|r}}$,

$$ \frac{(x+y)_n}{\underline{|n}} = \frac{x_n}{\underline{|n}} + \frac{x_{n-1}}{\underline{|n-1}} \cdot \frac{y_1}{\underline{|1}} + \frac{x_{n-2}}{\underline{|n-2}} \cdot \frac{y_2}{\underline{|2}} + \dots $$

$$ \dots + \frac{x_{n-r}}{\underline{|n-r}} \cdot \frac{y_r}{\underline{|r}} + \dots + \frac{y_n}{\underline{|n}}. $$

Multiply each side of the last equation by $\underline{|n}$, and we have

$$ (x+y)_n = x_n + nx_{n-1}y_1 + \frac{n(n-1)}{1.2} x_{n-2}y_2 + \dots $$

$$ \dots\dots + \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} x_{n-r}y_r + \dots\dots\dots + y_n. $$

The above has been proved on the supposition that x and y are positive integers such that $x+y$ is greater than n; and by means of the theorem of Art. 91, the proposition can be proved to be true for *all values* of x and y.

For the two expressions which are to be proved identical are only of the nth degree in x and y. But, if y has any particular integral value greater than n, the equation is known to be true for *any* positive integral value of x, and thus for *more than* n values; and hence it must be true for that value of y and any *value whatever* of x. Hence the proposition is true for any particular value whatever of x, and for more than n values of y; it must therefore be true for *all values* of x and for *all values* of y.

This proves Vandermonde's theorem, namely:—

If n be any positive integer, and x, y have any values whatever; then will

$$(x+y)_n = x_n + n\,.\,x_{n-1}\,y_1 + \frac{n(n-1)}{1\,.\,2}\,x_{n-2}\,y_2 + \dots$$

$$\dots + \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}\,x_{n-r}\,y_r + \dots\dots + y_n.$$

Homogeneous Products.

250. The number of different products each of r letters which can be made from n letters, when each letter may be repeated any number of times, is denoted by the symbol ${}_nH_r$.

For example, the homogeneous products of two dimensions formed by the three letters a, b, c are a^2, b^2, c^2, bc, ca, ab.

To find ${}_nH_r$.

Since in each of the r-dimensional products of n things there are r letters, the total number of letters in all the products will be ${}_nH_r \times r$; and, as each of the n letters occurs the same number of times, it follows that the

number of times any particular letter, a suppose, occurs is ${}_nH_r \times r \div n$.

Now consider all the terms which contain a at least once. If any one of these terms be divided by a the quotient will be of $r-1$ dimensions; and, when all the terms which contain a are so divided, we shall obtain without repetition all the possible homogeneous products of the n letters of $r-1$ dimensions. Now the homogeneous products of $r-1$ dimensions are in number ${}_nH_{r-1}$; and, by the above, the number of a's they contain is $\frac{r-1}{n} \times {}_nH_{r-1}$. Hence, taking into account the a which is a factor of each of the ${}_nH_{r-1}$ terms, the total number of a's which occur in all the r-dimensional products of the n letters is

$$ {}_nH_{r-1} + \frac{r-1}{n} \times {}_nH_{r-1}, \text{ that is } \frac{n+r-1}{n} {}_nH_{r-1}. $$

Hence equating the two expressions for the number of a's, we have

$$ \frac{r}{n} \times {}_nH_r = \frac{n+r-1}{n} \times {}_nH_{r-1}; $$

$$ \therefore \qquad {}_nH_r = \frac{n+r-1}{r} \times {}_nH_{r-1}. $$

Since the above relation is true for all values of n and r, we have in succession

$$ {}_nH_{r-1} = \frac{n+r-2}{r-1} \times {}_nH_{r-2}, $$

$$ {}_nH_{r-2} = \frac{n+r-3}{r-2} \times {}_nH_{r-3}, $$

$$ \ldots\ldots = \ldots\ldots\ldots\ldots\ldots $$

$$ {}_nH_2 = \frac{n+1}{2} \times {}_nH_1. $$

Also ${}_nH_1$ is obviously equal to n.

Hence, by multiplying and cancelling common factors, we have

$$_nH_r=\frac{n(n+1)\ldots\ldots(n+r-1)}{1.2\ldots\ldots\ldots r}\text{ [See also Art. 293].}$$

Ex. 1. Find the number of combinations three at a time of the letters a, b, c, d when the letters may be repeated. *Ans.* 20.

Ex. 2. Find the number of different combinations six at a time which can be formed from 6 a's, 6 b's, 6 c's and 6 d's. *Ans.* 84.

Ex. 3. Shew that $\quad {}_nH_r={}_{n-1}H_r+{}_nH_{r-1}$,
and deduce that

$${}_nH_r={}_nH_{r-1}+{}_{n-1}H_{r-1}+{}_{n-2}H_{r-1}+\ldots\ldots\ldots+{}_1H_{r-1}.$$

Ex. 4. Shew that

$${}_nH_r={}_{n-1}H_r+{}_{n-1}H_{r-1}+{}_{n-1}H_{r-2}+\ldots\ldots\ldots+{}_{n-1}H_1+1.$$

251. Many theorems relating to permutations and combinations are best proved by means of the binomial theorem: examples will be found in subsequent chapters. [See Art. 292.]

Ex. 1. Find the number of ways in which mn different things can be divided among n persons so that each may have m of them.

The number of ways in which the first set of m things can be given is ${}_{mn}C_m$; and, whatever set is given to the first, the second set can be given in ${}_{mn-m}C_m$ ways; so also, whatever sets are given to the first and second, the third set can be given in ${}_{mn-2m}C_m$ ways; and so on.

Hence the required number is

$$ {}_{mn}C_m\times{}_{m(n-1)}C_m\times{}_{m(n-2)}C_m\times\ldots\ldots\times{}_{2m}C_m\times{}_mC_m$$
$$=\frac{\lfloor mn}{\lfloor m\,\lfloor m(n-1)}\times\frac{\lfloor m(n-1)}{\lfloor m\,\lfloor m(n-2)}\times\frac{\lfloor m(n-2)}{\lfloor m\,\lfloor m(n-3)}\times\ldots\ldots\times\frac{\lfloor 2m}{\lfloor m\,\lfloor m}\times\frac{\lfloor m}{\lfloor m}$$
$$=\frac{\lfloor mn}{(\lfloor m)^n}.$$

Ex. 2. Prove that

$$1-{}_nC_1\cdot{}_nH_1+{}_nC_2\cdot{}_nH_2-{}_nC_3\cdot{}_nH_3+\ldots\ldots+(-1)^n{}_nC_n\cdot{}_nH_n=0.$$

Since $\displaystyle {}_nC_r\cdot{}_nH_r=\frac{n(n-1)n\ldots(n-r+1)}{\lfloor r}\cdot\frac{n(n+1)\ldots(n+r-1)}{\lfloor r}$

$$=\frac{n^2(n^2-1^2)\ldots\ldots(n^2-\overline{r-1}^2)}{1^2.2^2\ldots\ldots r^2},$$

we have to prove that

$$1-\frac{n^2}{1^2}+\frac{n^2(n^2-1^2)}{1^2.2^2}-\frac{n^2(n^2-1^2)(n^2-2^2)}{1^2.2^2.3^2}+\dots\dots$$

$$+(-1)^n\frac{n^2(n^2-1^2)\dots(n^2-\overline{n-1}^2)}{1^2.2^2\dots n^2}=0.$$

Now the first two terms $=-\dfrac{n^2-1^2}{1^2}$;

$\therefore$ ...... three ...... $=+\dfrac{(n^2-1^2)(n^2-2^2)}{1^2.2^2}$

$\therefore$ ...... four ...... $=-\dfrac{(n^2-1^2)(n^2-2^2)(n^2-3^2)}{1^2.2^2.3^2}$,

and so on.

Hence the sum of all the terms on the left

$$=(-1)^n\frac{(n^2-1^2)(n^2-2^2)\dots\dots(n^2-n^2)}{1^2.\,2^2\dots\dots\dots n^2}=0.$$

Ex. 3. Shew that n straight lines, no two of which are parallel and no three of which meet in a point, divide a plane into $\frac{1}{2}n(n+1)+1$ parts.

The nth straight line is cut by each of the other $n-1$ lines; and hence it is divided into n portions. Now there are two parts of the plane on the two sides of each of these portions of the nth line which would become one part if the nth line were away. Hence the plane is divided by n lines into n more parts than it is divided by $n-1$ lines.

Hence, if $F(x)$ be put for the number of parts into which the plane is divided by x straight lines, we have

$$F(n)=F(n-1)+n.$$

Similarly $\quad F(n-1)=F(n-2)+(n-1),$

$$\dots\dots=\dots\dots$$

and $\quad F(2)=F(1)+2.$

But obviously $\quad F(1)=2.$

Hence $\quad F(n)=2+2+3+4+\dots\dots+n$

$$=1+\tfrac{1}{2}n(n+1).$$

Ex. 4. Suppose n things to be given in a certain order of succession. Shew that the number of ways of taking a set of three things out of these, with the condition that no set shall contain any two things which were originally contiguous to each other is $\frac{1}{6}(n-2)(n-3)(n-4)$. Shew also that if the n given things are arranged cyclically, so that the nth is taken to be contiguous to the first, the number of sets is reduced to $\frac{1}{6}n(n-4)(n-5)$.

Consider the second case first.

Let the different things be represented by the letters $a, b, c, \ldots\ldots k, l$.

Suppose that a is taken first. Then, if either of the two letters next but one to a be taken second, any one of $n-5$ letters can be taken for the third of the set. If, however, the second letter is not next but one to a, but in either of the $n-5$ other possible places, there would be a choice of $n-6$ places for the third letter of the set. Hence the total number of ways of taking 3 letters *in order* a being first is $2(n-5)+(n-5)(n-6)$, that is $(n-4)(n-5)$. There is the same number when any one of the other letters is taken first; hence, as the order in which the three letters in a set are taken is indifferent, the total number of sets is $\frac{1}{6}n(n-4)(n-5)$.

In order to obtain the first case from the second, we have only to suppose that a and l are no longer contiguous. Hence the number in the first case is the same as that in the second with the addition of those sets which contain a and l, and there are $n-4$ of these. Hence the number in the first case is

$$\tfrac{1}{6}n(n-4)(n-5)+(n-4)=\tfrac{1}{6}(n-2)(n-3)(n-4).$$

Ex. 5. There are n letters and n directed envelopes; in how many ways could all the letters be put into the wrong envelopes?

Let the letters be denoted by the letters $a, b, c \ldots$ and the corresponding envelopes by $a', b', c', \ldots\ldots$

Let $F(n)$ be the required number of ways.

Then a can be put into any one of the $n-1$ envelopes $b', c', \ldots\ldots$ Suppose a is put into k'; then k may be put into a', in which case there will be $F(n-2)$ ways of putting all the others wrong. Also if a is put into k', the number of ways of disposing of the letters so that k is not put in a', b not in b', &c. is $F(n-1)$.

Hence the number of ways of satisfying the conditions when a is put into k' is $F(n-1)+F(n-2)$. The same is true when a is put into any other of the envelopes $b', c', \ldots$ Hence we have

$$F(n)=(n-1)\{F(n-1)+F(n-2)\};$$

$$\therefore \quad F(n)-nF(n-1)=-\{F(n-1)-(n-1)F(n-2)\}.$$

Similarly $F(n-1)-(n-1)F(n-2)=-\{F(n-2)-(n-2)F(n-3)\}$

$$\ldots\ldots=\ldots\ldots$$

$$F(3)-3F(2)=-\{F(2)-2F(1)\}.$$

But obviously $F(2)=1$ and $F(1)=0$;

$$\therefore \quad F(n)-nF(n-1)=(-1)^n.$$

Hence $$\frac{F(n)}{\lfloor n}-\frac{F(n-1)}{\lfloor n-1}=(-1)^n\cdot\frac{1}{\lfloor n}.$$

Similarly $$\frac{F(n-1)}{|\underline{n-1}} - \frac{F(n-2)}{|\underline{n-2}} = (-1)^{n-1}\frac{1}{|\underline{n-1}}.$$

$$\ldots\ldots = \ldots\ldots$$

and $$\frac{F(2)}{|\underline{2}} - \frac{F(1)}{|\underline{1}} = (-1)^2\frac{1}{|\underline{2}}.$$

Hence, by addition,

$$F(n) = |\underline{n}\left\{\frac{1}{|\underline{2}} - \frac{1}{|\underline{3}} + \frac{1}{|\underline{4}} - \ldots + \frac{(-1)^n}{|\underline{n}}\right\},$$

the number required.

EXAMPLES XXIII.

1. In how many different ways may twenty different things be divided among five persons so that each may have four?

2. A crew of an eight-oar has to be chosen out of eleven men, five of whom can row on the stroke side only, four on the bow-side only, and the remaining two on either side. How many different selections can be made?

3. There are three candidates for a certain office and twelve electors. In how many different ways is it possible for them all to vote; and in how many of these ways will the votes be equally divided between the candidates?

4. Shew that ${}_{4n}C_{2n} : {}_{2n}C_n$ is equal to

$$\frac{1 \,.\, 3 \,.\, 5 \ldots\ldots (4n-1)}{\{1 \,.\, 3 \,.\, 5 \ldots\ldots (2n-1)\}^2}.$$

5. Find the number of significant numbers which can be formed by using any number of the digits 0, 1, 2, 3, 4, but using each not more than once in each number.

6. Shew that in the permutations of n things r together, the number of permutations in which p particular things occur is ${}_{n-p}P_{r-p} \times {}_rP_p$.

7. There are n points in a plane, no three of which are in the same straight line; find the number of straight lines formed by joining them.

8. There are n points in a plane, of which no three are on a straight line except m which are all on the same straight line. Find the number of straight lines formed by joining the points.

9. There are n points in a plane, of which no three are on a straight line except m which are all on a straight line. Find the number of triangles formed by joining the points.

10. Shew that the number of different n-sided polygons formed by n straight lines in a plane, no three of which meet in a point, is $\frac{1}{2}\,\underline{|n-1}$.

11. There are n points in a plane which are joined in all possible ways by indefinite straight lines, and no two of these joining lines are parallel and no three of them meet in a point. Find the number of points of intersection, exclusive of the n given points.

12. Through each of the angular points of a triangle m straight lines are drawn, and no two of the $3m$ lines are parallel; also no three, one from each angular point, meet in a point. Find the number of points of intersection.

13. The streets of a city are arranged like the lines of a chess-board. There are m streets running north and south, and n east and west. Find the number of ways in which a man can travel from the N.W. corner to the S.E. corner, going the shortest possible distance.

14. How many triangles are there whose angular points are at the angular points of a given polygon of n sides but none of whose sides are sides of the polygon?

15. Shew that $2n$ persons may be seated at two round tables, n persons being seated at each, in $\dfrac{\underline{|2n}}{n^2}$ different ways.

16. A parallelogram is cut by two sets of m lines parallel to its sides: shew that the number of parallelograms thus formed is $\frac{1}{4}(m+1)^2(m+2)^2$.

17. Find the number of ways in which p positive signs and n negative signs may be placed in a row so that no two negative signs shall be together.

18. Shew that the number of ways of putting m things in $n+1$ places, there being no restriction as to the number in each place, is $(m+n)\,!/m\,!\,n\,!$

19. Shew that $2n$ things can be divided into groups of n pairs in $\dfrac{\lfloor 2n}{2^n \lfloor n}$ ways.

20. Find the number of ways in which mn things can be divided into m sets each of n things.

21. Shew that n planes through the centre of a sphere, no three of which pass through the same diameter, will divide the surface of the sphere into $n^2 - n + 2$ parts.

22. Shew that the number of parts into which an infinite plane is divided by $m + n$ straight lines, m of which pass through one point and the remaining n through another, is $mn + 2m + 2n - 1$, provided no two of the lines be parallel or coincident.

23. Find the number of parts into which a sphere is divided by $m + n$ planes through its centre, m of which pass through one diameter and the remaining n through another, no plane passing through both these diameters.

24. Find the number of parts into which a sphere is divided by $a + b + c + \ldots$ planes through the centre, a of the planes passing through one given diameter, b through a second, c through a third, and so on; and no plane passing through more than one of these given diameters.

25. Shew that n planes, no four of which meet in a point, divide infinite space into $\frac{1}{6}(n^3 + 5n + 6)$ different regions.

26. Prove that if each of m points in one straight line be joined to each of n points in another, by straight lines terminated by the points; then, excluding the given points, the lines will intersect $\frac{1}{4}mn(m-1)(n-1)$ times.

27. No four of n points lying in a plane are on the same circle. Through every three of the points a circle is drawn, and no three of the circles have a common point other than one of the original n points. Shew that the circles intersect in $\frac{1}{12}n(n-1)(n-2)(n-3)(n-4)(2n-1)$ points besides the original n points, assuming that every circle intersects every other circle in two points.

CHAPTER XX.

THE BINOMIAL THEOREM.

252. WE have already [Art. 67] proved that the continued product of any number of algebraical expressions is the sum of all the partial products which can be obtained by multiplying any term of the first, any term of the second, any term of the third, &c.

253. **Binomial Theorem.** Suppose that we have n factors each of which is $a+b$.

If we take a letter from each of the factors of

$$(a+b)(a+b)(a+b)\ldots\ldots$$

and multiply them all together, we shall obtain a term of the continued product; and if we do this in every possible way we shall obtain all the terms of the continued product. [Art. 67.]

Now we can take the letter a every time, and this can be done in only one way; hence a^n is a term of the product.

The letter b can be taken once, and a the remaining $(n-1)$ times, and the number of ways in which one b can be taken is the number of ways of taking 1 out of n things, so that the number is ${}_nC_1$: hence we have

$${}_nC_1 \cdot a^{n-1}b.$$

Again, the letter b can be taken twice, and a the remaining $(n-2)$ times, and the number of ways in which two b's can be taken is the number of ways of taking 2 out of n things, so that the number is ${}_nC_2$: hence we have

$$ {}_nC_2 \,.\, a^{n-2}b^2. $$

And, in general, b can be taken r times (where r is any positive integer not greater than n) and a the remaining $n-r$ times, and the number of ways in which r b's can be taken is the number of ways of taking r out of n things, so that the number is ${}_nC_r$: hence we have

$$ {}_nC_r \,.\, a^{n-r}b^r. $$

Thus $\quad (a+b)(a+b)(a+b)\ldots\ldots$ to n factors

$$ = a^n + {}_nC_1 \,.\, a^{n-1}b + {}_nC_2 \,.\, a^{n-2}b^2 + \ldots\ldots + {}_nC_r \,.\, a^{n-r}b^r + \ldots $$

the last term being ${}_nC_n a^{n-n} b^n$, i.e. b^n.

Hence, when n is any positive integer, we have

$$ (a+b)^n = a^n + {}_nC_1 \,.\, a^{n-1}b + {}_nC_2 \,.\, a^{n-2}b^2 + \ldots $$
$$ \ldots + {}_nC_r \,.\, a^{n-r}b^r + \ldots + b^n. $$

The above formula is called the *Binomial Theorem.*

If we substitute the known values [see Art. 244] of ${}_nC_1$, ${}_nC_2$, ${}_nC_3$,... in the series on the right, we obtain the form in which the theorem is usually given, namely

$$ (a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\,.\,2} a^{n-2}b^2 + \ldots $$
$$ \ldots + \frac{\underline{|n}}{\underline{|r}\;\underline{|n-r}} a^{n-r}b^r + \ldots + b^n. $$

The series on the right is called the *expansion* of $(a+b)^n$.

254. **Proof by Induction.** The Binomial Theorem may also be proved by induction, as follows.

We have to prove that, when n is any positive integer,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}b^2 + \ldots\ldots$$

$$\ldots + \frac{\lfloor n}{\lfloor r \, \lfloor n-r} a^{n-r}b^r + \ldots + b^n,$$

or that

$$(a+b)^n = a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + \ldots + {}_nC_r a^{n-r}b^r + \ldots + b^n.$$

Now if we *assume* that the theorem is true when the index is n, and multiply by another factor $a+b$, we have, when like terms of the product are collected,

$$(a+b)^{n+1} = a^{n+1} + (1 + {}_nC_1)\,a^nb + ({}_nC_1 + {}_nC_2)\,a^{n-1}b^2 + \ldots$$
$$\ldots + ({}_nC_{r-1} + {}_nC_r)\,a^{n-r+1}b^r + \ldots + b^{n+1}.$$

Now $\qquad 1 + {}_nC_1 = 1 + n = {}_{n+1}C_1,$

$${}_nC_1 + {}_nC_2 = n + \frac{n(n-1)}{1.2} = \frac{(n+1)n}{1.2} = {}_{n+1}C_2,$$

and, in general,

$${}_nC_{r-1} + {}_nC_r = {}_{n+1}C_r \text{ [Art. 247]}.$$

Hence

$$(a+b)^{n+1} = a^{n+1} + {}_{n+1}C_1 \,.\, a^nb + {}_{n+1}C_2 \,.\, a^{n-1}b^2 + \ldots$$
$$+ {}_{n+1}C_r \,.\, a^{n+1-r}b^r + \ldots + b^{n+1}.$$

Thus *if* the theorem be true for any value of n, it will be true for the next greater value.

Now the theorem is obviously true when $n=1$. Hence it must be true when $n=2$; and being true when $n=2$, it must be true when $n=3$; and so on indefinitely. The theorem is therefore true for all positive integral values of n.

Ex. 1. Expand $(a+b)^4$.

We have

$$(a+b)^4 = a^4 + 4a^3b + \frac{4.3}{1.2}a^2b^2 + \frac{4.3.2}{1.2.3}ab^3 + \frac{4.3.2.1}{1.2.3.4}b^4$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Ex. 2. Expand $(2x-y)^3$.

Put $2x$ for a, and $-y$ for b in the general formula: then

$$(2x-y)^3=(2x)^3+3(2x)^2(-y)+\frac{3.2}{1.2}(2x)(-y)^2+\frac{3.2.1}{1.2.3}(-y)^3$$
$$=8x^3-12x^2y+6xy^2-y^3.$$

Ex. 3. Expand $(a-b)^n$.

Change the sign of b in the general formula; then we have

$$(a-b)^n=a^n+na^{n-1}(-b)+\frac{n(n-1)}{1.2}a^{n-2}(-b)^2+\ldots\ldots$$
$$\ldots\ldots+\frac{\lfloor n}{\lfloor r\,\lfloor n-r}a^{n-r}(-b)^r+\ldots\ldots+(-b)^n$$
$$=a^n-na^{n-1}b+\frac{n(n-1)}{1.2}a^{n-2}b^2+\ldots\ldots$$
$$\ldots\ldots+(-1)^r\frac{\lfloor n}{\lfloor r\,\lfloor n-r}a^{n-r}b^r+\ldots\ldots+(-1)^nb^n.$$

255. **General term.** By the preceding articles we see that any term of the expansion of $(a+b)^n$ by the Binomial Theorem will be found by giving a suitable value to r in

$$\frac{\lfloor n}{\lfloor r\,\lfloor n-r}a^{n-r}b^r.$$

On this account the above is called the *general term* of the series. It should be noticed that the term is the $(r+1)$th term from the beginning. [See Note Art. 244.]

256. **Coefficients of terms equidistant respectively from the beginning and the end are equal.** In the expansion of $(a+b)^n$ by the Binomial Theorem, the $(r+1)$th term from the beginning and the $(r+1)$th term from the end are respectively

$${}_nC_r\,.\,a^{n-r}b^r \text{ and } {}_nC_{n-r}\,.\,a^rb^{n-r}.$$

But $${}_nC_r={}_nC_{n-r}. \quad \text{[Art. 245.]}$$

Hence, in the expansion of $(a+b)^n$, *the coefficients of any two terms equidistant respectively from the beginning and the end are equal.*

This result follows, however, at once from the fact that $(a+b)^n$, and therefore also its expansion, would be unaltered by an interchange of the letters a and b; and hence the coefficient of $a^{n-r}b^r$ must be equal to the coefficient of $b^{n-r}a^r$.

257. If, in the formula of Art. 253, we put $a=1$ and $b=x$, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}x^r + \ldots + x^n.$$

This is the most simple form of the Binomial Theorem, and the one which is generally employed.

The above form includes all possible cases: if, for example, we want to find $(a+b)^n$ by means of it, we have

$$(a+b)^n = \left\{a\left(1+\frac{b}{a}\right)\right\}^n = a^n\left(1+\frac{b}{a}\right)^n$$

$$= a^n\left\{1 + n\frac{b}{a} + \frac{n(n-1)}{1.2}\left(\frac{b}{a}\right)^2 + \ldots\right\}$$

$$= a^n + na^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}b^2 + \ldots.$$

258. **Greatest term of a binomial expansion.** In the expansion of $(1+x)^n$, the $(r+1)$th term is formed from the rth by multiplying by $\dfrac{n-r+1}{r}x$.

Now $\dfrac{n-r+1}{r}x = \left(\dfrac{n+1}{r}-1\right)x$, and $\dfrac{n+1}{r}$ clearly diminishes as r increases; hence $\dfrac{n-r+1}{r}x$ diminishes as r is increased. If $\dfrac{n-r+1}{r}x$ be less than 1 for any

value of r, the $(r+1)$th term will be less than the rth. In order therefore that the rth term of the expansion may be the *greatest* we must have

$$\frac{n-r+1}{r}x<1, \text{ and } \frac{n-\overline{r-1}+1}{r-1}x>1.$$

Hence $r>\dfrac{(n+1)x}{x+1}$, and $r<\dfrac{(n+1)x}{x+1}+1$.

The absolute values of the terms in the expansion of $(1+x)^n$ will not be altered by changing the sign of x; and hence the rth term of $(1-x)^n$ will also be greatest in absolute magnitude if

$$r>\frac{(n+1)x}{x+1}, \text{ and } r<\frac{(n+1)x}{x+1}+1.$$

If $r=\dfrac{(n+1)x}{x+1}$, then $\dfrac{n-r+1}{r}x=1$; and hence there is no *one* term which is the greatest, but the rth and $\overline{r+1}|$th terms are equal, and these are greater than any of the other terms.

Since $\qquad (a+x)^n=a^n\left(1+\dfrac{x}{a}\right)^n,$

the rth term of $(a+x)^n$ is the greatest when

$$r>\frac{(n+1)\frac{x}{a}}{\frac{x}{a}+1} \text{ and } <\frac{(n+1)\frac{x}{a}}{\frac{x}{a}+1}+1.$$

Ex. 1. Find the greatest term in the expansion of $(1+x)^{20}$, when $x=\dfrac{1}{4}$.

The rth term is the greatest, if $r>\dfrac{21}{5}$ and $r<1+\dfrac{21}{5}$. Hence the fifth term is the greatest.

Ex. 2. Find the greatest term in the expansion of $(1+x)^{10}$, when $x=\dfrac{5}{6}$.

The rth term is the greatest, if $r>5$ and <6. Thus there is no *one* term which is the greatest, but the 5th and 6th terms of the

expansion are equal to one another and greater than any of the other terms.

Ex. 3. Find the greatest term in the expansion of $(10+3x)^{15}$ when $x=4$. *Ans.* The ninth term.

The **greatest coefficient** of a binomial expansion can be found in a similar manner. For in the expansion of $(1 \pm x)^n$ the coefficient of the $(r+1)$th term is formed from that of the rth by multiplying by $\pm\dfrac{n-r+1}{r}$. Hence the rth coefficient will be the greatest in absolute magnitude,

if $$\frac{n-r+1}{r} < 1 \text{ and } \frac{n-\overline{r-1}+1}{r-1} > 1.$$

That is if $r > \dfrac{n+1}{2}$ and $< 1 + \dfrac{n+1}{2}$.

Hence when n is even, the coefficient of the rth term is the greatest when $r = \dfrac{n}{2}+1$; and when n is odd, the coefficients of the $\dfrac{n+1}{2}$th and $\dfrac{n+3}{2}$th terms are equal to one another and are greater than any of the other terms.

For example, in $(1+x)^{20}$ the coefficient of the 11th term is the greatest; and in $(1+x)^{11}$ the coefficients of the 6th and 7th terms are greater than any of the others.

EXAMPLES XXIV.

Write out the following expansions:

1. $(x+a)^5$. 2. $(2a-x)^5$. 3. $(1-x^2)^6$.

4. $(2a-3a^2)^4$. 5. $(2x^2-3)^4$. 6. $(x^2-2y^3)^5$.

7. Find the third term of $(x-3y)^{10}$.

8. Find the fifth term of $(3x-4)^{20}$.

9. Find the twenty-first term of $(2-x)^{22}$.

10 Find the fortieth term of $(x-y)^{42}$.

11. Find the middle term of $(1+x)^8$.

12. Find the middle terms of $(1+x)^{21}$.

13. Find the general term of $(x-3y)^n$.

14. Find the general term of $(x^2+y^3)^n$.

15. Write down the first three terms and the last three terms of $(3x-2y)^{15}$.

16. Find the term of $(1+x)^{12}$ which has the greatest coefficient.

17. Find the two terms of $(1+x)^{15}$ which have the greatest coefficients.

18. Shew that the coefficient of x^n in the expansion of $(1+x)^{2n}$ is double the coefficient of x^n in the expansion of

$$(1+x)^{2n-1}.$$

19. Shew that the middle term of $(1+x)^{2n}$ is

$$\frac{1\,.\,3\,.\,5\,...\,(2n-1)}{\underline{|n}}\,2^n x^n.$$

20. Employ the binomial theorem to find 99^4, 51^4 and 999^3.

21. Shew that the coefficient of x^r in the expansion of $\left(x+\frac{1}{x}\right)^n$ is $\dfrac{\underline{|n}}{\underline{|\tfrac{1}{2}(n+r)}\,\underline{|\tfrac{1}{2}(n-r)}}$.

22. Find the middle term of $\left(x-\frac{1}{x}\right)^{2n}$.

23. The coefficients of the 5th, 6th and 7th terms of the expansion of $(1+x)^n$ are in arithmetical progression: find n.

24. For what value of n are the coefficients of the second, third and fourth terms of the expansion of $(1+x)^n$ in arithmetical progression?

25. If a be the sum of the odd terms and b the sum of the even terms of the expansion of $(1+x)^n$, shew that

$$(1-x^2)^n = a^2 - b^2.$$

259. Properties of the coefficients of a binomial expansion.

It will be convenient to write the Binomial Theorem in the form

$$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \dots + c_r x^r + \dots c_n x^n \text{ ......(i)},$$

where, as we have seen, $c_0 = c_n = 1$; $c_1 = c_{n-1} = n$;

and, in general, $\quad c_r = c_{n-r} = \dfrac{\lfloor n}{\lfloor r \, \lfloor n-r}$.

I. Put $x = 1$ in (i); then

$$2^n = c_0 + c_1 + c_2 + \dots + c_n.$$

Thus *the sum of the coefficients in the expansion of* $(1+x)^n$ *is* 2^n.

II. Put $x = -1$ in (i); then

$$(1-1)^n = c_0 - c_1 + c_2 - \dots\dots + (-1)^n c_n;$$

$$\therefore 0 = (c_0 + c_2 + c_4 + \dots) - (c_1 + c_3 + c_5 + \dots).$$

Thus *the sum of the coefficients of the odd terms of a binomial expansion is equal to the sum of the coefficients of the even terms.*

III. Since $c_r = c_{n-r}$, we have

$$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \dots + c_r x^r + \dots + c_n x^n,$$

and $\quad (1+x)^n = c_n + c_{n-1} x + c_{n-2} x^2 + \dots c_{n-r} x^r + \dots + c_0 x^n.$

The coefficient of x^n in the product of the two series on the right is equal to

$$c_0^2 + c_1^2 + c_2^2 + \dots\dots + c_n^2.$$

Hence [Art. 91] $c_0^2 + c_1^2 + \dots + c_r^2 + \dots\dots + c_n^2$

is equal to the coefficient of x^n in $(1+x)^n \times (1+x)^n$, that is in $(1+x)^{2n}$; and this coefficient is $\dfrac{\lfloor 2n}{\lfloor n \, \lfloor n}$.

Hence the *sum of the squares* of the coefficients in the expansion of $(1+x)^n$ is $\dfrac{\lfloor 2n}{\lfloor n \lfloor n}$.

IV. As in III, we have

$$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n,$$

and $$(1-x)^n = c_n - c_{n-1}x + c_{n-2}x^2 + \dots + (-1)^n c_0 x^n.$$

The coefficient of x^n in the product of the two series on the right is equal to

$$(-1)^n \{c_0^2 - c_1^2 + c_2^2 - \dots\dots + (-1)^n c_n^2\}.$$

The coefficient of x^n in $(1+x)^n \times (1-x)^n$, that is in $(1-x^2)^n$, is zero if n be odd, and is equal to $(-1)^{\frac{n}{2}} \cdot \dfrac{\lfloor n}{\lfloor \frac{1}{2}n \lfloor \frac{1}{2}n}$ if n be even.

Hence $c_0^2 - c_1^2 + c_2^2 - \dots + (-1)^n c_n^2$ is zero or $(-1)^{\frac{n}{2}} n!/(\frac{1}{2}n!)^2$, according as n is odd or even.

Ex. 1. **Shew that**

$$c_1 + 2c_2 + 3c_3 + \dots + rc_r + \dots + nc_n = n2^{n-1}.$$

We have $$c_1 + 2c_2 + 3c_3 + \dots + nc_n$$

$$= n + 2\frac{n(n-1)}{1.2} + 3\frac{n(n-1)(n-2)}{1.2.3} + \dots\dots + r\frac{\lfloor n}{\lfloor r \lfloor n-r} + \dots + n$$

$$= n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{1.2} + \dots\dots + \frac{\lfloor n-1}{\lfloor r-1 \lfloor n-r} + \dots\dots + 1\right\}$$

$$= n(1+1)^{n-1} = n2^{n-1}.$$

Ex. 2. **Shew that** $c_0 - \frac{1}{2}c_1 + \frac{1}{3}c_2 - \dots\dots + (-1)^n \dfrac{c_n}{n+1} = \dfrac{1}{n+1}$.

We have $$c_0 - \frac{1}{2}c_1 + \frac{1}{3}c_2 - \&c. = 1 - \frac{1}{2}n + \frac{1}{3}\frac{n(n-1)}{1.2} - \&c.$$

$$=\frac{1}{n+1}\left\{n+1-\frac{(n+1)n}{1.2}+\frac{(n+1)n(n-1)}{1.2.3}-\ldots\ldots+(-1)^n\right\}$$

$$=\frac{1}{n+1}-\frac{1}{n+1}\left\{1-(n+1)+\frac{(n+1)n}{1.2}-\frac{(n+1)n(n-1)}{1.2.3}+\ldots\ldots+(-1)^{n+1}\right\}$$

$$=\frac{1}{n+1}-\frac{1}{n+1}(1-1)^{n+1}=\frac{1}{n+1}.$$

Ex. 3. Shew that, if n be any positive integer,

$$\frac{c_0}{x}-\frac{c_1}{x+1}+\frac{c_2}{x+2}-\ldots\ldots+(-1)^n\frac{c_n}{x+n}=\frac{\lfloor n}{x(x+1)\ldots(x+n)}.$$

Assume that

$$\frac{1}{x}-\frac{{}_nC_1}{x+1}+\frac{{}_nC_2}{x+2}-\ldots\ldots+(-1)^n\frac{{}_nC_n}{x+n}=\frac{\lfloor n}{x(x+1)\ldots(x+n)},$$

for *all values* of x, and for any particular value of n.

Change x into $x+1$; then

$$\frac{1}{x+1}-\frac{{}_nC_1}{x+2}+\frac{{}_nC_2}{x+3}-\ldots\ldots\ldots+(-1)^n\frac{{}_nC_n}{x+n+1}$$
$$=\frac{\lfloor n}{(x+1)(x+2)\ldots(x+n+1)}.$$

Hence, by subtraction,

$$\frac{1}{x}-\frac{{}_nC_1+1}{x+1}+\frac{{}_nC_2+{}_nC_1}{x+2}-\ldots\ldots+(-1)^r\frac{{}_nC_r+{}_nC_{r-1}}{x+r}+\ldots\ldots$$
$$+(-1)^{n+1}\frac{1}{x+n+1}=\frac{\lfloor n+1}{x(x+1)\ldots\ldots(x+n+1)}.$$

But ${}_nC_r+{}_nC_{r-1}={}_{n+1}C_r$, for all values of r [Art. 247].

Hence we have

$$\frac{1}{x}-\frac{{}_{n+1}C_1}{x+1}+\frac{{}_{n+1}C_2}{x+2}-\ldots\ldots+(-1)^{n+1}\frac{{}_{n+1}C_{n+1}}{x+n+1}=\frac{\lfloor n+1}{x(x+1)\ldots(x+n+1)}.$$

Hence *if* the theorem be true for any particular value of n it will be true for the next greater value. But the theorem is obviously true for all values of x when $n=1$: it is therefore true for all positive integral values of n. [See also Art. 297, Ex. 3.]

By giving particular values to x we obtain relations between c_0, c_1, &c. For example:

Put $x=1$; then we have

$$\frac{c_0}{1}-\frac{c_1}{2}+\frac{c_2}{3}-\ldots\ldots=\frac{1}{n+1}.$$

Put $x=\frac{1}{2}$; then $\frac{c_0}{1}-\frac{c_1}{3}+\frac{c_2}{5}-\ldots\ldots=\frac{2^n \lfloor n}{1.3.5\ldots(2n+1)}$.

Ex. 4. Shew that

$$c_0a-c_1(a-1)+c_2(a-2)-c_3(a-3)+\ldots\ldots+(-1)^n c_n(a-n)=0,$$

and that

$$c_0a^2-c_1(a-1)^2+c_2(a-2)^2-c_3(a-3)^2+\ldots\ldots+(-1)^n c_n(a-n)^2=0.$$

We have from II., if n be any positive integer,

$$1-n+\frac{n(n-1)}{1.2}-\frac{n(n-1)(n-2)}{1.2.3}+\ldots\ldots+(-1)^n=0\ldots\ldots\ldots\ldots\text{(i)}.$$

Hence, if $n>1$

$$1-(n-1)+\frac{(n-1)(n-2)}{1.2}-\ldots\ldots+(-1)^{n-1}=0\ldots\ldots\ldots\ldots\text{(ii)}.$$

Multiply (i) by a and (ii) by n and add; then

$$a-n(a-1)+\frac{n(n-1)}{1.2}(a-2)-\ldots\ldots+(-1)^n(a-n)=0\ \ldots\ldots\ldots\text{(iii)},$$

where n is >1.

Change a into $a-1$ and n into $n-1$ in (iii); then, n *being* >2, we have

$$a-1-(n-1)(a-2)+\ldots\ldots+(-1)^{n-1}(a-n)=0\ \ldots\ldots\ldots\text{(iv)}.$$

Now multiply (iii) by a and (iv) by n and add; then

$$a^2-n(a-1)^2+\frac{n(n-1)}{1.2}(a-2)^2-\ldots\ldots+(-1)^n(a-n)^2=0,$$

provided n is greater than 2.

By proceeding in this way we may prove that

$$a^p-n(a-1)^p+\frac{n(n-1)}{1.2}(a-2)^p-\ldots\ldots+(-1)^n(a-n)^p=0,$$

provided that p is any positive integer *less than n.*
[See also Art. 305.]

260. Continued product of n binomial factors of the form $x+a$, $x+b$, $x+c$, &c.

It will be convenient to use the following notation:

S_1 is written for $a+b+c+\ldots$, the sum of all the letters taken *one* at a time. S_2 is written for $ab+ac+\ldots$, the sum of all the products which can be obtained by

taking the letters *two* at a time. And, in general, S_r is written for the sum of all the products which can be obtained by taking the letters r at a time.

Now, if we take a letter from each of the binomial factors of

$$(x+a)(x+b)(x+c)(x+d)\ldots\ldots,$$

and multiply them all together, we shall obtain a term of the continued product; and, if we do this in every possible way, we shall obtain all the terms of the continued product.

We can take x every time, and this can be done in only one way; hence x^n is a term of the continued product.

We can take any *one* of the letters $a, b, c\ldots$, and x from all the remaining $n-1$ binomial factors; we thus have the terms ax^{n-1}, bx^{n-1}, cx^{n-1}, &c., and on the whole $S_1 . x^{n-1}$.

Again, we can take any *two* of the letters $a, b, c\ldots$, and x from all the remaining $n-2$ binomial factors; we thus have the terms abx^{n-2}, acx^{n-2}, &c., and on the whole $S_2 . x^{n-2}$.

And, in general, we can take any r of the letters $a, b, c\ldots$, and x from all the remaining $n-r$ binomial factors; and we thus have $S_r . x^{n-r}$.

Hence $(x+a)(x+b)(x+c)\ldots\ldots$

$$= x^n + S_1 . x^{n-1} + S_2 . x^{n-2} + \ldots + S_r . x^{n-r} + \ldots$$

the last term being $abcd\ldots\ldots$, the product of all the letters a, b, c, d, &c.

By changing the signs of a, b, c, &c., the signs of S_1, S_3, S_5, &c. will be changed, but the signs of S_2, S_4, S_6, &c. will be unaltered.

Hence we have

$$(x-c)\ldots\ldots$$

$$. x^{n-2} - \ldots + (-1)^r S_r . x^{n-r} \ldots + (-1)^n abcd\ldots$$

261. **Vandermonde's Theorem.** The following proof of Vandermonde's Theorem is due to Professor Cayley*. [See also Art. 249.]

We have to prove that if n be any positive integer, and a and b have any values whatever; then will

$$(a+b)_n = a_n + na_{n-1}b_1 + \frac{n(n-1)}{1.2}a_{n-2}b_2 + \dots$$
$$+ \frac{\lfloor n}{\lfloor r \lfloor n-r} a_{n-r}b_r + \dots + b_n.$$

Assume the theorem to be true for any particular value of n. Multiply the left side by $a+b-n$; it will then become $(a+b)_{n+1}$. Multiply the successive terms of the series on the right also by $a+b-n$ but arranged as follows:—

for the first term $\{(a-n)+b\}$;

for the second $\{(a-n+1)+(b-1)\}$;

and for the rth $\{(a-n+r-1)+(b-r+1)\}$.

We shall then have

$$\begin{aligned}(a+b)_{n+1} = a_n\{(a-n)+b\} &+ {}_nC_1 . a_{n-1}b_1\{(a-n+1)+(b-1)\} \\ &+ {}_nC_2 . a_{n-2}b_2\{(a-n+2)+(b-2)\} + \dots \\ &+ {}_nC_{r-1} . a_{n-r+1}b_{r-1}\{(a-n+r-1)+(b-r+1)\} \\ &+ {}_nC_r . a_{n-r}b_r\{(a-n+r)+(b-r)\} \\ &+ \dots + b_n\{a+(b-n)\}.\end{aligned}$$

Now $a_n\{(a-n)+b\} = a_{n+1} + a_n b_1$,

$${}_nC_1 . a_{n-1}b_1\{(a-n+1)+(b-1)\} = {}_nC_1(a_n b_1 + a_{n-1}b_2),$$

$${}_nC_{r-1} . a_{n-r+1}b_{r-1}\{(a-n+r-1)+(b-r+1)\}$$
$$= {}_nC_{r-1}(a_{n-r+2}b_{r-1} + a_{n-r+1}b_r),$$

$${}_nC_r . a_{n-r}b_r\{(a-n+r)+(b-r)\} = {}_nC_r(a_{n-r+1}b_r + a_{n-r}b_{r+1})$$

. . . .

$$b_n\{a+(b-n)\} = a_1 b_n + b_{n+1}.$$

* *Messenger of Mathematics*, Vol. v.

Hence $(a+b)_{n+1} = a_{n+1} + (1 + {}_nC_1)\, a_n b_1 + \ldots$

$$\ldots + ({}_nC_{r-1} + {}_nC_r)\, a_{n+1-r} b_r + \ldots + b_{n+1}$$

$$= a_{n+1} + {}_{n+1}C_1 a_n b_1 + \ldots + {}_{n+1}C_r a_{n+1-r} b_r + \ldots + b_{n+1},$$

since $${}_nC_{r-1} + {}_nC_r = {}_{n+1}C_r.$$

Thus, *if* the theorem be true for any particular value of n, it will also be true for the next greater value. But it is obviously true when $n=1$; it must therefore be true when $n=2$; and so on indefinitely. Thus the theorem is true for all positive integral values of n.

262. **The Multinomial Theorem.** The expansion of the nth power of the multinomial expression $a+b+c+\ldots$ can be found by means of the Binomial Theorem.

For the general term in the expansion of

$(a+b+c+d+\ldots)^n$, that is of $\{a+(b+c+d+\ldots)\}^n$,

by the Binomial Theorem is

$$\frac{\lfloor n}{\lfloor r \lfloor n-r} a^r (b+c+d+\ldots)^{n-r}.$$

Similarly the general term in the expansion of

$$(b+c+d+\ldots)^{n-r}$$

by the Binomial Theorem is

$$\frac{\lfloor n-r}{\lfloor s \lfloor n-r-s} b^s (c+d+\ldots)^{n-r-s}.$$

The general term in the expansion of $(c+d+\ldots)^{n-r-s}$ by the Binomial Theorem is

$$\frac{\lfloor n-r-s}{\lfloor t \lfloor n-r-s-t} c^t (d+\ldots)^{n-r-s-t}.$$

Hence the general term in the expansion of

$$(a+b+c+d+\ldots)^n$$

is

$$\frac{\lfloor n}{\lfloor r \lfloor n-r} \times \frac{\lfloor n-r}{\lfloor s \lfloor n-r-s} \times \frac{\lfloor n-r-s}{\lfloor t \lfloor n-r-s-t} \ldots a^r b^s c^t \ldots,$$

that is
$$\frac{\lfloor n}{\lfloor r \lfloor s \lfloor t \ldots} a^r b^s c^t \ldots,$$

where each of r, s, $t \ldots$ is zero or a positive integer, and
$$r + s + t + \ldots = n.$$

The above result can however be at once obtained by the method of Art. 253, as follows.

We know [Art. 67] that the continued product
$$(a + b + c + \ldots)(a + b + c + \ldots)(a + b + c + \ldots)\ldots$$
is the sum of all the different partial products which can be obtained by multiplying any term from the first multinomial factor, any term from the second, any term from the third, &c.

The term $a^r b^s c^t \ldots$ will therefore be obtained by taking a from any r of the n factors, which can be done in ${}_nC_r$ different ways; then taking b from any s of the remaining $n-r$ factors, which can be done in ${}_{n-r}C_s$ different ways; then taking c from any t of the remaining $n-r-s$ factors, which can be done in ${}_{n-r-s}C_t$ different ways; and so on. Hence the total number of ways in which the term $a^r b^s c^t \ldots$ will be obtained, which is the coefficient of the term in the required expansion, must be
$${}_nC_r \times {}_{n-r}C_s \times {}_{n-r-s}C_t \times \ldots,$$
that is
$$\frac{\lfloor n}{\lfloor r \lfloor n-r} \times \frac{\lfloor n-r}{\lfloor s \lfloor n-r-s} \times \frac{\lfloor n-r-s}{\lfloor t \lfloor n-r-s-t} \times \ldots = \frac{\lfloor n}{\lfloor r \lfloor s \lfloor t \ldots}.$$

Hence the general term in the expansion of $(a + b + c + \ldots)^n$ is
$$\frac{\lfloor n}{\lfloor r \lfloor s \lfloor t \ldots} a^r b^s c^t \ldots.$$

Ex. 1. Find the coefficient of abc in the expansion of $(a+b+c)^3$.

The required coefficient $=\dfrac{|\underline{3}}{|\underline{1}\,|\underline{1}\,|\underline{1}}=6.$

Ex. 2. Find the coefficients of a^2b^2, bcd^2 and $abcd$ in the expansion of $(a+b+c+d)^4$.

We have the terms

$$\frac{|\underline{4}}{|\underline{2}\,|\underline{2}}\,a^2b^2,\quad \frac{|\underline{4}}{|\underline{1}\,|\underline{1}\,|\underline{2}}\,bcd^2 \text{ and } \frac{|\underline{4}}{|\underline{1}\,|\underline{1}\,|\underline{1}\,|\underline{1}}\,abcd.$$

Thus the required coefficients are 6, 12 and 24 respectively.

263. By the previous Article, the general term of the expansion of $(a + bx + cx^2 + dx^3 + \ldots\ldots)^n$ is

$$\frac{|\underline{n}}{|\underline{r}\,|\underline{s}\,|\underline{t}\,|\underline{u}\ldots}\,a^r\,(bx)^s\,(cx^2)^t\,(dx^3)^u\ldots\ldots,$$

that is
$$\frac{|\underline{n}}{|\underline{r}\,|\underline{s}\,|\underline{t}\,|\underline{u}\ldots}\,a^rb^sc^td^u\ldots\ldots x^{s+2t+3u\ldots\ldots}.$$

Hence to find the coefficient of any particular power of x, say of x^a, in the expansion, we must find all the different sets of positive integral values of $r, s, t,\ldots$ which satisfy the equations

$$s + 2t + 3u + \ldots\ldots = a,$$

$$r + s + t + u + \ldots\ldots = n.$$

The required coefficient will then be the sum of the coefficients corresponding to each set of values.

Ex. 1. Find the coefficient of x^5 in the expansion of $(1+2x+3x^2)^4$.

The general term is $\dfrac{|\underline{4}}{|\underline{r}\,|\underline{s}\,|\underline{t}}\,2^s3^tx^{s+2t}$, and the terms required are those for which $s+2t=5$ and $r+s+t=4$.

Since each of the quantities r, s and t must be zero or a positive integer, the only possible sets of values are $t=2$, $s=1$, $r=1$ and $t=1$, $s=3$, $r=0$, the corresponding coefficients being $\dfrac{|\underline{4}}{|\underline{1}\,|\underline{1}\,|\underline{2}}\,.\,2\,.\,3^2$

and $\frac{\lfloor 4}{\lfloor 0 \, \lfloor 3 \, \lfloor 1} \cdot 2^3 \cdot 3$, that is 216 and 96 respectively. Hence the required coefficient is 312.

In simple cases the result can be readily obtained by actual expansion. We have

$(1+2x+3x^2)^4 = 1+4(2x+3x^2)+6(2x+3x^2)^2+4(2x+3x^2)^3+(2x+3x^2)^4.$

Only the last two terms will contain x^5 and the coefficients of x^5 in these terms will be found to be 216 and 96 respectively, so that the required coefficient is 312.

Ex. 2. Find the coefficient of x^4 in the expansion of $(1+x+x^2)^3$.

Ans. 6.

Ex. 3. Find the coefficient of x^5 in the expansion of $(1+x+x^2)^4$.

Ans. 16.

Ex. 4. Find the coefficient of x^8 in the expansion of $(2+x-x^2)^5$.

Ans. 0.

Ex. 5. Find the coefficient of x^{10} in the expansion of

$(7+x+x^2+x^3+x^4+x^5)^3$. *Ans.* 39.

Ex. 6. Find the coefficient of the middle term of the expansion of

$(1+x+x^2+x^3+x^4)^5$. *Ans.* 381.

EXAMPLES XXV.

1. Prove that

$$c_0 - 2c_1 + 3c_2 - \ldots\ldots + (-1)^n (n+1) c_n = 0.$$

2. Prove that

$$c_1 - 2c_2 + 3c_3 - \ldots\ldots + (-1)^{n-1} nc_n = 0.$$

3. Prove that

$$c_0 + 2c_1 + 3c_2 + \ldots\ldots\ldots + (n+1) c_n = 2^{n-1}(n+2).$$

4. Prove that

$$c_2 + 2c_3 + 3c_4 + \ldots\ldots + (n-1) c_n = 1 + (n-2) 2^{n-1}.$$

5. Prove that

$$c_0 + 3c_1 + 5c_2 + \ldots\ldots + (2n+1) c_n = (n+1) 2^n.$$

6. Prove that

$$3c_1 + 7c_2 + 11c_3 + \ldots\ldots + (4n-1) c_n = 1 + (2n-1) 2^n.$$

7. Prove that

$$\frac{c_0}{1}+\frac{c_1}{2}+\frac{c_2}{3}+\ldots\ldots+\frac{c_n}{n+1}=\frac{2^{n+1}-1}{n+1}.$$

8. Prove that

$$\frac{c_0}{1}+\frac{c_2}{3}+\frac{c_4}{5}+\frac{c_6}{7}+\ldots\ldots=\frac{2^n}{n+1}.$$

9. Prove that

$$\frac{c_1}{2}+\frac{c_3}{4}+\frac{c_5}{6}+\ldots\ldots=\frac{2^n-1}{n+1}.$$

10. Prove that

$$\frac{c_0}{2}+\frac{c_1}{3}+\frac{c_2}{4}+\ldots\ldots+\frac{c_n}{n+2}=\frac{1+n\,2^{n+1}}{(n+1)(n+2)}.$$

11. Prove that

$$\frac{c_1}{1}-\frac{c_2}{2}+\frac{c_3}{3}-\ldots\ldots+(-1)^{n-1}\frac{c_n}{n}=\frac{1}{1}+\frac{1}{2}+\ldots\ldots+\frac{1}{n}.$$

12. Prove that

$$\frac{c_0}{1}-\frac{c_1}{4}+\frac{c_2}{7}-\ldots\ldots+(-1)^n\frac{c_n}{3n+1}=\frac{3^n\,\underline{|n}}{1\,.\,4\,.\,7\ldots(3n+1)}.$$

13. Prove that

$$c_0c_r+c_1c_{r+1}+\ldots\ldots+c_{n-r}c_n=\frac{\underline{|2n}}{\underline{|n+r}\,\underline{|n-r}}.$$

14. Prove that, if

$$(1+x)^n=c_0+c_1x+c_2x^2+\ldots\ldots\ldots\ldots+c_nx^n,$$

then $n(1+x)^{n-1}=c_1+2c_2x+3c_3x^2+\ldots\ldots\ldots\ldots+nc_nx^{n-1}$,

and $\{1+(n+1)x\}(1+x)^{n-1}=c_0+2c_1x+\ldots\ldots+(n+1)c_nx^n$.

Hence prove that,

$$c_1^2+2c_2^2+3c_3^2+\ldots\ldots\ldots\ldots+nc_n^2=\frac{\underline{|2n-1}}{\underline{|n-1}\,\underline{|n-1}},$$

and $c_0^2+2c_1^2+3c_2^2+\ldots\ldots\ldots\ldots+(n+1)c_n^2=\dfrac{(n+2)\,\underline{|2n-1}}{\underline{|n}\,\underline{|n-1}}$.

15. Shew, by expanding $\{(1+x)^n-1\}^m$, where m and n are positive integers, that

$$ {}_mC_1 \cdot {}_nC_m - {}_mC_2 \cdot {}_{2n}C_m + {}_mC_3 \cdot {}_{3n}C_m - \ldots\ldots = (-1)^{m-1}n^m. $$

16. Prove that, if $n > 3$,

(i) $a - n(a-1) + \dfrac{n(n-1)}{1 \,.\, 2}(a-2) - \ldots\ldots + (-1)^n(a-n) = 0.$

(ii) $ab - n(a-1)(b-1) + \dfrac{n(n-1)}{1 \,.\, 2}(a-2)(b-2) - \ldots\ldots$

$\ldots\ldots + (-1)^n(a-n)(b-n) = 0.$

(iii) $abc - n(a-1)(b-1)(c-1) + \dfrac{n(n-1)}{1 \,.\, 2}(a-2)(b-2)(c-2)$

$\ldots\ldots + (-1)^n(a-n)(b-n)(c-n) = 0.$

17. Shew that, if there be a middle term in a binomial expansion, its coefficient will be even.

18. Shew that the coefficient of x^n in the nth power of $x^2 + (a+b)x + ab$ is

$$ a^n + {}_nC_1{}^2a^{n-1}b + {}_nC_2{}^2a^{n-2}b^2 + \ldots\ldots + b^n. $$

19. If n be a positive integer and P_n denote the product of all the coefficients in the expansion of $(1+x)^n$, shew that

$$ \frac{P_{n+1}}{P_n} = \frac{(n+1)^n}{\lfloor n}. $$

20. Shew that

$$ (1-x)^n = (1+x)^n - 2nx(1+x)^{n-1} + \frac{2n(2n-2)}{1 \,.\, 2}x^2(1+x)^{n-2} - \ldots\ldots $$

21. Shew that, if n be a positive integer,

$$ 1 - n\frac{1+x}{1+nx} + \frac{n(n-1)}{1 \,.\, 2}\frac{1+2x}{(1+nx)^2} - \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3}\frac{1+3x}{(1+nx)^3} + \ldots\ldots = 0. $$

22. Shew that

$$ (a+b+c+d+e)^5 = \Sigma a^5 + 5\Sigma a^4b + 10\Sigma a^3b^2 + 20\Sigma a^3bc + 30\Sigma a^2b^2c + 60\Sigma a^2bcd + 120abcde. $$

23. If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots,$

prove that $a_r - na_{r-1} + \dfrac{n(n-1)}{\lfloor 2}a_{r-2} - \ldots\ldots + \dfrac{(-1)^r \lfloor n}{\lfloor r \lfloor n-r}a_0 = 0,$

unless r is a multiple of 3.

24. Shew that, in the expansion of $(1+x+x^2+\ldots\ldots+x^r)^n$, where n is a positive integer, the coefficients of terms equidistant from the beginning and the end are equal.

25. If $a_0, a_1, a_2, \ldots\ldots$ be the coefficients in the expansion of $(1+x+x^2)^n$ in ascending powers of x, prove that

$$a_0^2 - a_1^2 + a_2^2 - \ldots\ldots + a_{2n}^2 = a_n, \text{ and that}$$

$$a_0^2 - a_1^2 + a_2^2 - \ldots\ldots + (-1)^{n-1}a_{n-1}^2 = \tfrac{1}{2}\{a_n - (-1)^n a_n^2\}.$$

26. If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots,$

prove that

$$a_0a_1 - a_1a_2 + a_2a_3 - \ldots\ldots\ldots = 0.$$

27. Shew that, in the expansion of $(a_1 + a_2 + a_3 + \ldots + a_r)^n$, where n is a whole number less than r, the coefficient of any term in which none of the quantities a_1, a_2, &c. appears more than once is $n!$

28. Shew that, if the quantities $(1+x)$, $(1+x+x^2)$, ..., $(1+x+x^2+\ldots\ldots+x^n)$ be multiplied together, the coefficients of terms equidistant from the beginning and end will be equal; and that the sum of all the odd coefficients will be equal to the sum of all the even, each being $\frac{1}{2}(n+1)!$

29. Shew that the coefficient of x^n in the expansion of $(1+x+x^2)^n$ is

$$1 + \frac{n(n-1)}{1^2} + \frac{n(n-1)(n-2)(n-3)}{1^2 \cdot 2^2}$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1^2 \cdot 2^2 \cdot 3^2} + \ldots\ldots$$

30. Shew that 18 can be made up of 8 odd numbers in 792 different ways, where repetitions are allowed and the order of addition is taken into account.

CHAPTER XXI.

CONVERGENCY AND DIVERGENCY OF SERIES.

264. A *series* is a succession of quantities which are formed in order according to some definite law. When a series terminates after a certain number of terms it is said to be a *finite* series, and when there is an endless succession of terms the series is said to be *infinite*.

We have already found that when the common ratio of a geometrical progression is numerically less than unity the sum of n terms will not increase indefinitely, but that the sum will become more and more nearly equal to a fixed finite quantity as n is increased without limit. Thus the sum of an infinite series is not in all cases infinitely great.

When the sum of the first n terms of a series tends to a finite limit S, so that the sum can, by sufficiently increasing n, be made to differ from S by less than any assignable quantity, however small, the series is said to be *convergent*, and S is called its *sum*. Thus $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ is a convergent series whose sum is 2.

When the sum of the first n terms of a series increases numerically without limit as n is increased indefinitely, the series is said to be *divergent*. Thus $1 + 2 + 3 + 4 + \dots$ is a divergent series.

When the sum of the n first terms of a series does not increase indefinitely as n is increased without limit, and yet does not approach to any determinate limit, the series is neither convergent nor divergent. Such a series is sometimes called an *indeterminate* or a *neutral* series*.

For example, the series $1-1+1-1+\ldots$ is an indeterminate series, for the sum of n terms is 1 or 0 according as n is odd or even.

It is clear that a series whose terms are all of the same sign cannot be indeterminate, but must either be convergent or divergent. For unless the sum of n terms increases without limit as n is increased without limit, there must be some finite limit which the sum can never exceed, but to which it approaches indefinitely near.

265. If each term of a series be finite, and all the terms have the same sign, the series must be divergent. For, if each term be not less than a, the sum of n terms will be not less than na, and na can be made greater than any finite quantity, however large, by sufficiently increasing n.

266. The successive terms of a series will be denoted by $u_1, u_2, u_3, \ldots$; and, since it is impossible to write down all the terms of an infinite series, it is necessary to know how to express the *general term*, u_n, in terms of n.

The sum of the n first terms will be denoted by U_n; and the sum of the whole series, supposed convergent, in which case alone it has a sum, will be denoted by U.

Thus $\quad U = u_1 + u_2 + u_3 + \ldots + u_n + u_{n+1} + \ldots,$

and $\quad U_n = u_1 + u_2 + u_3 + \ldots + u_n.$

267. In order that the series u_1, u_2, u_3, u_4, ..., u_n, u_{n+1}, &c. may be convergent it is by definition necessary and sufficient that the sum

$$U_n = u_1 + u_2 + u_3 + \ldots + u_n$$

* These series are however called *divergent* series by Cauchy, Bertrand, Laurent and others.

should converge indefinitely to some finite limit U as n is indefinitely increased.

Hence U_n, U_{n+1}, U_{n+2}, &c. ... must differ from U, and therefore from one another, by quantities which diminish indefinitely as n is increased without limit.

Now
$$U_{n+1} - U_n = u_{n+1},$$
$$U_{n+2} - U_n = u_{n+1} + u_{n+2},$$
$$\dots\dots\dots = \dots\dots\dots$$
$$U - U_n = u_{n+1} + u_{n+2} + u_{n+3} + \dots$$

Hence, in order that a series may be convergent, the $(n+1)$th term must decrease indefinitely as n is increased indefinitely, and also the sum of *any number of terms* beginning at the $(n+1)$th must become less than any assignable quantity, however small, when n is indefinitely increased.

For example, the series $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}+\dots$ cannot be convergent, although the nth term diminishes indefinitely as n is increased indefinitely; for the sum of n terms beginning at the $(n+1)$th is $\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}$, which is greater than $\frac{1}{2n}\times n$, that is, greater than $\frac{1}{2}$.

268. We shall for the present consider series in which all the terms have the same sign; and as it is clear that the convergency or divergency of such a series does not depend on whether the signs are all positive or all negative, we shall consider all the signs to be positive.

The convergency or divergency of series can generally be determined by means of the following theorems.

269. **Theorem I.** *A series is convergent if all its terms are less than the corresponding terms of a second series which is known to be convergent.*

Let the two series be respectively

$$U = u_1 + u_2 + u_3 + \ldots\ldots$$

and
$$V = v_1 + v_2 + v_3 + \ldots\ldots$$

Then, since $u_r < v_r$ for all values of r, it follows that U is less than V. Hence, as V is finite, U must also be finite: this proves the theorem, for a series must be convergent when its sum is finite and all the terms have the same sign.

It can be proved in a similar manner that a series is divergent if all its terms are greater than the corresponding terms of a divergent series.

Ex. (i). To shew that the series $\frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \ldots$ is convergent.

The terms of the series are less than the terms of the series $\frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.2} + \frac{1}{1.2.2.2} + \ldots$, and this latter series is a geometrical progression whose common ratio is $\frac{1}{2}$, which is therefore known to be a convergent series. The given series must therefore also be convergent.

Ex. (ii). Shew that the series

$$\frac{(a+x)}{(b+x)} + \frac{(a+x)(2a+x)}{(b+x)(2b+x)} + \frac{(a+x)(2a+x)(3a+x)}{(b+x)(2b+x)(3b+x)} + \ldots\ldots$$

is convergent if a, b and x are all positive, and $a < b$.

The terms of the given series are less than the corresponding terms of the series $\quad \frac{a+x}{b+x} + \frac{(a+x)^2}{(b+x)^2} + \frac{(a+x)^3}{(b+x)^3} + \ldots$,

[since $\frac{ra+x}{rb+x} < \frac{a+x}{b+x}$ if $r > 1$, a, b and x being positive and $b > a$].

The latter series is convergent, and therefore also the given series.

To ensure the convergency of the first series it is not necessary that *all* its terms should be less than the corresponding terms of the second series, it will be sufficient if all the terms except a *finite number* of them

be less than the corresponding terms of the second, for the sum of a finite number of terms of any series must be finite.

Ex. Shew that the series $1+\frac{4}{\lfloor 2}+\frac{4^2}{\lfloor 3}+\frac{4^3}{\lfloor 4}+\frac{4^4}{\lfloor 5}+\frac{4^5}{\lfloor 6}+\frac{4^6}{\lfloor 7}+\ldots$ is convergent.

From the sixth term onwards, each term is less than the corresponding term of the series $\frac{4^5}{5\lfloor 5}+\frac{4^6}{5^2\lfloor 5}+\ldots$. Hence the series beginning at the sixth term is convergent, and therefore the whole series is convergent.

270. **Theorem II.** *If the ratio of the corresponding terms of two series be always finite, the series will both be convergent or both divergent.*

Let the series be respectively

$$U=u_1+u_2+u_3+\ldots\ldots,$$

and $$V=v_1+v_2+v_3+\ldots\ldots$$

Then, since the quantities are all positive, $\frac{U}{V}$ must lie between the greatest and least of the fractions $\frac{u_r}{v_r}$ [Art. 113]. Hence $U:V$ is finite. It therefore follows that if U is finite so also is V, and if U is infinite so also is V.

For example, the two series $\frac{8}{2.3}+\frac{16}{3.4}+\ldots\ldots+\frac{8n}{(n+1)(n+2)}+\ldots$ and $\frac{1}{1}+\frac{1}{2}+\ldots\ldots+\frac{1}{n}+\ldots$ are both convergent or both divergent. For the ratio of the rth terms, namely $\frac{8r}{(r+1)(r+2)}:\frac{1}{r}$ is equal to $\frac{8r^2}{(r+1)(r+2)}$, which is >1 and <8 for all values of r. Now we have already proved that the second series is divergent: the first series is therefore also divergent.

271. **Theorem III.** *A series is convergent if, after any particular term, the ratio of each term to the preceding is always less than some fixed quantity which is itself less than unity.*

Let the ratio of each term after the r^{th} to the preceding term be less than k, where $k < 1$.

Then, since $\dfrac{u_{r+1}}{u_r} < k$, $\dfrac{u_{r+2}}{u_{r+1}} < k, \ldots\ldots,$
we have

$$u_r + u_{r+1} + u_{r+2} + \ldots\ldots < u_r(1 + k + k^2 + \ldots\ldots)$$

$$< \frac{u_r}{1-k}, \text{ since } k \text{ is less than } 1.$$

Hence the sum of the series beginning at the r^{th} term is finite, and the sum of any finite number of terms is finite; therefore the whole series must be convergent.

272. **Theorem IV.** *A series is divergent if, after any particular term, the ratio of each term to the preceding is either equal to unity or greater than unity.*

First, let all the terms after the r^{th} be equal to u_r; then $u_{r+1} + u_{r+2} + \ldots + u_{n+r} = nu_r$, and nu_r can be made greater than any finite quantity by sufficiently increasing n. The series must therefore be divergent.

Next, let the ratio of each term, after the r^{th}, to the preceding term be greater than 1.

Then $u_{r+1} > u_r$, $u_{r+2} > u_{r+1} > u_r$, &c.

Hence $u_{r+1} + u_{r+2} + \ldots + u_{n+r} > nu_r$; the series must therefore be divergent.

Ex. 1. In the series $\dfrac{1}{1} + \dfrac{2}{2} + \dfrac{2^2}{3} + \dfrac{2^3}{4} + \ldots\ldots + \dfrac{2^{n-1}}{n} + \ldots\ldots$, the ratio $\dfrac{u_{n+1}}{u_n} = \dfrac{2n}{n+1}$, which is greater than 1; the series is therefore divergent.

Ex. 2. In the series $1^2 + 2^2x + 3^2x^2 + \ldots\ldots$, the test ratio is $\dfrac{(n+1)^2}{n^2}x$, that is $\left(1+\dfrac{1}{n}\right)^2 x$. Now, if x be less than 1, and any fixed quantity k be chosen between x and 1, the test ratio will be less than k for all terms after the first which makes

$$\left(1+\frac{1}{n}\right)\sqrt{x} < \sqrt{k}, \text{ i.e. } n > \frac{\sqrt{x}}{\sqrt{k}-\sqrt{x}}.$$

Hence the series is convergent if $x < 1$.

If $x=1$ the series is $1^2+2^2+3^2+\ldots\ldots$ which is obviously divergent, and if $x>1$ the series is greater than $1^2+2^2+3^2+\ldots\ldots.$

Thus the series $1^2+2^2x+3^2x^2+\ldots\ldots$ is divergent except when x is less than unity.

273. When a series is such that after a finite number of terms the ratio $\frac{u_{n+1}}{u_n}$ is always less than unity but becomes indefinitely nearly equal to unity as n is indefinitely increased, the test contained in Theorem III. fails to give any result; and in this case, which is a very common one, it is often difficult to determine whether a series is convergent or divergent.

For example, in the series

$$\frac{1}{1^k}+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\ldots\ldots,$$

the ratio $$\frac{u_{n+1}}{u_n}=\frac{n^k}{(n+1)^k}=\frac{1}{\left(1+\frac{1}{n}\right)^k}.$$

Hence, if k be positive, the test ratio is less than unity, but becomes more and more nearly equal to unity as n is increased.

We cannot therefore determine from Theorem III. whether the series in question is convergent or divergent.

274. *To shew that the series* $\frac{1}{1^k}+\frac{1}{2^k}+\frac{1}{3^k}+\ldots$ *is convergent when k is greater than unity, and is divergent when k is equal to unity or less than unity.*

First, let k be greater than unity.

Since each term of the series is less than the preceding term, we have the following relations:

$$\frac{1}{2^k}+\frac{1}{3^k}<\frac{2}{2^k},$$

$$\frac{1}{4^k}+\frac{1}{5^k}+\frac{1}{6^k}+\frac{1}{7^k}<\frac{4}{4^k},$$

and $$\frac{1}{2^{nk}}+\frac{1}{(2^n+1)^k}+\ldots\ldots\ldots+\frac{1}{(2^{n+1}-1)^k}<\frac{2^n}{2^{nk}}.$$

Hence the whole series is less than

$$\frac{1}{1^k}+\frac{2}{2^k}+\frac{4}{4^k}+\frac{8}{8^k}+\ldots\ldots+\frac{2^n}{2^{nk}}+\ldots,$$

that is, less than

$$\frac{1}{1}+\frac{1}{2^{k-1}}+\frac{1}{2^{2(k-1)}}+\frac{1}{2^{3(k-1)}}+\ldots\ldots+\frac{1}{2^{n(k-1)}}+\ldots.$$

But this latter series is a geometrical progression whose common ratio, $\frac{1}{2^{k-1}}$, is less than unity, since $k>1$. Hence the given series is convergent.

Next, let $k=1$; then we can group the series as follows:

$$\frac{1}{1}+\frac{1}{2}+\left[\frac{1}{3}+\frac{1}{4}\right]+\left[\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right]+\ldots\ldots$$
$$+\left[\frac{1}{2^{n-1}+1}+\frac{1}{2^{n-1}+2}+\ldots\ldots+\frac{1}{2^n}\right]+\ldots\ldots;$$

therefore, as each group of terms in brackets is greater than $\frac{1}{2}$, the given series taken to 2^n terms is greater than $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots\ldots$ taken to $n+1$ terms, that is, greater than $1+\frac{1}{2}n$, which increases indefinitely with n.

Hence $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots\ldots$ is divergent.

Lastly, let k be less than unity; then each term of the series $\frac{1}{1^k}+\frac{1}{2^k}+\ldots\ldots$ is greater than the corresponding term of the divergent series $\frac{1}{1}+\frac{1}{2}+\ldots\ldots$; the series is therefore divergent when $k<1$.

275. The convergency or divergency of many series can be determined by means of Theorems I. and II., using the series of the last Article as a standard series. The method will be seen from the following examples.

Ex. 1. Is the series whose general term is $\frac{2n}{n^2+1}$ convergent or divergent?

Since $\frac{2n}{n^2+1} > \frac{1}{n}$, if $n > 1$, it follows that $\Sigma \frac{2n}{n^2+1} > \Sigma \frac{1}{n}$. But $\Sigma \frac{1}{n}$ is divergent; therefore $\Sigma \frac{2n}{n^2+1}$ is also divergent.

Ex. 2. Is the series whose general term is $\frac{n+2}{n^3+1}$ convergent or divergent?

Now $\frac{n+2}{n^3+1} < \frac{n+2}{n^3} < \frac{3n}{n^3} < \frac{3}{n^2}$. Hence $\Sigma \frac{n+2}{n^3+1} < 3\Sigma \frac{1}{n^2}$. But $\Sigma \frac{1}{n^2}$ is convergent [Art. 274]; therefore $\Sigma \frac{n+2}{n^3+1}$ is also convergent.

276. We have hitherto supposed that the terms of the series whose convergency or divergency was to be determined were all of the same sign. When, however, some terms are positive and others negative, we first see whether the series which would be obtained by making all the signs positive is convergent; and, if this is the case, it follows that the given series is also convergent; for a convergent series, all of whose terms are positive, would clearly remain convergent when the signs of some of its terms were changed. If, however, the series obtained by making all the signs positive is a divergent series it does not necessarily follow that the given series is divergent. For example, it will be proved in the next Article that the series $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is convergent, although the series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is divergent.

277. Many series whose terms are alternately positive and negative are at once seen to be convergent by means of

Theorem V. *A series is convergent when its terms are alternately positive and negative, provided each term is less than the preceding, and that the terms decrease without limit in absolute magnitude.*

Let the series be

$$u_1 - u_2 + u_3 - u_4 + \dots \pm u_n \mp u_{n+1} \pm u_{n+2} \mp \dots\dots$$

By writing the series in the forms

$$u_1 - u_2 + (u_3 - u_4) + (u_5 - u_6) + \dots,$$

and $$u_1 - (u_2 - u_3) - (u_4 - u_5) - \dots\dots,$$

we see that, since each term is less than the preceding, the sum of the series must be intermediate to $u_1 - u_2$ and u_1; and hence the sum of the series is *finite*. It is also similarly clear that the absolute value of $U - U_n$ is intermediate to the absolute values of $u_{n+1} - u_{n+2}$ and u_{n+1}, and therefore $U - U_n$ becomes indefinitely small when n is increased without limit. The series must therefore be convergent.

For example, the series $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\dots$ is convergent, since the terms are alternately positive and negative and decrease without limit. The series $\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \dots\dots$ is not however a convergent series although its sum is a finite quantity between $\frac{1}{2}$ and 2, for the nth term, namely $\frac{n+1}{n}$, does not diminish indefinitely as n is indefinitely increased.

278. We will now apply the preceding tests of convergency to three series of very great importance.

I. **The Binomial Series.** In the binomial series, namely

$$1 + mx + \frac{m(m-1)}{1.2}x^2 + \dots\dots$$
$$+ \frac{m(m-1)\dots(m-n+1)}{\underline{|n}}x^n + \dots,$$

the number of terms is finite when m is a positive integer; but when m is not a positive integer no one of the factors m, $m-1$, $m-2$, &c. can be zero, and hence the series must be endless.

To determine the convergency of the series when m is not a positive integer we must consider the ratio $u_{n+1}:u_n$. Now $\frac{u_{n+1}}{u_n}=\frac{m-n+1}{n}x=-x\left(1-\frac{m+1}{n}\right)$.

Hence, for all values of n greater than $m+1$, u_{n+1}, and u_n have different signs when x is positive, and have the same sign when x is negative. Moreover, as n is increased, the absolute value of u_{n+1}/u_n becomes more and more nearly equal to x. If therefore x be numerically less than unity, the ratio u_{n+1}/u_n will, either from the beginning, or after a finite number of terms, be numerically less than unity. Hence by Art. 271 the series formed by adding the absolute values of the successive terms will be convergent, and therefore also the series itself must be convergent, whether its terms have all the same sign or are alternately positive and negative.

Thus the binomial series is convergent, if x is numerically less than unity*.

II. **The Exponential Series.** In the exponential series, namely

$$1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots+\frac{x^n}{\underline{|n}}+\ldots,$$

the ratio u_{n+1}/u_n is x/n. Hence the ratio u_{n+1}/u_n is numerically less than unity for all terms after the first for which n is numerically greater than x. The series is therefore convergent for all values of x.

* The series is also convergent when $x=1$, provided $n>-1$; and it is convergent when $x=-1$, provided $n>0$. [See Art. 338.]

III. **The Logarithmic Series.** In the logarithmic series, namely

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots - (-1)^n \frac{x^n}{n} + \ldots,$$

the ratio u_{n+1}/u_n is $-\frac{xn}{n+1} = -x\left(1 - \frac{1}{n+1}\right)$; and hence u_{n+1}/u_n will be numerically less than unity provided x is numerically less than unity. The logarithmic series is therefore convergent when x has any value between -1 and $+1$.

If $x = 1$, the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \ldots$, which is convergent by Theorem V.

If $x = -1$, the series becomes $-(1 + \frac{1}{2} + \frac{1}{3} + \ldots)$, which is known to be divergent. [Art. 274.]

279. The condition for the convergency of the product of an infinite number of factors, and also some other theorems in convergency, will be proved in a subsequent chapter. [See Art. 337.] The two important theorems which follow cannot however be deferred.

280. If the two series

$$U = u_0 + u_1 x + u_2 x^2 + \ldots\ldots + u_n x^n + \ldots,$$

and
$$V = v_0 + v_1 x + v_2 x^2 + \ldots\ldots + v_n x^n + \ldots,$$

be both convergent, and the third series

$$P = u_0 v_0 + (u_0 v_1 + u_1 v_0)\, x + (u_0 v_2 + u_1 v_1 + u_2 v_0)\, x^2$$
$$+ \ldots\ldots + (u_0 v_n + u_1 v_{n-1} + \ldots\ldots + u_n v_0)\, x^n + \ldots$$

be formed, in which the coefficient of any power of x is the same as in the product of the two first series; then P will be a convergent series equal to $U \times V$, provided (1) that the series U and V have all their terms positive, or (2) that the series U and V would not lose their convergency if the signs were all made positive*.

* This Article, and in fact the whole of this Chapter, is taken with slight modifications from Cauchy's *Analyse Algébrique.*

First, suppose that all the terms in U and V are positive.

Then $U_{2n} \times V_{2n} = P_{2n} +$ terms containing x^{2n} and higher powers of x. Hence $U_{2n} \times V_{2n} > P_{2n}$.

Also $P_{2n} = U_n \times V_n +$ other terms. Hence $P_{2n} > U_n \times V_n$.

Hence P_{2n} is intermediate to $U_n \times V_n$ and $U_{2n} \times V_{2n}$.

Now, the series U and V being convergent, U_{2n} and U_n both approach indefinitely near to U, also V_{2n} and V_n both approach indefinitely near to V, when n is indefinitely increased. Hence $U_{2n} \times V_{2n}$ and $U_n \times V_n$, and therefore also P_{2n} which is intermediate to them, will in the limit be equal to $U \times V$. Hence, when all the terms are positive, $P = U \times V$.

Next, let the signs in the two series be not all positive, and let U' and V' be the series obtained by making all the signs positive in U and V; and let P' be the series formed from U' and V' in the same way as P is formed from U and V.

Then $U_{2n} \times V_{2n} - P_{2n}$ cannot be numerically greater than $U'_{2n} \times V'_{2n} - P'_{2n}$, for the terms in the latter expression are the same as those in the former but with all the signs positive.

Now, provided the series U and V do not lose their convergency when the signs of all the terms are made positive, it follows from the first case that $U'_{2n} \times V'_{2n} - P'_{2n}$, and therefore also $U_{2n} \times V_{2n} - P_{2n}$ which is not numerically greater, must diminish indefinitely when n is increased without limit. Hence the limit of P_{2n} is equal to the limit of $U_{2n} \times V_{2n}$; so that P must be a convergent series equal to the product of U and V.

If the series U and V are convergent, but are such that they would lose their convergency by making the signs of all the terms positive, the series P may or may not be convergent; and, when P is not convergent, the relation $U \times V = P$ does not hold good, for P has no definite value and cannot therefore be equal to $U \times V$,

although the coefficient of any particular power of x in the series P is always equal to the coefficient of the same power of x in the product of the series U and V*.

281. If the two series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots,$$

and

$$b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots\ldots,$$

be equal to one another for all values of x for which they are convergent; then will $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, &c.

For if the series are both convergent, their difference will be convergent. Hence

$$a_0 - b_0 + (a_1 - b_1)\,x + (a_2 - b_2)\,x^2 + \ldots\ldots = 0\ldots\ldots\ldots\text{(i)},$$

for all values of x for which the series is convergent.

The last series is clearly convergent when $x = 0$; and putting $x = 0$ we have $a_0 - b_0 = 0$. Hence $a_0 = b_0$.

We now have

$$x\,\{a_1 - b_1 + (a_2 - b_2)\,x + (a_3 - b_3)\,x^2 + \ldots\ldots\} = 0\ldots\ldots\text{(ii)}.$$

Now for any value x_1 for which the series in (i) is convergent, $a_2 - b_2 + (a_3 - b_3)\,x_1 + \ldots\ldots$ is equal to a finite limit, L_1 suppose.

Hence (ii) may be written $x_1\,\{a_1 - b_1 + x_1 L_1\} = 0$; and, since this is true for all values of x_1, however small, it follows that $a_1 - b_1$, must be numerically indefinitely small compared with L_1; that is, $a_1 - b_1$ must be zero. It can now be proved in a similar manner that $a_2 - b_2 = 0$, $a_3 - b_3 = 0$, &c.

Hence *if two series which contain x be equal to one another for all values of x for which the series are convergent, we may equate the coefficients of the same powers of x in the two series.*

The particular case of two series which have a finite number of terms was proved in Art. 91.

* It can be proved that P is convergent if *either* U or V is absolutely convergent. See Chrystal's *Algebra*, Part II., p. 127.

EXAMPLES XXVI.

Determine whether the following series are convergent or divergent:

1. $\dfrac{1}{1\,.\,2}+\dfrac{1}{3\,.\,4}+\dfrac{1}{5\,.\,6}+\ldots+\dfrac{1}{(2n+1)(2n+2)}+\ldots$

2. $\dfrac{1}{a(a+b)}+\dfrac{1}{(a+2b)(a+3b)}+\dfrac{1}{(a+4b)(a+5b)}+\ldots$

3. $\dfrac{3}{4}+\dfrac{3\,.\,4}{4\,.\,6}+\dfrac{3\,.\,4\,.\,5}{4\,.\,6\,.\,8}+\ldots+\dfrac{3\,.\,4\ldots(n+2)}{4\,.\,6\ldots(2n+2)}+\ldots$

4. $\dfrac{3}{4}+\dfrac{3\,.\,5}{4\,.\,7}+\dfrac{3\,.\,5\,.\,7}{4\,.\,7\,.\,10}+\ldots+\dfrac{3\,.\,5\,.\,7\ldots(2n+1)}{4\,.\,7\,.\,10\ldots(3n+1)}+\ldots$

5. $\dfrac{1}{3}+\dfrac{1\,.\,3}{3\,.\,6}+\dfrac{1\,.\,3\,.\,5}{3\,.\,6\,.\,9}+\ldots+\dfrac{1\,.\,3\,.\,5\ldots(2n-1)}{3\,.\,6\,.\,9\ldots3n}+\ldots$

6. $\dfrac{1}{x}+\dfrac{1}{x+1}+\dfrac{1}{x+2}+\dfrac{1}{x+3}+\ldots$

7. $\dfrac{1}{1+x}+\dfrac{1}{1+2x}+\dfrac{1}{1+3x}+\ldots$

8. $\dfrac{1}{1+x}+\dfrac{1}{1+x^2}+\dfrac{1}{1+x^3}+\ldots$

9. $\dfrac{1}{1+x}+\dfrac{x}{1+x^2}+\dfrac{x^2}{1+x^4}+\ldots+\dfrac{x^n}{1+x^{2n}}+\ldots$

10. $\dfrac{1}{1+x}+\dfrac{1}{1+2x^2}+\dfrac{1}{1+3x^3}+\ldots+\dfrac{1}{1+nx^n}+\ldots$

11. $\dfrac{1}{1\,.\,2}+\dfrac{x}{2\,.\,3}+\dfrac{x^2}{3\,.\,4}+\ldots+\dfrac{x^n}{(n+1)(n+2)}+\ldots$

12. $1-\dfrac{x}{1+a}+\dfrac{x^2}{1+2a}-\ldots+(-1)^n\dfrac{x^n}{1+na}+\ldots$

13. $1+\dfrac{1+2}{1+2^2}+\dfrac{1+3}{1+3^2}+\ldots+\dfrac{1+n}{1+n^2}+\ldots$

14. $1+\dfrac{2^2-1^2}{2^2+1^2}+\dfrac{3^2-2^2}{3^2+2^2}+\ldots+\dfrac{n^2-(n-1)^2}{n^2+(n-1)^2}+\ldots$

15. $\dfrac{m}{x+m}+\dfrac{m^2}{x+2m}+\dfrac{m^3}{x+3m}+\ldots$

16. $\dfrac{1}{x+1}+\dfrac{m}{x+m}+\dfrac{m^2}{x+m^2}+\ldots$

17. $\dfrac{(1+a)(1+b)}{1\,.\,2\,.\,3}+\dfrac{(2+a)(2+b)}{2\,.\,3\,.\,4}+\ldots+\dfrac{(n+a)(n+b)}{n(n+1)(n+2)}+\ldots$

18. $\dfrac{1}{1+\sqrt{2}}+\dfrac{2}{1+2\sqrt{3}}+\dfrac{3}{1+3\sqrt{4}}+\ldots+\dfrac{n}{1+n\sqrt{n+1}}+\ldots$

19. $\dfrac{\sqrt{2}}{2+\sqrt{2}}+\dfrac{\sqrt{3}}{3+\sqrt{3}}+\dfrac{\sqrt{4}}{4+\sqrt{4}}+\ldots+\dfrac{\sqrt{n}}{n+\sqrt{n}}+\ldots$

20. $\dfrac{1}{3}\left(1-\dfrac{1}{\sqrt{2}}\right)+\dfrac{1}{4}\left(1-\dfrac{1}{\sqrt{3}}\right)+\ldots+\dfrac{1}{n+1}\left(1-\dfrac{1}{\sqrt{n}}\right)+\ldots$

21. $(\sqrt{2}-1)+(\sqrt{5}-2)+\ldots+(\sqrt{n^2+1}-n)+\ldots$

22. $\dfrac{1}{1^k}+\dfrac{x}{3^k}+\dfrac{x^2}{5^k}+\ldots+\dfrac{x^n}{(2n+1)^k}+\ldots$

23. $\dfrac{2}{2}+\dfrac{4x}{5}+\dfrac{6x^2}{10}+\ldots+\dfrac{2nx^n}{n^2+1}+\ldots$

24. $\dfrac{3}{4}+\dfrac{1}{2}x+\dfrac{1}{12}x^2+\ldots\ldots+\dfrac{2n-5}{n^3-5n}x^{n-1}+\ldots$

25. Shew that the series

$$\frac{1}{1^2-x}+\frac{1}{2^2-x}+\frac{1}{3^2-x}+\ldots+\frac{1}{n^2-x}+\ldots$$

is convergent for all values of x, except only when x is the square of an integer.

26. $\Sigma\left(\dfrac{1}{\sqrt{(n-1)}}-\dfrac{1}{\sqrt{n}}\right)n^m x^n.$

27. $\Sigma n^k\{\sqrt{(n-1)}-2\sqrt{(n-2)}+\sqrt{(n-3)}\}\,x^n.$

CHAPTER XXII.

THE BINOMIAL THEOREM. ANY INDEX.

282. IT was proved in Chapter XX. that, when n is any positive integer,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \dots\dots$$
$$\dots\dots + \frac{n(n-1)\dots(n-r+1)}{\lfloor r}x^r + \dots\dots$$

We now proceed to prove that the above formula is true for all values of n, provided that the series on the right is *convergent*.

When n is a positive integer the above series stops, as we have already seen, at the $(n+1)$th term; but when n is not a positive integer the series is endless, for no one of the factors n, $n-1$, $n-2$, &c. can in this case be zero.

It should be noticed that the general term of the binomial series, namely $\dfrac{n(n-1)(n-2)\dots(n-r+1)}{\lfloor r}x^r$, cannot be written in the shortened form $\dfrac{\lfloor n}{\lfloor r \lfloor n-r}x^r$ unless n is a positive integer; we may however employ the notation of Art. 241, and write the series in the form

$$1 + n_1 x + \frac{n_2}{\lfloor 2}x^2 + \frac{n_3}{\lfloor 3}x^3 + \dots + \frac{n_r}{\lfloor r}x^r + \dots\dots$$

283. Proof of the Binomial Theorem. Represent, for shortness, any series of the form $1 + \frac{m_1}{\lfloor 1}x + \frac{m_2}{\lfloor 2}x^2 + \dots + \frac{m_r}{\lfloor r}x^r + \dots$ by $f(m)$. Thus

$$f(m) \equiv 1 + \frac{m_1}{\lfloor 1}x + \frac{m_2}{\lfloor 2}x^2 + \dots\dots + \frac{m_r}{\lfloor r}x^r + \dots\dots,$$

$$f(n) \equiv 1 + \frac{n_1}{\lfloor 1}x + \frac{n_2}{\lfloor 2}x^2 + \dots\dots + \frac{n_r}{\lfloor r}x^r + \dots\dots,$$

and

$$f(m+n) \equiv 1 + \frac{(m+n)_1}{\lfloor 1}x + \frac{(m+n)_2}{\lfloor 2}x^2 + \dots + \frac{(m+n)_r}{\lfloor r}x^r \dots$$

Now the coefficient of x^r in the product $f(m) \times f(n)$ is

$$\frac{m_r}{\lfloor r} + \frac{m_{r-1}n_1}{\lfloor r-1 \lfloor 1} + \frac{m_{r-2}n_2}{\lfloor r-2 \lfloor 2} + \dots + \frac{m_{r-s}n_s}{\lfloor r-s \lfloor s} + \dots + \frac{n_r}{\lfloor r},$$

that is

$$\frac{1}{\lfloor r}\left\{m_r + \dots + \frac{\lfloor r}{\lfloor r-s \lfloor s} m_{r-s}n_s + \dots + n_r\right\}.$$

And, by Vandermonde's Theorem [Art. 249 or 261], this coefficient is equal to $\frac{(m+n)_r}{\lfloor r}$, which is the coefficient of x^r in $f(m+n)$.

Thus the coefficient of any power of x in $f(m+n)$ is equal to the coefficient of the same power of x in the product $f(m) \times f(n)$; also the series $f(m)$, $f(n)$ and $f(m+n)$ are convergent, for all values of m and n, when x is numerically less than unity [Art. 278]. It therefore follows from Art. 280 that

$$f(m) \times f(n) = f(m+n) \dots\dots\dots (\alpha),$$

for *all values* of m and n, provided that x is numerically less than unity.

Now it is obvious that $f(0) = 1$, and that $f(1) = (1+x)$; we also know that if r be a *positive integer* $f(r) = (1+x)^r$.

Hence, by continued application of (α), we have

$$f(m) \times f(n) \times f(p) \times \ldots = f(m+n) \times f(p) \times \ldots$$
$$= f(m+n+p+\ldots).$$

Now let $m = n = p = \ldots = \dfrac{r}{s}$, where r and s are positive integers; then taking s factors, we have

$$\left\{f\left(\frac{r}{s}\right)\right\}^s = f\left(\frac{r}{s} \times s\right) = f(r).$$

But, since r is a positive integer, $f(r)$ is $(1+x)^r$;

$$\therefore \left\{f\left(\frac{r}{s}\right)\right\}^s = (1+x)^r;$$

$$\therefore (1+x)^{\frac{r}{s}} = f\left(\frac{r}{s}\right).$$

This proves the Binomial Theorem for a positive fractional exponent: the theorem is therefore true for any *positive* index.

And, assuming that the binomial theorem is true for any positive index, it can be proved to be true also for any negative index. For, from (α),

$$f(-n) \times f(n) = f(-n+n) = f(0).$$

Hence, as $f(0) = 1$, we have

$$f(-n) = \frac{1}{f(n)} = \frac{1}{(1+x)^n}, \text{ since } n \text{ is positive,}$$
$$= (1+x)^{-n}.$$

Hence $(1+x)^{-n} = f(-n)$, which proves the theorem for any negative index.

284. **Euler's Proof.** Euler's proof of the Binomial Theorem is as follows.

Represent, for shortness, any series of the form

$$1 + mx + \frac{m(m-1)}{1 \cdot 2} x^2 + \ldots + \frac{m(m-1)\ldots(m-r+1)}{\lfloor r} x^r + \ldots$$

by $f(m)$: thus

$$f(m) \equiv 1 + mx + \frac{m_2}{\lfloor 2} x^2 + \ldots + \frac{m_r}{\lfloor r} x^r + \ldots\ldots\text{(i)},$$

$$f(n) \equiv 1 + nx + \frac{n_2}{\lfloor 2} x^2 + \ldots + \frac{n_r}{\lfloor r} x^r + \ldots\ldots\text{(ii)},$$

and,

$$f(m+n) \equiv 1 + (m+n)x + \frac{(m+n)_2}{\lfloor 2} x^2 + \ldots + \frac{(m+n)_r}{\lfloor r} x^r + \ldots$$

Now, if the series on the right of (i) and (ii) be multiplied, and the product be arranged according to ascending powers of x, the result must involve m and n in the same way whatever their values may be. But, when m and n are positive integers, we know that $f(m)$ is $(1+x)^m$, and that $f(n)$ is $(1+x)^n$, and the product $f(m) \times f(n)$ is therefore $(1+x)^{m+n}$, which again, as $m+n$ is a positive integer, is $f(m+n)$. Hence when m and n are positive integers the product $f(m) \times f(n)$ is $f(m+n)$; and, as the *form* of the product is the same for *all values* of m and n it follows that

$$f(m) \times f(n) = f(m+n) \ldots\ldots\ldots\ldots(\alpha),$$

for *all values* of m and n. [See however Art. 280.]

From this point the proof is the same as in Art. 288.

Ex. 1. Expand $(1+x)^{-1}$.

Put $n = -1$ in the above formula; then we have

$$(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{1 \cdot 2} x^2 + \frac{(-1)(-2)(-3)}{1 \cdot 2 \cdot 3} x^3 + \ldots$$

$$\ldots + \frac{(-1)(-2)\ldots\ldots(-r)}{\lfloor r} x^r + \ldots\ldots$$

$$= 1 - x + x^2 - x^3 + \ldots\ldots + (-1)^r x^r + \ldots\ldots$$

This example illustrates the necessity of some limitation in the value of x; for we know [Art. 229] that $1-x+x^2-\ldots\ldots$ is not equal to $\dfrac{1}{1+x}$ unless x is between -1 and $+1$.

Ex. 2. Expand $(1-x)^{-2}$.

We have

$$(1-x)^{-2}=1+(-2)(-x)+\frac{(-2)(-3)}{1\,.\,2}(-x)^2+\frac{(-2)(-3)(-4)}{1\,.\,2\,.\,3}(-x)^3$$

$$+\ldots\ldots+\frac{(-2)(-3)\ldots\ldots(-\overline{r+1})}{\underline{|r}}(-x)^r+\ldots\ldots$$

$$=1+2x+3x^2+4x^3+\ldots\ldots+(r+1)\,x^r+\ldots\ldots$$

Here again it is clear that the result cannot be true for *all values* of x; if $x=2$, for example, we should have

$$1=1+2\,.\,2+3\,.\,2^2+4\,.\,2^3+\ldots\ldots,$$

which is absurd.

Ex. 3. Expand $(1+x)^{\frac{1}{2}}$.

We have $(1+x)^{\frac{1}{2}}=1+\dfrac{1}{2}x+\dfrac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\,.\,2}x^2+\dfrac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\,.\,2\,.\,3}x^3+\ldots\ldots,$

the general term being

$$\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\ldots\left(\frac{1}{2}-\overline{r-1}\right)}{\underline{|r}}x^r, \text{ that is } (-1)^{r-1}\frac{1\,.\,3\,.\,5\ldots(2r-3)}{2^r\,\underline{|r}}x^r.$$

Hence $(1+x)^{\frac{1}{2}}=1+\dfrac{1}{2}x-\dfrac{1}{2\,.\,4}x^2+\dfrac{1\,.\,3}{2\,.\,4\,.\,6}x^3+\ldots\ldots$

$$+(-1)^{r-1}\frac{1\,.\,3\,.\,5\ldots(2r-3)}{2\,.\,4\,.\,6\ldots2r}x^r+\ldots\ldots$$

Ex. 4. Expand $(1-x)^{-\frac{1}{2}}$.

We have $(1-x)^{-\frac{1}{2}}=1+(-\tfrac{1}{2})(-x)+\dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\,.\,2}(-x)^2+\ldots\ldots$

$$+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\ldots\left(-\frac{2r-1}{2}\right)}{\underline{|r}}(-x)^r+\ldots\ldots$$

Hence

$$(1-x)^{-\frac{1}{2}}=1+\tfrac{1}{2}x+\frac{1\,.\,3}{2\,.\,4}x^2+\ldots\ldots+\frac{1\,.\,3\,.\,5\ldots\ldots(2r-1)}{2\,.\,4\,.\,6\ldots\ldots2r}x^r+\ldots\ldots$$

All the terms are positive, for in the general term there are $2r$ negative factors.

Ex. 5. Expand $(a^3-3a^2x)^{\frac{5}{3}}$ according to ascending powers of x.

$$(a^3-3a^2x)^{\frac{5}{3}}=\left\{a^3\left(1-\frac{3x}{a}\right)\right\}^{\frac{5}{3}}=a^5\left(1-\frac{3x}{a}\right)^{\frac{5}{3}}$$

$$=a^5\left[1+\frac{5}{3}\left(-\frac{3x}{a}\right)+\frac{\frac{5}{3}\cdot\frac{2}{3}}{1.2}\left(-\frac{3x}{a}\right)^2+\frac{\frac{5}{3}\cdot\frac{2}{3}\left(-\frac{1}{3}\right)}{1.2.3}\left(-\frac{3x}{a}\right)^3\right.$$

$$\left.+\ldots\ldots+\frac{\frac{5}{3}\cdot\frac{2}{3}\left(-\frac{1}{3}\right)\ldots\ldots\left(\frac{5}{3}-r+1\right)}{\lfloor r}\left(-\frac{3x}{a}\right)^r+\ldots\ldots\right]$$

$$=a^5\left[1-\frac{5}{\lfloor 1}\cdot\frac{x}{a}+\frac{5.2}{\lfloor 2}\left(\frac{x}{a}\right)^2+\frac{5.2.1}{\lfloor 3}\left(\frac{x}{a}\right)^3\right.$$

$$\left.+\ldots\ldots+\frac{5.2.1.4.7\ldots(3r-8)}{\lfloor r}\left(\frac{x}{a}\right)^r+\ldots\ldots\right].$$

After the second, all the signs are positive; for in the general term there are $r-2+r$, that is an *even* number, of negative factors.

285. The $(r+1)$th term of the expansion of $(1+x)^n$ is obtained from the rth by multiplying by $\frac{n-r+1}{r}x$, that is by $\left(-1+\frac{n+1}{r}\right)x$. Now $-1+\frac{n+1}{r}$ is always negative if $n+1$ is negative; and, whatever $n+1$ may be, $-1+\frac{n+1}{r}$ will be negative for all terms after the first for which $r>n+1$.

Hence, if x be positive, the ratio of the $r+1$th and rth terms will be always negative when $r>n+1$. The terms of the expansion of $(1+x)^n$ will therefore be *alternately positive and negative* after r terms, where r is the first positive integer greater than $n+1$.

If x be negative, the ratio of the $(r+1)$th and rth terms will be always positive when $r>n+1$. The terms of the expansion of $(1-x)^n$ will therefore be *all of the same sign* as the rth term, where r is the first positive integer greater than $n+1$; and, as a particular case, all the terms of the expansion of $(1-x)^n$ are positive when n is negative.

For example, all the terms in the expansion of $(1-x)^{\frac{3}{2}}$ are of the same sign as the rth, where r is the integer next greater than $\frac{3}{2}+1$, so that r is 3. Also, after the ninth, the terms of the expansion of $(1+x)^{\frac{15}{2}}$ are alternately positive and negative.

286. **Greatest Term.** In the expansion of $(1 \pm x)^n$ by the binomial theorem, we know that the ratio of the $(r+1)$th term to the rth is $\pm \dfrac{n-r+1}{r}x$, that is $\mp x\left(1-\dfrac{n+1}{r}\right)$; we also know that x must be numerically less than 1, unless n is a positive integer.

First suppose that $n+1$ is negative, and equal to $-m$. Then the absolute value of the ratio of the $(r+1)$th term to the rth term is $x\left(1+\dfrac{m}{r}\right)$. Hence the rth term is $\stackrel{>}{\stackrel{=}{<}}(r+1)$th term according as $x\left(1+\dfrac{m}{r}\right)\stackrel{<}{\stackrel{=}{>}}1$; that is, according as $r\stackrel{>}{\stackrel{=}{<}}\dfrac{mx}{1-x}$, that is $\stackrel{>}{\stackrel{=}{<}}\dfrac{-(1+n)x}{1-x}$.

Hence, if $\dfrac{-(1+n)x}{1-x}$ be an integer, r suppose, the rth term will be equal to the $(r+1)$th term, and these will be greater than any other terms. But, if $\dfrac{-(1+n)x}{1-x}$ be not an integer, the rth term will be the greatest when r is the integer next above $\dfrac{-(1+n)x}{1-x}$.

Next, suppose that $n+1$ is positive, and let k be the integer next greater than $n+1$. Then, if r be equal or greater than k, $\dfrac{n+1}{r}-1$ will be negative and less than unity; hence, as x must be less than unity, each term after the kth will be less than the one before it, and therefore the greatest term must precede the kth. And since, for values of r less than $n+1$, $\dfrac{n+1}{r}-1$ will be

positive; the rth term will be $\gtreqless (r+1)$th according as $\left(\frac{n+1}{r}-1\right)x \lesseqgtr 1$; that is, according as $r \gtreqless \frac{(n+1)x}{1+x}$.

Hence, if $\frac{(n+1)x}{1+x}$ be an integer, r suppose, the rth term will be equal to the $(r+1)$th, and these will be greater than any other terms. But, if $\frac{(n+1)x}{1+x}$ be not an integer, the rth term will be the greatest when r is the integer next above $\frac{(n+1)x}{x+1}$.

Ex. 1. Find the greatest term in the expansion of $(1-x)^{-\frac{3}{2}}$, when $x=\frac{8}{9}$. Here $n+1$ is negative, and $\frac{-(1+n)x}{1-x}=\frac{\frac{1}{2}\times\frac{8}{9}}{1-\frac{8}{9}}=4$. Hence the fourth and fifth terms are equal to one another, and are greater than any other terms.

Ex. 2. Find when the expansion of $(1-x)^{-\frac{17}{2}}$ begins to converge, if $x=\frac{3}{4}$.

Here $n+1$ is negative, and $\frac{-(1+n)x}{1-x}=\frac{\frac{15}{2}\times\frac{3}{4}}{\frac{1}{4}}=22\frac{1}{2}$. Hence the convergence begins after the 23rd term.

Ex. 3. Find the greatest term in the expansion of $(a+x)^{\frac{19}{2}}$, when $4x=3a$.

Since $(a+x)^{\frac{19}{2}}=a^{\frac{19}{2}}\left(1+\frac{x}{a}\right)^{\frac{19}{2}}$, the greatest term required is the term corresponding to the greatest term in $\left(1+\frac{x}{a}\right)^{\frac{19}{2}}$. Now $(n+1)\frac{x}{a}\div\left(1+\frac{x}{a}\right)=\frac{21}{2}\cdot\frac{3}{4}\div\frac{7}{4}=\frac{9}{2}$; hence r must be the integer next greater than $\frac{9}{2}$, so that the 5th term is the greatest.

EXAMPLES XXVII.

1. Find the general term in the expansion of each of the following expressions by the binomial theorem.

(i) $(1-x)^{-2}$, (ii) $(1-x)^{-3}$, (iii) $(1-x)^{-n}$.

(iv) $(1+x)^{-\frac{2}{3}}$, (v) $(1+x)^{\frac{2}{3}}$, (vi) $(1+x)^{\frac{5}{3}}$,
(vii) $(1-5x)^{-\frac{2}{5}}$, (viii) $(1-5x)^{\frac{2}{5}}$, (ix) $(1-x)^{-\frac{q}{p}}$,
(x) $(2a+3x)^{-\frac{2}{5}}$, (xi) $(a^2-2ax)^{\frac{5}{2}}$, and
(xii) $(4-7x)^{\frac{2}{7}}$.

2. Find the first negative term in the expansion (i) of $(1+\frac{4}{3}x)^{\frac{27}{4}}$, and (ii) of $(1+\frac{2}{3}x)^{\frac{21}{4}}$.

3. Find the greatest term in the expansion of $(1+x)^{-12}$ when $x=\frac{7}{9}$.

4. Find the greatest term in the expansion of $(1-\frac{2}{3}x)^{-2}$ when $x=\frac{3}{4}$.

5. After what term will the expansion of $(1-x)^{\frac{122}{5}}$ begin to converge, when $x=\frac{5}{6}$?

6. Shew that the coefficients of the first 19 terms in the expansion of $\dfrac{19-21x}{(1-x)^3}$ are all positive, and that the greatest of them is 100.

7. If a_1, a_2, a_3, a_4 be any four coefficients of consecutive terms of an expanded binomial, prove that

$$\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}.$$

8. Find the general term in the expansion by the binomial theorem of each of the following expressions according to ascending powers of x:

(i) $\dfrac{a}{\sqrt[3]{(a^2-x^2)}}$, (ii) $\dfrac{a+x}{a-x}$, (iii) $\left(\dfrac{a+x}{a-x}\right)^2$,

(iv) $(a+x)^{\frac{1}{2}}(a-x)^{-\frac{1}{2}}$, (v) $(a+x)^2(a-x)^{-3}$, and
(vi) $(a-x)^4(a+x)^{-2}$.

9. Shew that the coefficient of x^{3n} in the expansion of $(1+x^2)^3(1-x^3)^{-2}$ is $2n$.

10. Shew that the coefficient of x^n in the expansion of $(1+2x)^3(1-x)^{-2}$ is $27(n-1)$, $n \nless 3$.

287. **Sum of coefficients.** The sum of the first $r+1$ coefficients of the expansion of $(1-x)^n$ can be obtained as follows.

We have

$$(1-x)^n = 1 - \frac{n_1}{\lfloor 1}x + \frac{n_2}{\lfloor 2}x^2 - \dots + (-1)^r \frac{n_r}{\lfloor r}x^r + \dots,$$

also $$(1-x)^{-1} = 1 + x + x^2 + \dots + x^r + \dots$$

From [Art. 281] the coefficient of x^r in the product of the two series is equal to the coefficient of x^r in $(1-x)^n \times (1-x)^{-1}$, that is in $(1-x)^{n-1}$; hence we have

$$1 - \frac{n_1}{\lfloor 1} + \frac{n_2}{\lfloor 2} - \dots + (-1)^r \frac{n_r}{\lfloor r}$$

$$= \text{coefficient of } x^r \text{ in } (1-x)^{n-1} = (-1)^r \frac{(n-1)_r}{\lfloor r}.$$

Similarly, if $\phi(x) = a_0 + a_1x + a_2x^2 + \dots + a_rx^r + \dots$, the sum $a_0 + a_1 + \dots + a_r$ will be the coefficient of x^r in $\dfrac{\phi(x)}{1-x}$. Thus, to find the sum of the first $r+1$ coefficients in the expansion of $\phi(x)$, we have only to find the coefficient of x^r in the expansion of $\dfrac{\phi(x)}{1-x}$.

Ex. 1. Find the sum of the first r coefficients in the expansion of $(1-x)^{-3}$. *Ans.* $\frac{1}{6}r(r+1)(r+2)$.

The sum required is the coefficient of x^{r-1} in $(1-x)^{-4}$.

Ex. 2. Find the sum of n terms of the series

$$1.2.3 + 2.3.4 + 3.4.5 + \dots\dots$$

Since $(1-x)^{-4} = \dfrac{1}{1.2.3}[1.2.3 + 2.3.4x + 3.4.5x^2 + \dots\dots]$; the sum required $= 6 \times$ sum of the first n coefficients in the expansion of $(1-x)^{-4} = 6 \times$ coefficient of x^{n-1} in $(1-x)^{-5} = \frac{1}{4}n(n+1)(n+2)(n+3)$.

Ex. 3. Find the sum of the first $n+r$ coefficients in the expansion of $\dfrac{(1+x)^n}{(1-x)^2}$.

The sum required $=$ coefficient of x^{n+r-1} in the expansion of $\frac{(1+x)^n}{(1-x)^3}$. Now $(1+x)^n=(2-\overline{1-x})^n=2^n-n\,.\,2^{n-1}(1-x)$

$$+\frac{n(n-1)}{1\,.\,2}2^{n-2}(1-x)^2+\text{higher powers of }(1-x).$$

Hence $\frac{(1+x)^n}{(1-x)^3}=\frac{2^n}{(1-x)^3}-\frac{n2^{n-1}}{(1-x)^2}+\frac{n(n-1)\,2^{n-3}}{1-x}+$ an integral expression of the $(n-3)$th degree.

The coefficients of x^{n+r-1} in $(1-x)^{-3}$, $(1-x)^{-2}$ and $(1-x)^{-1}$ respectively are $\frac{1}{2}(n+r)(n+r+1)$, $n+r$, and 1; hence the coefficient of x^{n+r-1} in $\frac{(1+x)^n}{(1-x)^3}$ is

$$2^{n-1}(n+r)(n+r+1)-2^{n-1}n(n+r)+2^{n-3}n(n-1).$$

Ex. 4. Find the sum of n terms of the series

$$1+n+\frac{n(n+1)}{1\,.\,2}+\frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}+\dots\dots$$

Ans. $(2n-1)!/n!\,(n-1)!$.

288. **Binomial Series.** Series which are derived from the expansion of $(1+x)^n$ by giving particular values to x and n are of frequent occurrence: it is therefore of importance to be able to determine at once when a given series is a binomial series.

The case in which the index is a positive integer needs no remark.

When the index is a negative integer, we have

$$(1-x)^{-n}=1+nx+\frac{n(n+1)}{1\,.\,2}x^2+\dots$$
$$\dots+\frac{n(n+1)\dots(n+r-1)}{\underline{|r}}x^r+\dots,$$

and it should be carefully noticed that this expansion can be written in the form

$$(1-x)^{-n}=\frac{1}{\underline{|n-1}}[1\,.\,2\dots(n-1)+\{2\,.\,3\dots n\}x+\dots$$
$$\dots+\{(r+1)\dots(r+n-1)\}x^r+\dots].$$

When the index is fractional, $-p/q$ suppose, we have

$$(1 \pm x)^{-\frac{p}{q}} = 1 \mp \frac{p}{\lfloor 1} \frac{x}{q} + \frac{p(p+q)}{\lfloor 2} \left(\frac{x}{q}\right)^2$$

$$\mp \frac{p(p+q)(p+2q)}{\lfloor 3} \left(\frac{x}{q}\right)^3 + \dots\dots\dots\dots \text{(A)}.$$

Here we notice that (i) there is an additional factor both in the numerator and in the denominator for every successive term, (ii) the successive factors of the numerator are in an A.P. whose *common difference is the denominator of the index*, (iii) the successive factors of the denominator are 1, 2, 3, 4, &c., or multiples of these.

Bearing in mind the above laws, there will be no difficulty in determining the expression which will produce a given binomial series.

Ex. 1. Find the sum of the series

$$\frac{1}{3} + \frac{1 \cdot 3}{3 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{3 \cdot 6 \cdot 9} + \dots\dots \text{ to infinity.}$$

Writing the series in the form

$$\frac{1}{\lfloor 1} \cdot \frac{1}{3} + \frac{1 \cdot 3}{\lfloor 2} \cdot \frac{1}{3^2} + \frac{1 \cdot 3 \cdot 5 \cdot 1}{\lfloor 3} \cdot \frac{1}{3^3} + \dots\dots \equiv S,$$

we see from (A) that it is obtained from the expansion of $(1-x)^{-\frac{1}{2}}$ by giving to x the value found from $\frac{x}{2} = \frac{1}{3}$.

Thus $\left(1 - \frac{2}{3}\right)^{-\frac{1}{2}} = 1 + \frac{1}{2} \cdot \frac{2}{3} + \frac{\frac{1}{2} \cdot \frac{3}{2}}{1 \cdot 2} \left(\frac{2}{3}\right)^2 + \dots\dots = 1 + S$; therefore $S = \sqrt{3} - 1$.

Ex. 2. Find the sum of the series

$$1 + \frac{2}{6} + \frac{2 \cdot 5}{6 \cdot 12} + \frac{2 \cdot 5 \cdot 8}{6 \cdot 12 \cdot 18} + \dots\dots \text{ to infinity.}$$

Writing the series in the form

$$1 + \frac{2}{\lfloor 1} \cdot \frac{1}{6} + \frac{2 \cdot 5}{\lfloor 2} \frac{1}{6^2} + \frac{2 \cdot 5 \cdot 8}{\lfloor 3} \cdot \frac{1}{6^3} + \dots\dots,$$

we see from (A) that it is obtained from the expansion of $(1-x)^{-\frac{2}{3}}$

by giving to x the value $\frac{x}{3}=\frac{1}{6}$. Hence the sum required is $\left(1-\frac{1}{2}\right)^{-\frac{2}{3}}=\sqrt[3]{4}$.

Ex. 3. Find the sum of the series $\frac{3}{18}+\frac{3.7}{18.24}+\frac{3.7.11}{18.24.30}+\ldots\ldots$ to infinity.

In this case the factors of the denominator, although multiples of 1, 2, 3, 4, &c., do not begin at the beginning. Additional factors must therefore be introduced in the denominator, and corresponding additional factors in the numerator. We then have

$$\frac{(-5)(-1)3}{\underline{|3}}\frac{1}{6^3}+\frac{(-5)(-1)3.7}{\underline{|4}}\frac{1}{6^4}+\ldots\ldots$$

Now the terms of this latter series are terms of (A), if $q=4, p=-5$, and $\frac{x}{4}=\frac{1}{6}$.

We can therefore find the required sum, as follows:

$$\left(1-\frac{4}{6}\right)^{\frac{5}{4}}=1-\frac{5}{1}.\frac{1}{6}+\frac{5.1}{\underline{|2}}\frac{1}{6^2}+\frac{5.1.3}{\underline{|3}}\frac{1}{6^3}+\frac{5.1.3.7}{\underline{|4}}\frac{1}{6^4}+\ldots\ldots$$

$$=1-\frac{5}{6}+\frac{5}{6.12}+\frac{5}{6.12}\left[\frac{3}{18}+\frac{3.7}{18.24}+\frac{3.7.11}{18.24.30}+\ldots\ldots\right];$$

$$\therefore \left(\frac{1}{3}\right)^{\frac{5}{4}}=1-\frac{5}{6}+\frac{5}{72}+\frac{5}{72}S.$$

Whence $S=\frac{1}{5}\{8\sqrt[4]{27}-17\}$.

Ex. 4. Find the sum of the series $\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\ldots\ldots$ to infinity.

$\left[\text{From } \left(1-\frac{1}{2}\right)^{-\frac{3}{2}}\right]$. *Ans.* $2\sqrt{2}-1$.

Ex. 5. Find the sum to infinity of the series

$$\frac{1}{2^3\underline{|3}}-\frac{1.3}{2^4\underline{|4}}+\frac{1.3.5}{2^5\underline{|5}}-\ldots\ldots$$

[From $(1+1)^{\frac{3}{2}}$]. *Ans.* $\frac{23}{24}-\frac{2}{3}\sqrt{2}$.

Ex. 6. Shew that $1+\frac{1}{4}+\frac{1.4}{4.8}+\frac{1.4.7}{4.8.12}+\frac{1.4.7.10}{4.8.12.16}+\ldots\ldots$ to infinity $=1+\frac{2}{6}+\frac{2.5}{6.12}+\frac{2.5.8}{6.12.18}+\frac{2.5.8.11}{6.12.18.24}+\ldots\ldots$ to infinity.

$\left[\text{Since } \left(1-\frac{3}{4}\right)^{-\frac{1}{3}}=\left(1-\frac{1}{2}\right)^{-\frac{2}{3}}\right]$.

289. We know from Art. 281 that if any expression containing x be expanded in two different convergent series arranged according to ascending powers of x, the coefficients of like powers of x in the two series will be equal. By means of this very important principle many theorems can be proved.

Ex. 1. Shew that, if n be any positive integer,

$$1-\frac{n^2}{1^2}+\frac{n^2(n^2-1^2)}{1^2.2^2}-\frac{n^2(n^2-1^2)(n^2-2^2)}{1^2.2^2.3^2}+\ldots\ldots=0.$$

We have $(1-x)^n=1-nx+\frac{n(n-1)}{1.2}x^2-\frac{n(n-1)(n-2)}{1.2.3}x^3+\ldots\ldots$

$$+(-1)^n\frac{n(n-1)\ldots(n-\overline{n-1})}{1.2\ldots n}x^n.$$

Also, provided $x>1$, we have

$$\left(1-\frac{1}{x}\right)^{-n}=1+n\frac{1}{x}+\frac{n(n+1)}{1.2}\frac{1}{x^2}+\frac{n(n+1)(n+2)}{1.2.3}\frac{1}{x^3}+\ldots\ldots$$

$$+\frac{n(n+1)\ldots(n+\overline{n-1})}{1.2\ldots n}\frac{1}{x^n}+\ldots\ldots$$

Hence $1-n^2+\frac{n^2(n^2-1^2)}{1^2.2^2}-\ldots\ldots+(-1)^n\frac{n^2(n^2-1^2)\ldots\{n^2-(n-1)^2\}}{1^2.2^2\ldots n^2}$

is equal to the coefficient of x^0 in $(1-x)^n\times\left(1-\frac{1}{x}\right)^{-n}$, that is equal to the coefficient of x^0 in $(-1)^nx^n$, which is zero. [See also Art. 251, Ex. 3.]

Ex. 2. Find the sum of

$$(n+1)+n.\frac{1}{2}+(n-1)\frac{1.3}{2.4}+\ldots\ldots+1.\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}.$$

[Equate coefficients of x^n in $(1-x)^{-\frac{1}{2}}\times(1-x)^{-2}$ and in $(1-x)^{-\frac{5}{2}}$.]

Ans. $\frac{5.7\ldots(2n+3)}{2.4\ldots 2n}$.

Ex. 3. Shew that $1-3n+\frac{(3n-1)(3n-2)}{1.2}-\ldots\ldots=(-1)^n$.

We have $\frac{1+x}{1+x^3}=\frac{1}{1-x+x^2}=\frac{1}{1-x(1-x)}$.

Hence $(1+x)\{1-x^3+x^6\ldots+(-1)^nx^{3n}+\ldots\}$

$$=1+x(1-x)+x^2(1-x)^2+\ldots+x^{3n+1}(1-x)^{3n+1}+\ldots$$

The coefficient of x^{3n+1} on the left is $(-1)^n$.

The terms on the right which give x^{3n+1} are

$$x^{3n+1}(1-x)^{3n+1}+x^{3n}(1-x)^{3n}+x^{3n-1}(1-x)^{3n-1}+\ldots;$$

and hence the coefficient of x^{3n+1} will be found to be

$$1-3n+\frac{(3n-1)(3n-2)}{1.2}-\frac{(3n-2)(3n-3)(3n-4)}{1.2.3}+\ldots\ldots$$

290. **Expansion of Multinomials.** Any multinomial expression can be expanded by means of the binomial theorem.

Since $(p+qx+rx^2+\ldots)^n$ may be written in the form $p^n\left(1+\frac{q}{p}x+\frac{r}{p}x^2+\ldots\right)^n$, it is only necessary to consider expressions in which the first term is unity.

Now in the expansion of $\{1+ax+bx^2+cx^3+\ldots\}^n$, that is of $\{1+(ax+bx^2+cx^3+\ldots)\}^n$, by the binomial theorem, the general term is

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}}(ax+bx^2+cx^3+\ldots)^r;$$

also in the expansion of $(ax+bx^2+cx^3+\ldots)^r$, r being a positive integer, the general term is by Art. 262

$$\frac{\underline{|r}}{\underline{|\alpha}\,\underline{|\beta}\,\underline{|\gamma}\ldots}a^\alpha b^\beta c^\gamma\ldots x^{\alpha+2\beta+3\gamma+\ldots},$$

where each of α, β, γ,... is zero or a positive integer, and $\alpha+\beta+\gamma+\ldots=r$.

Hence the general term of the expansion of the multinomial is

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|\alpha}\,\underline{|\beta}\,\underline{|\gamma}\ldots}a^\alpha b^\beta c^\gamma\ldots x^{\alpha+2\beta+3\gamma+\ldots}.$$

To find the coefficient of any particular power of x, say of x^k, we must therefore find all the different sets of positive integral values (including zero) of α, β, γ,... which satisfy the equation $\alpha+2\beta+3\gamma+\ldots=k$; the corresponding value of r is then given by $r=\alpha+\beta+\gamma+\ldots$, and the corresponding coefficient is found by substituting

in the formula for the general term. The required coefficient will then be the sum of the coefficients corresponding to each set of values of α, β, γ....

Ex. 1. Find the coefficient of x^5 in $(1-x+2x^2-3x^3)^{-\frac{1}{2}}$.

The values of α, β, γ which satisfy $\alpha+2\beta+3\gamma=5$ will be found to be 0, 1, 1; 2, 0, 1; 1, 2, 0; 3, 1, 0; and 5, 0, 0. The corresponding values of r will be 2, 3, 3, 4 and 5 respectively; and the corresponding coefficients will be

$$\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\lfloor 1 \lfloor 1}(2)^1(-3)^1, \qquad \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{\lfloor 2 \lfloor 1}(-1)^2(-3)^1,$$

$$\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{\lfloor 1 \lfloor 2}(-1)^1(2)^2, \qquad \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{\lfloor 3 \lfloor 1}(-1)^3(2)^1,$$

and $$\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right)}{\lfloor 5}(-1)^5;$$

that is $-\frac{9}{2}$, $\frac{45}{16}$, $\frac{15}{4}$, $-\frac{35}{16}$ and $\frac{63}{256}$.

Hence the required coefficient is $\frac{31}{256}$.

291. From the above example it will be seen that the process of finding even the first six terms in the expansion of a multinomial is very laborious; in many cases, however, the work can be much shortened, as in the following examples.

Ex. 2. Find the coefficient of x^{13} in the expansion of

$$(1+x+x^2+x^3+x^4)^{-2}.$$

We have $(1+x+x^2+x^3+x^4)^{-2}=\left(\frac{1-x^5}{1-x}\right)^{-2}=(1-x)^2(1-x^5)^{-2}$

$$=(1-2x+x^2)(1+2x^5+3x^{10}+4x^{15}+\ldots).$$

Hence the coefficient required is zero.

Ex. 3. Find the coefficient of x^n in the expansion of $(1+x+x^2+x^3)^{-1}$.

We have $(1+x+x^2+x^3)^{-1}=\frac{1}{1+x+x^2+x^3}=\frac{1-x}{1-x^4}$

$$=(1-x)(1+x^4+x^8+\ldots+x^{4r}+\ldots).$$

Hence the coefficient of x^{4r} is 1, the coefficient of x^{4r+1} is -1, the coefficient of x^{4r+2} is zero, and the coefficient of x^{4r+3} is zero.

Thus the coefficient of x^n is 1 when n is of the form $4r$, it is -1 when n is of the form $4r+1$, and it is zero when n is of either of the forms $4r+2$ or $4r+3$.

Ex. 4. Find the coefficient of x^r in the expansion of

$$(1+2x+3x^2+4x^3+\ldots\ldots \text{ to infinity})^n.$$

Since $1+2x+3x^2+\ldots\ldots=(1-x)^{-2}$, the required expansion is that of $(1-x)^{-2n}$; the coefficient of x^r is therefore

$$\frac{2n(2n+1)\ldots(2n+r-1)}{\underline{|r}}.$$

292. **Combinations with repetitions.** The number of combinations of n things α together of which p are of one kind, q of a second, r of a third, and so on, can be found in the following manner.

Let the different things be represented by the letters $a, b, c, \ldots$; and consider the continued product

$$(1+ax+a^2x^2+\ldots+a^px^p)(1+bx+\ldots+b^qx^q)(1+cx+\ldots+c^rx^r)\ldots$$

It is clear that all the terms in the continued product are of the same degree in the letters $a, b, c, \ldots$ as in x; it is also clear that the coefficient of x^α is the sum of all the different ways of taking α of the letters $a, b, c, \ldots$ with the restriction that there are to be not more than p a's, not more than q b's, &c.; so that the coefficient of x^α in the continued product gives the actual combinations required. Hence the *number* of the combinations will be given by putting $a=b=c=\ldots=1$. Thus the number of the combinations of the n things α together is the coefficient of x^α in

$$(1+x+x^2+\ldots+x^p)(1+x+\ldots+x^q)(1+x+\ldots+x^r)\ldots$$

Permutations. The number of permutations of the n things α together being represented by P_α, it is easily seen that

$$\left\{1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\ldots+\frac{x^p}{\underline{|p}}\right\}\left\{1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\ldots\ldots+\frac{x^q}{\underline{|q}}\right\}\times\ldots\ldots$$
$$=1+\frac{P_1}{\underline{|1}}x+\frac{P_2}{\underline{|2}}x^2+\ldots+\frac{P_n}{\underline{|n}}x^n.$$

For $\lfloor\underline{\alpha}\times$ the coefficients of x^{α} in

$$\left\{1+\frac{ax}{\lfloor\underline{1}}+\frac{a^2x^2}{\lfloor\underline{2}}+\ldots+\frac{a^px^p}{\lfloor\underline{p}}\right\}$$

$$\times\left\{1+\frac{bx}{\lfloor\underline{1}}+\frac{b^2x^2}{\lfloor\underline{2}}+\ldots+\frac{b^qx^q}{\lfloor\underline{q}}\right\}\times\ldots$$

is the sum of all possible terms of the form

$$\frac{\lfloor\underline{\alpha}}{\lfloor\underline{l}\lfloor\underline{m}\ldots}a^lb^m,\ldots$$

for which $l+m+\ldots=\alpha$, and the number of permutations α together formed by taking l of the a's, m of the b's, &c. is

$$\frac{\lfloor\underline{\alpha}}{\lfloor\underline{l}\lfloor\underline{m}\ldots}.$$

Ex. 1. Find the number of combinations 7 together of 5 a's, 4 b's and 2 c's.

The number required is the coefficient of x^7 in $(1+x+\ldots+x^5)(1+x+\ldots x^4)(1+x+x^2)$, that is in $(1-x^6)(1-x^5)(1-x^3)(1-x)^{-3}$. Rejecting terms of higher than the seventh degree in the continued product of the first three factors, we have

$$(1-x^3-x^5-x^6)(1+3x+6x^2+10x^3+15x^4+21x^5+28x^6+36x^7+\ldots);$$

and the coefficient of x^7 is $36-15-6-3=12$.

Ex. 2. Find the total number of ways in which a selection can be made from n things of which p are alike of one kind, q alike of a second kind, and so on.

The total number of the combinations is the sum of the coefficients of $x^1, x^2,\ldots, x^n$ in $(1+x+\ldots+x^p)(1+x+\ldots+x^q)\ldots$; and this sum is obtained by putting $x=1$ in the product and subtracting 1 for the coefficient of x^0. Hence the required number is

$$(p+1)(q+1)\ldots-1.$$

The above result can, however, be obtained at once from the consideration that there are $p+1$ ways of selecting from the a's, namely by taking 0, or 1, or 2,... or p of them; and, when this is done, there are $q+1$ ways of selecting from the b's; and so on.

Hence the total number of ways, *excluding* the case in which no letter at all is selected, is $(p+1)(q+1)\ldots-1$. [Whitworth's *Choice and Chance*, Prop. XIII.]

Ex. 3. A candidate is examined in three papers to each of which m marks are assigned as a maximum. His total in the three papers is $2m$; shew that there are $\frac{1}{2}(m+1)(m+2)$ ways in which this may occur.

The number of ways is the coefficient of x^{2m} in $(1+x+x^2+\ldots x^m)^3$, that is in $(1-x^{m+1})^3(1-x)^{-3}=(1-3x^{m+1}+\ldots)\times$

$\frac{1}{2}\{1.2+2.3x+\ldots+m(m+1)x^{m-1}+\ldots+(2m+1)(2m+2)x^{2m}+\ldots\}$.

Hence the number required

$$=\frac{1}{2}\{(2m+1)(2m+2)-3m(m+1)\}=\frac{1}{2}(m+1)(m+2).$$

Ex. 4. Shew that the number of permutations four at a time which can be made of n groups of things of which each consists of three things like one another but unlike all the rest is n^4-n.

The number required is equal to $\underline{|4}\times$ the coefficient of x^4 in

$$\left(1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}\right)^n.$$

293. **Homogeneous Products.** We have already [Art. 250] found the number of homogeneous products of r dimensions which can be formed with n letters, where each letter may be repeated any number of times. We now give another method of obtaining the result. Suppose the letters to be $a, b, c,\ldots$; then if the continued product

$$(1+ax+a^2x^2+a^3x^3+\ldots)\times(1+bx+b^2x^2+b^3x^3+\ldots)$$
$$\times(1+cx+c^2x^2+c^3x^3+\ldots)\ldots$$

be formed, the coefficient of x^r will clearly be of r dimensions in the letters $a, b, c,\ldots$, and will be the sum of all the possible ways of taking r of the letters*. Hence the *number* of the products each of r dimensions will be given by putting $a=b=c=\ldots=1$ in the continued product. Thus the number required is the coefficient of x^r in $(1+x+x^2+\ldots)^n$, that is in $(1-x)^{-n}$. Hence

$${}_nH_r=\frac{n(n+1)\ldots(n+r-1)}{\underline{|r}}=\frac{\underline{|n+r-1}}{\underline{|r}\,\underline{|n-1}}.$$

This result can be expressed in the form ${}_nH_r={}_{n+r-1}C_r$.

* An expression for the sum of the homogeneous products will be found in Art. 300, Ex. 4.

COR. The number of terms in the expansion of $(a_1+a_2+a_3+\ldots+a_n)^r$ is $\dfrac{|\underline{n+r-1}}{|\underline{r}\,|\underline{n-1}}$.

294. We shall conclude this chapter by solving the following examples.

Ex. 1. Find $\sqrt{14}$, by the binomial theorem, to six places of decimals.

$$\sqrt{14}=\sqrt{(16-2)}=4\left(1-\frac{1}{8}\right)^{\frac{1}{2}}=4\left\{1-\frac{1}{2}\cdot\frac{1}{8}-\frac{1}{2\,.\,4}\frac{1}{8^2}-\frac{1\,.\,3}{2\,.\,4\,.\,6}\frac{1}{8^3}-\ldots\right\}$$
$$=4\,\{1-\cdot0625-\cdot001953-\cdot0001220-\cdot0000095-\cdot0000010\}$$
$$=3\cdot741657.$$

Ex. 2. Shew that, when x is small,

$$\frac{(1-3x)^{-\frac{2}{3}}+(1-4x)^{-\frac{3}{4}}}{(1-3x)^{-\frac{1}{3}}+(1-4x)^{-\frac{1}{4}}}=1+\frac{3}{2}x \text{ approximately.}$$

Since x is small, its square and higher powers may be rejected; and when all powers of x except the first are neglected the given expansion becomes equal to

$$\frac{1+\frac{2}{3}\,.\,3x+1+\frac{3}{4}\,.\,4x}{1+\frac{1}{3}\,.\,3x+1+\frac{1}{4}\,.\,4x}=\frac{2+5x}{2+2x}=\frac{1+\frac{5}{2}x}{1+x}$$
$$=\left(1+\frac{5}{2}x\right)(1+x)^{-1}=\left(1+\frac{5}{2}x\right)(1-x)=1+\frac{3}{2}x.$$

Ex. 3. Shew that the integral part of

$$(\sqrt{3}+1)^{2n+1} \text{ is } (\sqrt{3}+1)^{2n+1}-(\sqrt{3}-1)^{2n+1}.$$

Since $\sqrt{3}-1$ is a proper fraction, $(\sqrt{3}-1)^{2n+1}$ must also be a proper fraction. It therefore follows that if $(\sqrt{3}+1)^{2n+1}-(\sqrt{3}-1)^{2n+1}$ be an integer, it must be the integral part of $(\sqrt{3}+1)^{2n+1}$.

Now $\qquad (\sqrt{3}+1)^{2n+1}-(\sqrt{3}-1)^{2n+1}$

$$=\{3^n\sqrt{3}+(2n+1)\,3^n+\frac{(2n+1)\,2n}{1\,.\,2}3^{n-1}\sqrt{3}+\ldots\ldots+(2n+1)\sqrt{3}+1\}$$
$$-\{3^n\sqrt{3}-(2n+1)\,3^n+\frac{(2n+1)\,2n}{1\,.\,2}3^{n-1}\sqrt{3}-\ldots\ldots+(2n+1)\sqrt{3}-1\}$$
$$=2\,\{(2n+1)\,3^n+\frac{(2n+1)\,2n\,(2n-1)}{1.2.3}3^{n-1}+\ldots\ldots+1\},$$

all the irrational terms disappearing.

Since the coefficients of all the different powers of 3 in the last expression are integers, it follows that $(\sqrt{3}+1)^{2n+1}-(\sqrt{3}-1)^{2n+1}$ is an integer, and is moreover an *even* integer.

By the following method it can be proved that

$(\sqrt{3}+1)^{2n+1}-(\sqrt{3}-1)^{2n+1}$ is an integer divisible by 2^{n+1}.

Represent $(\sqrt{3}+1)^{2n+1}-(\sqrt{3}-1)^{2n+1}$ by I_{2n+1}.

Then $I_1=2$; and it will be found that $I_3=20$, and also that

$$(\sqrt{3}+1)^2+(\sqrt{3}-1)^2=8.$$

Hence

$$8I_{2n+1}=\{(\sqrt{3}+1)^{2n+1}-(\sqrt{3}-1)^{2n+1}\}\{(\sqrt{3}+1)^2+(\sqrt{3}-1)^2\}$$
$$=(\sqrt{3}+1)^{2n+3}-(\sqrt{3}-1)^{2n+3}+4\{(\sqrt{3}+1)^{2n-1}-(\sqrt{3}-1)^{2n-1}\};$$
$$\therefore\ I_{2n+3}=8I_{2n+1}-4I_{2n-1}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}.$$

It follows from the last relation that I_{2n+3} will be an integer if I_{2n+1} and I_{2n-1} are integers. Now we know that I_1 and I_3 are integers; hence by induction I_{2n+1} is always an integer.

The relation (A) also shews that I_{2n+3} will be divisible by 2^{n+2} provided I_{2n+1} is divisible by 2^{n+1} and I_{2n-1} by 2^n. Now we know that I_1 is divisible by 2^1 and I_3 by 2^2; hence I_5 must be divisible by 2^3; and it will then follow that I_7 must be divisible by 2^4; and so on, so that I_{2n+1} is always divisible by 2^{n+1}.

Ex. 4. To shew that, if n be any positive integer,

$$a^n-n(a+b)^n+\frac{n(n-1)}{1\,.\,2}(a+2b)^n-\ldots\ldots=(-b)^n\,\underline{|n}.$$

Put $\dfrac{y+a}{b}$ for x in the identity proved in Art. 259, Ex. 3; then, after reduction, we have

$$\frac{\underline{|n}\,b^n}{(y+a)(y+a+b)\ldots(y+a+nb)}=\frac{c_0}{y+a}-\frac{c_1}{y+a+b}+\ldots$$
$$\ldots+(-1)^r\frac{c_r}{y+a+rb}+\ldots$$

Now expand the expressions on the two sides in powers of $\dfrac{1}{y}$.

Left side $=\dfrac{\underline{|n}\,b^n}{y^{n+1}\left(1+\dfrac{a}{y}\right)\ldots\left(1+\dfrac{a+nb}{y}\right)}=\dfrac{\underline{|n}\,b^n}{y^{n+1}}+$ higher negative powers of y.

Right side $=\dfrac{c_0}{y}\left(1+\dfrac{a}{y}\right)^{-1}-\ldots+(-1)^r\dfrac{c_r}{y}\left(1+\dfrac{a+rb}{y}\right)^{-1}+\ldots\ldots;$

hence the coefficient of $\dfrac{1}{y^{k+1}}$ on the right is

$$(-1)^k[c_0a^k-c_1(a+b)^k+\ldots\ldots+(-1)^rc_r(a+rb)^k+\ldots].$$

Hence $\Sigma(-1)^rc_r(a+rb)^k$ is zero if $k<n$, and is equal to $(-1)^nb^n\,\underline{|n}$ if $k=n$.

EXAMPLES XXVIII.

1. Find the sum to infinity of each of the following series:

(i) $1 + \frac{1}{\underline{|1}}\frac{3}{2^3} + \frac{1.3}{\underline{|2}}\frac{3^2}{2^6} + \frac{1.3.5}{\underline{|3}}\frac{3^3}{2^9} + \dots$

(ii) $1 - \frac{1}{2}\frac{1}{2} + \frac{1.3}{2.4}\frac{1}{2^2} - \frac{1.3.5}{2.4.6}\frac{1}{2^3} + \dots$

(iii) $1 + \frac{4}{\underline{|1}}\frac{1}{4} + \frac{4.7}{\underline{|2}}\frac{1}{4^2} + \frac{4.7.10}{\underline{|3}}\frac{1}{4^3} + \dots$

(iv) $\frac{3.5}{3.6} + \frac{3.5.7}{3.6.9} + \frac{3.5.7.9}{3.6.9.12} + \dots$

(v) $\frac{3}{2.4} + \frac{3.4}{2.4.6} + \frac{3.4.5}{2.4.6.8} + \frac{3.4.5.6}{2.4.6.8.10} + \dots$

(vi) $1 + \frac{2}{6} + \frac{2.5}{6.12} + \frac{2.5.8}{6.12.18} + \dots$

(vii) $1 - \frac{3}{4} + \frac{3.5}{4.8} - \frac{3.5.7}{4.8.12} + \dots$

(viii) $\frac{4}{18} + \frac{4.12}{18.27} + \frac{4.12.20}{18.27.36} + \dots$

(ix) $1 + \frac{2}{9} + \frac{2.5}{9.18} + \frac{2.5.8}{9.18.27} + \dots$

(x) $\frac{1}{9.18} - \frac{1.3}{9.18.27} + \frac{1.3.5}{9.18.27.36} - \dots$

(xi) $\frac{1}{2.4.6} + \frac{1.3}{2.4.6.8} + \frac{1.3.5}{2.4.6.8.10} + \dots$

(xii) $\frac{7}{72} + \frac{7.28}{72.96} + \frac{7.28.49}{72.96.120} + \dots$

2. Shew that

$$\frac{1 + n\frac{a}{a+b} + \frac{n(n+1)}{1.2}\left(\frac{a}{a+b}\right)^2 + \dots}{1 + n\frac{b}{a+b} + \frac{n(n+1)}{1.2}\left(\frac{b}{a+b}\right)^2 + \dots} = \frac{a^n}{b^n}.$$

3. Shew that

$$(1+x)^n = 2^n\left\{1 - n\frac{1-x}{1+x} + \frac{n(n+1)}{1\,.\,2}\left(\frac{1-x}{1+x}\right)^2 - \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}\left(\frac{1-x}{1+x}\right)^3 + \dots\right\}.$$

4. Shew that, if x be greater than $-\frac{1}{2}$,

$$\frac{x}{\sqrt{(x+1)}} = \frac{x}{1+x} + \frac{1}{2}\left(\frac{x}{1+x}\right)^2 + \frac{1\,.\,3}{2\,.\,4}\left(\frac{x}{1+x}\right)^3 + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\left(\frac{x}{1+x}\right)^4 + \dots\dots$$

5. Shew that

$$(1-x^2)^n = (1+x)^{2n} - 2nx(1+x)^{2n-1} + \frac{2n(2n-2)}{1\,.\,2}x^2(1+x)^{2n-2} - \dots$$

6. Shew that

$$1 + n\frac{a-x}{a+x} + \frac{n(n+1)}{1\,.\,2}\left(\frac{a-x}{a+x}\right)^2 + \dots\dots = \left(\frac{a+x}{2x}\right)^n.$$

7. Shew that

$$(1+x)^{2n} = (1+x)^n + nx(1+x)^{n-1} + \frac{n(n+1)}{1\,.\,2}x^2(1+x)^{n-2} + \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}x^3(1+x)^{n-3} + \dots$$

8. Shew that, if $a < b$,

$$a^2b^2 = (a+b)^4\left\{\frac{a^2}{b^2} - \frac{4}{1}\cdot\frac{a^3}{b^3} + \frac{4\,.\,5}{1\,.\,2}\frac{a^4}{b^4} - \frac{4\,.\,5\,.\,6}{1\,.\,2\,.\,3}\frac{a^5}{b^5} + \dots\dots\right\}.$$

9. Shew that

$$1 - \frac{n+x}{1+x} + \frac{(n+2x)(n-1)}{\lfloor 2\,(1+x)^2} - \frac{(n+3x)(n-1)(n-2)}{\lfloor 3\,(1+x)^3} + \dots = 0.$$

10. Shew that, if the numerical value of y be less than one-third of that of x,

$$1 + n\frac{2y}{x+y} + \frac{n(n+1)}{1\,.\,2}\left(\frac{2y}{x+y}\right)^2 + \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}\left(\frac{2y}{x+y}\right)^3 + \dots$$

$$= 1 + n\frac{2y}{x-y} + \frac{n(n-1)}{1\,.\,2}\left(\frac{2y}{x-y}\right)^2 + \dots\dots$$

11. Find the value of

$$r-(r-1)n+(r-2)\frac{n(n-1)}{1.2}-(r-3)\frac{n(n-1)(n-2)}{1.2.3}+\ldots$$

to r terms.

12. Shew that, if n be a positive integer,

$$n-\frac{n^2(n-1)}{\lfloor 1 \lfloor 2}+\frac{n^2(n^2-1^2)(n-2)}{\lfloor 2 \lfloor 3} - \frac{n^2(n^2-1^2)(n^2-2^2)(n-3)}{\lfloor 3 \lfloor 4}+\ldots=0.$$

13. Shew that, if n be a positive integer,

$$n-\frac{n(n^2-1^2)}{\lfloor 1 \lfloor 2}+\frac{n(n^2-1^2)(n^2-2^2)}{\lfloor 2 \lfloor 3}-\ldots\ldots + (-1)^r\frac{n(n^2-1^2)\ldots(n^2-r^2)}{\lfloor r \lfloor r+1}+\ldots=(-1)^{n+1}.$$

14. Shew that if n be a positive integer $\nless 4$

$$1-4n+\frac{4.5}{1.2}\,\frac{n(n-1)}{1.2}-\frac{4.5.6}{1.2.3}\,\frac{n(n-1)(n-2)}{1.2.3}+\ldots=0.$$

15. Shew that

$$1.n(n+1)+2(n-1)n+3(n-2)(n-1)+\ldots+n.1.2 = \frac{1}{12}n(n+1)(n+2)(n+3).$$

16. Prove that

$$1.n(n+1)+\frac{n}{1}.(n-1)n+\frac{n(n+1)}{1.2}(n-2)(n-1) + \frac{n(n+1)(n+2)}{1.2.3}(n-3)(n-2)+\ldots=2\frac{\lfloor 2n+1}{\lfloor n-1 \lfloor n+2}.$$

17. Shew that, if $p_r=\dfrac{1.3.5\ldots(2r-1)}{2.4.6\ldots 2r}$,

$$p_n+p_{n-1}p_1+p_{n-2}p_2+\ldots+p_1p_{n-1}+p_n=1.$$

18. If $p_r = \dfrac{1 \cdot 3 \cdot 5 \dots (2r-1)}{2 \cdot 4 \cdot 6 \dots 2r}$, and $q_r = \dfrac{5 \cdot 7 \dots (2r+3)}{2 \cdot 4 \dots 2r}$, prove that
$$p_r + p_{r-1}q_1 + p_{r-2}q_2 + \dots + q_r = \tfrac{1}{2}(r+1)(r+2).$$

19. Shew that
$$1 + 2(n-1) + 2^2\frac{(n-2)(n-3)}{1 \cdot 2} + 2^3\frac{(n-3)(n-4)(n-5)}{1 \cdot 2 \cdot 3} + \dots$$
$$= \tfrac{1}{3}\{2^{n+1} + (-1)^n\}.$$

20. Shew that $\dfrac{a^n - b^n}{a-b} = (a+b)^{n-1} - (n-2)ab(a+b)^{n-3}$
$$+ \frac{(n-3)(n-4)}{1 \cdot 2}a^2b^2(a+b)^{n-5} - \dots$$

21. From the expansion of $(1 + 2x + x^2)^{2n}$ prove that
$$2^{2n} + \frac{2n(2n-1)}{\lfloor 1 \lfloor 1} 2^{2n-2} + \frac{2n(2n-1)(2n-2)(2n-3)}{\lfloor 2 \lfloor 2} 2^{2n-4} + \dots$$
$$+ \frac{\lfloor 2n}{\lfloor n \lfloor n} = \frac{\lfloor 4n}{\lfloor 2n \lfloor 2n}.$$

22. Shew that
$$\frac{n(n+1)\dots(n+m-1)}{\lfloor m} - n\frac{n(n+1)\dots(n+m-4)}{\lfloor m-3}$$
$$+ \frac{n(n-1)}{1 \cdot 2}\,\frac{n(n+1)\dots(n+m-7)}{\lfloor m-6} - \dots = 0,$$
if $m > 2n$, and $= 1$ if $m = 2n$.

23. Find the coefficient of x^n in
$$(1+x)(1+x^2)(1+x^4)(1+x^8)\dots$$

24. Shew that, if x be a proper fraction,
$$\frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)\dots} = (1+x)(1+x^2)(1+x^3)\dots$$

25. In how many ways can 12 pennies be distributed among 6 children so that each may receive one at least, and none more than three?

26. There are n things of which p are alike and the rest unlike; prove that the total number of combinations that can be formed of them is $(p+1)\,2^{n-p}-1$.

27. Shew that the number of ways in which n like things can be allotted to r different persons, blank lots being admissible, is ${}_{n+r-1}C_{r-1}$.

28. Shew that the number of combinations n together of $2n$ things, n of which are alike and the rest are all different, is 2^n.

29. The number of combinations n together of $3n$ things, of which n are alike and the rest all different, is

$$2^{2n-1}+\underline{|2n-1}\Big/\underline{|n}\,\underline{|n-1}.$$

30. A man goes in for an examination in which there are four papers with a maximum of m marks for each paper; shew that the number of ways of getting half marks on the whole is

$$\tfrac{1}{3}(m+1)(2m^2+4m+3).$$

31. Find the coefficient of x^4 in $(1-2x-2x^2)^{\frac{7}{2}}$.

32. Find the coefficients of x^5 in the expansions of

$$(1+x+x^2+x^3+x^4)^6 \text{ and } (1+x+x^2+x^3+x^4+x^5)^8.$$

33. In a shooting competition a man can score 5, 4, 3, 2, 1 or 0 points for each shot. Find the number of different ways in which he can score 30 in 7 shots.

34. In how many ways can 20 be thrown with 4 dice, each of which has six faces marked 1, 2, 3, 4, 5, 6 respectively?

35. Find the coefficient of x^r in the expansion, according to ascending powers of x, of $(4a^2+6ax+9x^2)^{-1}$.

36. Shew that the coefficient of x^{3m} in the expansion of $\dfrac{1+x}{(1+x+x^2)^2}$ is $2m+1$.

37. Shew that the coefficient of x^r in the expansion of

$$(1+2x+3x^2+\dots)^2 \text{ is } \tfrac{1}{6}(r+1)(r+2)(r+3).$$

38. Find the coefficient of x^n in the expansion of

$$\{1 . 2 + 2 . 3x + 3 . 4x^2 + \dots \text{ to infinity}\}^2.$$

39. Find the coefficient of x^r in the expansion of

$$(1 . 2 + 2 . 3 . 2x + 3 . 4 . 2^2x^2 + \dots\dots + (n+1)(n+2)2^n x^n + \dots \text{ to infinity})^n.$$

40. Shew that the coefficient of x^r in the expansion of

$$(1 + x + 2x^2 + 3x^3 + \dots)^2 \text{ is } \tfrac{1}{6}r(r^2 + 11).$$

41. Shew that if $p - q$ be small compared with p or q, then will

$$\sqrt[n]{\frac{p}{q}} = \frac{(n+1)p + (n-1)q}{(n-1)p + (n+1)q} \text{ nearly.}$$

42. If $(6\sqrt{6} + 14)^{2n+1} = N$, and F be its fractional part; then will $NF = 20^{2n+1}$.

43. If $(3\sqrt{3} + 5)^{2r+1} = I + F$, where I is an integer and F a proper fraction, then will $F(I + F) = 2^{2r+1}$.

44. Shew that the integer next greater than $(3 + \sqrt{7})^{2m}$ is divisible by 2^{m+1}.

45. If m be a positive integer, the integer next greater than $(3 + \sqrt{5})^m$ is divisible by 2^m.

46. Shew that the general term in the expansion of

$$\frac{1 + x + y + xy}{1 + x + y}$$

is

$$(-1)^{m+n} \frac{\lfloor m + n - 2}{\lfloor m - 1 \lfloor n - 1} x^m y^n.$$

47. Shew that the coefficient of x^r in the expansion of

$$\frac{x}{(1-x)^2 - cx} \text{ is } r\left\{1 + \frac{r^2 - 1^2}{\lfloor 3} c + \frac{(r^2 - 1^2)(r^2 - 2^2)}{\lfloor 5} c^2 + \frac{(r^2 - 1^2)(r^2 - 2^2)(r^2 - 3^2)}{\lfloor 7} c^3 + \dots\right\}.$$

48. Shew that

$$1 \cdot 2 \cdot n + 3 \cdot 4 \frac{n(n-1)}{1 \cdot 2} + 5 \cdot 6 \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \dots$$
$$+ (2n-3)(2n-2) \cdot n + (2n-1)\, 2n \cdot 1 = 2^n n^2.$$

49. Shew that the coefficient of x^{n+r-1} in the expansion of $\dfrac{(1+x)^n}{(1-x)^2}$ is $2^{n-1}(n+2r)$.

50. Shew that the coefficient of x^{n+r-1} in the expansion of $\dfrac{(1-3x)^n}{(1-2x)^2}$ is $(-1)^n (r-2n)\, 2^{r-1}$.

51. Shew that

$$n^n - n(n-2)^n + \frac{n(n-1)}{1 \cdot 2}(n-4)^n - \dots \text{ to } n+1 \text{ terms}$$
$$= 2 \cdot 4 \cdot 6 \cdot 8 \dots 2n.$$

52. Shew that

$$a^{n+1} - n(a+b)^{n+1} + \frac{n(n-1)}{1 \cdot 2}(a+2b)^{n+1} - \dots$$
$$= \tfrac{1}{2} \, \underline{|n+1} \, (2a+nb)(-b)^n.$$

53. If three consecutive coefficients in the expansion of any power of a binomial be in arithmetical progression, prove that the index, when rational, must be of the form $q^2 - 2$, where q is an integer.

54. Shew that the sum of the squares of the coefficients in the expansion of $(1+x+x^2)^n$, where n is a positive integer, is

$$\Sigma_0^n \frac{\underline{|2n}}{\underline{|r}\, \underline{|r}\, \underline{|2n-2r}}.$$

55. Shew that, if n is any positive integer,

$$1 + \frac{n(n-1)}{2(2r+1)} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2r+1)(2r+3)} + \dots$$
$$= 2^n \frac{r(r+1)(r+2)\dots(r+n-1)}{2r(2r+1)(2r+2)\dots(2r+n-1)}.$$

CHAPTER XXIII.

PARTIAL FRACTIONS. INDETERMINATE COEFFICIENTS.

295. IN Chapter VIII. it was shewn how to express as a single fraction the algebraic sum of any number of given fractions. It is often necessary to perform the converse operation, namely that of finding a number of fractions, called *partial fractions*, whose denominators are of lower dimensions than the denominator of a given fraction and whose algebraic sum is equal to the given fraction.

296. We may always suppose that the numerator of any fraction which is to be expressed in partial fractions is of lower dimensions in some chosen letter than the denominator. For, if this be not the case to begin with, the numerator can be divided by the denominator until the remainder is of lower dimensions: the given fraction will then be expressed as the sum of an integral expression and a fraction whose numerator is of lower dimensions than its denominator.

297. Any fraction whose denominator is expressed as the product of a number of different factors of the first degree can be reduced to a series of partial fractions whose denominators are those factors of the first degree.

For let the denominator be the product of the n factors $x-a$, $x-b$, $x-c, \ldots$; and let the numerator be represented by $F(x)$, where $F(x)$ is any expression which is not higher than the $(n-1)$th degree in x.

We have to find values of A, B, C,... which are independent of x and which will make

$$\frac{F(x)}{(x-a)(x-b)(x-c)\dots} \equiv \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \dots\dots;$$

or, multiplying by $(x-a)(x-b)(x-c)\dots\dots$,

$$F(x) \equiv A(x-b)(x-c)\dots\dots + B(x-a)(x-c)\dots\dots$$
$$+ C(x-a)(x-b)\dots\dots \qquad \dots\dots\dots\dots\dots\dots\text{(i)}.$$

In order that (i) may be an identity it is necessary and sufficient that the coefficients of like powers of x on the two sides should be equal. Now $F(x)$ is of the $(n-1)$th degree at most, and the terms on the right of (i) are all of the $(n-1)$th degree; hence, by equating the coefficients of x^0, x^1,... x^{n-1} on the two sides of (i), we have n equations which are sufficient to determine the n quantities A, B, C,......

The values of A, B, C,... can however be obtained *separately* in the following manner. Since (i) is to be true for all values of x, it must be true when $x=a$; and, putting $x=a$, we have $F(a) = A(a-b)(a-c)\dots\dots$; and therefore $A = F(a)/(a-b)(a-c)\dots\dots$ Similarly we have $B = F(b)/(b-a)(b-c)\dots$; and so for C, D,....

We have thus found values of A, B, C, ... which make the relation (i) true for the n values a, b, c, ... of x; and as the expressions on the two sides of (i) are of not higher degree than the $(n-1)$th, it follows [Art. 91] that the relation (i) is true for *all values* of x.

Thus

$$\frac{F(x)}{(x-a)(x-b)(x-c)\dots} = \Sigma \frac{F(a)}{(a-b)(a-c)\dots}\,\frac{1}{x-a}.$$

Ex. 1. Resolve $\dfrac{3x+7}{(x-1)(x-2)}$ into partial fractions.

Assume $$\frac{3x+7}{(x-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x-2};$$

then $$3x+7 \equiv A(x-2) + B(x-1).$$

In this identity put $x=1$; then $10=-A$. Now put $x=2$; then $13=B$.

Thus $$\frac{3x+7}{(x-1)(x-2)}=\frac{13}{x-2}-\frac{10}{x-1}.$$

Ex. 2. Resolve $\dfrac{(b-c)(c-a)(a-b)}{(x-a)(x-b)(x-c)}$ into partial fractions.

Let $$\frac{(b-c)(c-a)(a-b)}{(x-a)(x-b)(x-c)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c};$$

then $$(b-c)(c-a)(a-b)=A(x-b)(x-c)+B(x-c)(x-a)+C(x-a)(x-b).$$

Putting $x=a$, we have $(b-c)(c-a)(a-b)=A(a-b)(a-c)$; therefore $A=c-b$; and the values of B and C can be written down from symmetry.

Thus $$\frac{(b-c)(c-a)(a-b)}{(x-a)(x-b)(x-c)}=\frac{c-b}{x-a}+\frac{a-c}{x-b}+\frac{b-a}{x-c}.$$

Ex. 3. Resolve $\dfrac{1}{x(x+1)(x+2)\ldots(x+n)}$ into partial fractions.

Assume
$$\frac{1}{x(x+1)(x+2)\ldots(x+n)}=\frac{A_0}{x}+\frac{A_1}{x+1}+\ldots+\frac{A_r}{x+r}+\ldots+\frac{A_n}{x+n}.$$

Then, we have
$$1\equiv A_0\{(x+1)(x+2)\ldots(x+n)\}+A_1\{x(x+2)(x+3)\ldots(x+n)\}+\ldots+A_r\{x(x+1)\ldots(x+r-1)(x+r+1)\ldots(x+n)\}+\ldots+A_n\{x(x+1)\ldots(x+n-1)\}.$$

If we put $x=0$, all the terms on the right will vanish except the first, and we shall have $1=A_0\times \lfloor n$, so that $A_0=1/\lfloor n$.

To find the general term, put $x=-r$; we then have
$$1=A_r\{(-r)(-r+1)\ldots(-1)(1)(2)\ldots(n-r)\},$$
that is $1=(-1)^rA_r\lfloor r\lfloor n-r$; hence $A_r=(-1)^r/\lfloor r\lfloor n-r$.

Hence the required result is
$$\frac{1}{x(x+1)\ldots(x+n)}=\frac{1}{\lfloor n}\left\{\frac{1}{x}-\ldots+(-1)^r\frac{\lfloor n}{\lfloor r\lfloor n-r}\,\frac{1}{x+r}+\ldots+(-1)^n\frac{1}{x+n}\right\}.$$

[See Art. 259. Ex. 3.]

Ex. 4. Express $\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$ in partial fractions.

Ans. $\Sigma\dfrac{pa^2+qa+r}{(a-b)(a-c)}\,\dfrac{1}{x-a}$.

Ex. 5. Resolve $\dfrac{x^2+15}{(x-1)(x^2+2x+5)}$ into partial fractions.

The factors of x^2+2x+5 are the complex expressions $x+1+2i$ and $x+1-2i$, where i is written for $\sqrt{-1}$.

Assume $\dfrac{x^2+15}{(x-1)(x^2+2x+5)} \equiv \dfrac{A}{x-1}+\dfrac{B}{x+1+2i}+\dfrac{C}{x+1-2i}$;

$$\therefore\ x^2+15 = A(x+1+2i)(x+1-2i)+B(x-1)(x+1-2i) + C(x-1)(x+1+2i).$$

Put $x=1$; then $16=8A$, so that $A=2$.

Put $x=-1-2i$, then $(1+2i)^2+15=B(-2-2i)(-4i)$, that is $12+4i=B(-8+8i)$; therefore $B=-\dfrac{3+i}{2-2i}$. Change the sign of i in the value of B, and we have $C=-\dfrac{3-i}{2+2i}$.

Thus $\dfrac{x^2+15}{(x-1)(x^2+2x+5)}=\dfrac{2}{x-1}-\dfrac{3+i}{2-2i}\,\dfrac{1}{x+1+2i}-\dfrac{3-i}{2+2i}\,\dfrac{1}{x+1-2i}$.

298. We have in the last example resolved the given fraction into three partial fractions whose denominators are all of the first degree, two of the factors of the denominator being imaginary. Although it is for most purposes necessary to do this, the reduction into partial fractions, of a fraction whose denominator has imaginary factors, is often left in a more incomplete state. Take, for example, the fraction just considered, and assume

$$\frac{x^2+15}{(x-1)(x^2+2x+5)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+2x+5}.$$

[It is to be noticed that we must now assume for the numerator of the second fraction an expression containing x but of lower degree than the denominator.]

Then $x^2+15=A(x^2+2x+5)+(Bx+C)(x-1)$.

Putting $x=1$, we have $16=8A$, so that $A=2$.

Put $A=2$ in the above identity; then after transposition $-x^2-4x+5=(Bx+C)(x-1)$;

or, dividing by $x-1$, $Bx+C=-x-5$.

Thus $\dfrac{x^2+15}{(x-1)(x^2+2x+5)}=\dfrac{2}{x-1}-\dfrac{x+5}{x^2+2x+5}$.

299. We have hitherto supposed that the factors of the denominator of the fraction which is to be expressed in partial fractions, were all different from one another. The method of procedure when this is not the case will be seen from the following examples.

Ex. 1. Express $\dfrac{2x+5}{(x-1)^3(x-3)}$ in partial fractions.

We may assume that

$$\frac{2x+5}{(x-1)^3(x-3)} \equiv \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)} + \frac{D}{x-3};$$

or, clearing from fractions,

$$2x+5 \equiv A(x-3) + B(x-1)(x-3) + C(x-1)^2(x-3) + D(x-1)^3.$$

By equating the coefficients of x^0, x^1, x^2, x^3 on the two sides of the last equation, we shall have four equations to determine the four quantities A, B, C, D, so that the assumption made is a legitimate one. The actual values of A, B, C, D are not however generally best found from the equations obtained by equating the coefficients of the different powers of x. In the present case, the following method is more expeditious.

Put $x-1=y$; then we have

$$2+2y+5 = A(y-2) + By(y-2) + Cy^2(y-2) + Dy^3.$$

Now equate coefficients of y^0, y^1, y^2, y^3, and we have $7=-2A$; $2=A-2B$; $0=B-2C$; and $0=D+C$.

Whence $\quad A=-\dfrac{7}{2}$, $B=-\dfrac{11}{4}$, $C=-\dfrac{11}{8}$ and $D=\dfrac{11}{8}$.

Hence $\dfrac{2x+5}{(x-1)^3(x-3)} = \dfrac{11}{8(x-3)} - \dfrac{7}{2(x-1)^3} - \dfrac{11}{4(x-1)^2} - \dfrac{11}{8(x-1)}$.

Ex. 2. Express the fractional part of $\dfrac{(1+x)^n}{(1-2x)^3}$ in partial fractions.

Assume

$$\frac{(1+x)^n}{(1-2x)^3} = \frac{A}{(1-2x)^3} + \frac{B}{(1-2x)^2} + \frac{C}{(1-2x)} + \text{an integral expression.}$$

Then

$$(1+x)^n = A + B(1-2x) + C(1-2x)^2 + (1-2x)^3 \times \text{integral expression.}$$

Now put $1-2x=y$; then

$(1+x)^n = \left(\dfrac{3}{2} - \dfrac{y}{2}\right)^n = \dfrac{1}{2^n}\left(3^n - n\,3^{n-1}y + \dfrac{n(n-1)}{1\,.\,2}3^{n-2}y^2 + \right.$ terms containing higher powers of y).

Also right side $=A+By+Cy^2+y^3\times$ integral expression in y.
Hence, equating coefficients of y^0, y^1, y^2, we have

$$A=\frac{3^n}{2^n},\ B=-\frac{n\,3^{n-1}}{2^n},\ C=\frac{n(n-1)\,3^{n-2}}{2^{n+1}}.$$

300. The following examples will illustrate the use of partial fractions.

Ex. 1. Find the coefficient of x^n in the expansion of $\dfrac{1}{1-5x+6x^2}$ according to ascending powers of x.

We have
$$\frac{1}{1-5x+6x^2}=\frac{3}{1-3x}-\frac{2}{1-2x}$$
$$=3\{1+3x+(3x)^2+\ldots+(3x)^n+\ldots\}$$
$$-2\{1+2x+(2x)^2+\ldots+(2x)^n+\ldots\}.$$

Hence the required coefficient is $3^{n+1}-2^{n+1}$.

Ex. 2. Find the coefficient of x^{n+r} in the expansion of $\dfrac{(1+x)^n}{(1-2x)^3}$.

From Ex. 2, Art. 299, we have
$$\frac{(1+x)^n}{(1-2x)^3}=\frac{3^n}{2^n}\,\frac{1}{(1-2x)^3}-\frac{n3^{n-1}}{2^n}\,\frac{1}{(1-2x)^2}+\frac{n(n-1)\,3^{n-2}}{2^{n+1}}\,\frac{1}{1-2x}$$
+ an integral expression of the $(n-3)$th degree. Whence the required result.

Ex. 3. Shew that the sum of all the homogeneous products of n dimensions of the three letters a, b, c is equal to
$$\frac{a^{n+2}(c-b)+b^{n+2}(a-c)+c^{n+2}(b-a)}{(b-c)(c-a)(a-b)}.$$

The sum of all the homogeneous products of n dimensions is the coefficient of x^n in the product

$(1+ax+a^2x^2+\ldots)(1+bx+b^2x^2+\ldots)(1+cx+c^2x^2+\ldots)$ [See Art. 293];

that is in $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$, which will be found to be equal to
$$\frac{a^2}{(a-b)(a-c)}\,\frac{1}{1-ax}+\frac{b^2}{(b-c)(b-a)}\,\frac{1}{1-bx}+\frac{c^2}{(c-a)(c-b)}\,\frac{1}{1-cx};$$
and the coefficients of x^n in the expansions of these partial fractions is easily seen to be
$$\frac{a^{n+2}}{(a-b)(a-c)}+\frac{b^{n+2}}{(b-c)(b-a)}+\frac{c^{n+2}}{(c-a)(c-b)},$$
which equals
$$\frac{a^{n+2}(c-b)+b^{n+2}(a-c)+c^{n+2}(b-a)}{(b-c)(c-a)(a-b)}.$$

Ex. 4. To find the sum of all the homogeneous products of n dimensions which can be formed from the r letters $a_1, a_2, a_3, \dots\dots, a_r$.

As in the previous example, the sum required will be the coefficient of x^n in $\dfrac{1}{(1-a_1x)(1-a_2x)(1-a_3x)\dots\dots}$, which will be found to be equivalent to $\Sigma \dfrac{a_1^{r-1}}{(a_1-a_2)(a_1-a_3)\dots} \dfrac{1}{1-a_1x}$.

Hence the required sum is $\Sigma \dfrac{a_1^{n+r-1}}{(a_1-a_2)(a_1-a_3)\dots(a_1-a_r)}$.

301. Indeterminate coefficients. We shall conclude this Chapter by giving two examples to illustrate a method, called *the method of indeterminate coefficients,* which depends upon the theorems established in Articles 91 and 281.

Ex. 1. Find the coefficient of x^r in the expansion, according to ascending powers of x, of $(1+cx)(1+c^2x)(1+c^3x)\dots(1+c^nx)$.

The continued product is of the nth degree in x; we may therefore assume that

$$(1+cx)(1+c^2x)\dots(1+c^nx) \equiv A_0 + A_1x + A_2x^2 + \dots + A_rx^r + \dots + A_nx^n,$$

where $A_0, A_1, A_2, \dots$ do not contain x.

Now change x into cx; then, since A_0, A_1, A_2, &c. do not contain x, we have

$$(1+c^2x)(1+c^3x)\dots(1+c^{n+1}x) \equiv A_0 + A_1cx + A_2c^2x^2 + \dots + A_rc^rx^r + \dots + A_nc^nx^n.$$

Hence

$$(1+c^{n+1}x)(A_0 + A_1x + A_2x^2 + \dots + A_rx^r + \dots + A_nx^n)$$
$$\equiv (1+cx)(A_0 + A_1cx + A_2c^2x^2 + \dots + A_rc^rx^r + \dots + A_nc^nx^n).$$

Now equate the coefficients of x^r on the two sides of the last identity, and we have

$$A_r + c^{n+1}A_{r-1} = A_rc^r + A_{r-1}c^r;$$

$$\therefore A_r = \frac{c^{n+1}-c^r}{c^r-1} A_{r-1} = c^r \frac{c^{n-r+1}-1}{c^r-1} A_{r-1} \dots\dots(a).$$

By continued application of (a) we have

$$A_r = c^r \frac{c^{n-r+1}-1}{c^r-1} A_{r-1} = c^r \frac{c^{n-r+1}-1}{c^r-1} \cdot c^{r-1} \frac{c^{n-r+2}-1}{c^{r-1}-1} A_{r-2}$$

$$\dots = c^r \cdot c^{r-1} \dots c^2 \cdot c^1 \frac{(c^{n-r+1}-1)(c^{n-r+2}-1)\dots(c^{n-1}-1)(c^n-1)}{(c^r-1)(c^{r-1}-1)\dots(c^2-1)(c-1)} A_0,$$

$$= c^{\frac{1}{2}r(r+1)} \frac{(c^n-1)(c^{n-1}-1)\dots(c^{n-r+1}-1)}{(c^r-1)(c^{r-1}-1)\dots(c-1)}, \text{ for } A_0 \text{ is obviously } 1.$$

Ex. 2. To find the sum of the series $1^2+2^2+3^2+\ldots+n^2$.

Let $\quad 1^2+2^2+3^2+\ldots+n^2=A_1n+A_2n^2+A_3n^3\ldots\ldots(a)$

for some particular value of n, where A_1, A_2, A_3 do not contain n.

The relation (a) will be true for $n+1$ as well as for n, provided

$1^2+2^2+3^2+\ldots+n^2+(n+1)^2=A_1(n+1)+A_2(n+1)^2+A_3(n+1)^3$;

or, subtracting (a), provided

$$(n+1)^2=A_1+(2n+1)A_2+(3n^2+3n+1)A_3.$$

Now the last relation will be true for *all values* of n if we give to A_1, A_2, A_3 the values which satisfy the equations found by equating the coefficients of n^2, n^1 and n^0, namely, the equations

$$3A_3=1,\quad 3A_3+2A_2=2,\quad \text{and}\quad A_3+A_2+A_1=1,$$

from which we obtain $6A_1=2A_2=3A_3=1$.

Hence, if the relation $1^2+2^2+\ldots+n^2=\frac{1}{6}n+\frac{1}{2}n^2+\frac{1}{3}n^3$, be true for any value of n, it will be true for the next greater value. But it is obviously true when $n=1$; it will therefore be true when $n=2$; and, being true when $n=2$, it must be true when $n=3$; and so on indefinitely.

The sum of the cubes, or of any other integral powers, of the first n integers can be found in a similar manner. [See also Art. 321.]

EXAMPLES XXIX.

Resolve into partial fractions:

1. $\dfrac{3x}{x^2+7x+6}$.

2. $\dfrac{x+1}{x^2-5x+6}$.

3. $\dfrac{x}{(x+1)(x+3)(x+5)}$.

4. $\dfrac{x^2+1}{x(x+1)^2}$.

5. $\dfrac{8-x}{(2-x)^2(1+x)}$.

6. $\dfrac{x^2+x+1}{x^3-4x^2+x+6}$.

7. $\dfrac{x^2-3}{(x+2)(x^2+1)}$.

8. $\dfrac{1+7x-x^2}{(1+3x)^2(1-10x)}$.

9. $\dfrac{x^2-x+1}{(x^2+1)(x-1)^2}$.

10. $\dfrac{5-9x}{(1-3x)^3(1+x)}$.

11. $\dfrac{6x^2+x-1}{(x^2+1)(x-2)(x+3)}$. 12. $\dfrac{x^3+2}{(x-2)^2(x^2+1)}$.

13. $\dfrac{x^2+x}{(x-1)^2(x^2+4)}$. 14. $\dfrac{x^2-x+1}{(x-1)^2(x-2)(x^2+1)}$.

15. $\dfrac{1+2x}{x^2(x+2)^3(x+1)}$. 16. $\dfrac{1+2x}{x^2(x+2)^3(x-1)}$.

17. Find the coefficient of x^n in the expansion of

$$\frac{x+4}{x^2+5x+6}.$$

18. Find the coefficient of x^n in the expansion of

$$\frac{x-2}{(x+2)(x-1)^2}.$$

19. Shew that the coefficient of x^{2n-1} in the expansion of $\dfrac{x+5}{(x^2-1)(x+2)}$ is $1-\dfrac{1}{2^{2n}}$.

20. Find the sum of the n first coefficients in the expansion of $\dfrac{3-2x}{1-2x-3x^2}$.

21. Find the sum of the n first coefficients in the expansion of $\dfrac{2-5x}{(1-5x)(1-3x)(1-2x)}$.

22. Find the coefficient of x^n in the expansion of $\dfrac{(1+x)^n}{(1-x)^3}$. Find also the sum of the n first coefficients.

23. Shew that the coefficient of x^{n+r} in the expansion of $\dfrac{(1+3x)^n}{(1+2x)^2}$ is $(-2)^r(r-2n+1)$.

24. Shew that $\dfrac{x^{n+1}}{(x-a_1)(x-a_2)\dots(x-a_n)}$

$$=x+a_1+a_2+\dots+a_n+\Sigma\frac{a_1{}^{n+1}}{(a_1-a_2)(a_1-a_3)\dots}\frac{1}{x-a_1}.$$

25. Shew that the coefficient of z^{n-1} in the expansion of

$$\{(1-z)(1-cz)(1-c^2z)(1-c^3z)\}^{-1} \text{ is}$$

$$(1-c^n)(1-c^{n+1})(1-c^{n+2})/(1-c)(1-c^2)(1-c^3).$$

26. Prove that

$$\frac{a(b-c)(bc-aa')(a^m-a'^m)}{a-a'}+\frac{b(c-a)(ca-bb')(b^m-b'^m)}{b-b'}$$
$$+\frac{c(a-b)(ab-cc')(c^m-c'^m)}{c-c'}$$
$$=-\frac{1}{abc}(b-c)(c-a)(a-b)(bc-aa')(ca-bb')(ab-cc')H_{m-3},$$

where $aa'=bb'=cc'$, and H_{m-3} is the sum of the homogeneous products of a, b, c, a', b', c' of $m-3$ dimensions.

27. Shew that the product of any r consecutive terms of the series $1-c$, $1-c^2$, $1-c^3$,... is divisible by the first r of them.

28. Shew that, if c be numerically less than unity,

$$(1+cx)(1+c^2x)(1+c^3x)\dots\text{to infinity}$$
$$=1+\frac{c}{1-c}x+\frac{c^3}{(1-c)(1-c^2)}x^2+\dots+\frac{c^{\frac{1}{2}n(n+1)}}{(1-c)\dots(1-c^n)}x^n+\dots$$

29. Shew that, if c be numerically less than unity,

$$(1+cx)(1+c^3x)(1+c^5x)\dots\text{ to infinity}$$
$$=1+\frac{c}{1-c^2}x+\frac{c^4}{(1-c^2)(1-c^4)}x^2+\frac{c^9}{(1-c^2)(1-c^4)(1-c^6)}x^3+\dots$$

30. Shew that, if c be less than unity,

$$\frac{1}{(1-x)(1-cx)(1-c^2x)\dots}=1+\frac{x}{1-c}+\frac{x^2}{(1-c)(1-c^2)}$$
$$+\frac{x^3}{(1-c)(1-c^2)(1-c^3)}+\dots \quad \text{[Gauss.]}$$

31. Shew that, if c be less than unity,

$$\frac{(1+cx)(1+c^2x)(1+c^3x)\ldots}{(1-x)(1-cx)(1-c^2x)\ldots}=1+\frac{1+c}{1-c}x+\frac{(1+c)(1+c^2)}{(1-c)(1-c^2)}x^2+\ldots$$

[Gauss.]

32. Shew that the coefficient of x^r in the expansion of

$$\frac{(1+cx)(1+c^2x)(1+c^3x)\ldots}{(1-cx)(1-c^2x)(1-c^3x)\ldots}$$

is

$$c^r\frac{(1+1)(1+c)\ldots(1+c^{r-1})}{(1-c)(1-c^2)\ldots(1-c^r)},$$

c being less than unity.

33. Shew that

$$\frac{1}{1-x}+\frac{y}{1-ax}+\frac{y^2}{1-a^2x}+\frac{y^3}{1-a^3x}+\ldots$$

$$=\frac{1}{1-y}+\frac{x}{1-ay}+\frac{x^2}{1-a^2y}+\frac{x^3}{1-a^3y}+\ldots$$

34. Shew that

$$\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\frac{4x^4}{1-x^4}+\ldots$$

$$=\frac{x}{(1-x)^2}+\frac{x^2}{(1-x^2)^2}+\frac{x^3}{(1-x^3)^2}+\ldots$$

35. Shew that Lambert's series, namely,

$$\frac{x}{1-x}+\frac{x^2}{1-x^2}+\frac{x^3}{1-x^3}+\frac{x^4}{1-x^4}+\ldots$$

is equivalent to

$$x\frac{1+x}{1-x}+x^4\frac{1+x^2}{1-x^2}+x^9\frac{1+x^3}{1-x^3}+\ldots$$ [Clausen.]

CHAPTER XXIV.

EXPONENTIAL THEOREM. LOGARITHMS. LOGARITHMIC SERIES.

302. **The Exponential Theorem.** If $1/n$ be numerically less than unity, $\left(1+\frac{1}{n}\right)^{nx}$ can be expanded by the Binomial Theorem; and we have

$$\left(1+\frac{1}{n}\right)^{nx}=1+nx\frac{1}{n}+\frac{nx(nx-1)}{1.2}\frac{1}{n^2}+$$

$$\frac{nx(nx-1)(nx-2)}{1.2.3}\frac{1}{n^3}+\ldots+\frac{nx(nx-1)\ldots(nx-r+1)}{\lfloor r}\frac{1}{n^r}+\ldots,$$

which may be written

$$\left(1+\frac{1}{n}\right)^{nx}=1+x+\frac{x\left(x-\frac{1}{n}\right)}{1.2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{1.2.3}+\ldots$$

$$\ldots+\frac{x\left(x-\frac{1}{n}\right)\ldots\left(x-\frac{r-1}{n}\right)}{\lfloor r}+\ldots$$

Putting $x=1$, we have

$$\left(1+\frac{1}{n}\right)^{n}=1+1+\frac{1-\frac{1}{n}}{\lfloor 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}+\ldots$$

$$+\frac{\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{r-1}{n}\right)}{\lfloor r}+\ldots.$$

But $\left(1+\frac{1}{n}\right)^{nx}=\left\{\left(1+\frac{1}{n}\right)^n\right\}^x$; hence

$$1+x+\frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3}+\dots$$

$$=\left\{1+1+\frac{1-\frac{1}{n}}{\lfloor 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}+\dots\right\}^x.$$

The above relation is true for all values of n however great, and therefore when n is infinite; but when n is infinite, $1/n$ is zero, and the relation becomes*

$$1+x+\frac{x^2}{\lfloor 2}+\dots\dots+\frac{x^r}{\lfloor r}+\dots=\left(1+1+\frac{1}{\lfloor 2}+\dots+\frac{1}{\lfloor r}+\dots\right)^x.$$

Denoting the series $1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\dots+\frac{1}{\lfloor r}+\dots$ by e, we have the Exponential Theorem, namely

$$e^x=1+x+\frac{x^2}{\lfloor 2}+\dots+\frac{x^r}{\lfloor r}+\dots$$

It should be remarked that the above series for e^x is convergent for all values of x [Art. 278].

303. The quantity e is of very great importance in mathematics.

It is obvious that it is greater than 2 and it is clearly less than $1+1+2^{-1}+2^{-2}+2^{-3}+\dots$, and therefore less than 3. Its actual value can be found to be $2{\cdot}71828\dots$.

* This requires more careful examination not only to find the limit of each term, but also because the limit of a sum is not necessarily equal to the sum of the limits of its terms unless the number of the terms is *finite*. This examination is however omitted here for the investigation in Art. 304 is preferable.

To prove that e *is an incommensurable number.*

If possible, let $e = m/n$, where m and n are integers; then we should have

$$\frac{m}{n} = 1 + 1 + \frac{1}{\underline{|2}} + \ldots\ldots + \frac{1}{\underline{|n}} + \frac{1}{\underline{|n+1}} + \frac{1}{\underline{|n+2}} + \frac{1}{\underline{|n+3}} + \ldots.$$

Multiply both sides by $\underline{|n}$; then all the terms will become integral except

$$\frac{1}{n+1} + \frac{1}{(n+2)(n+1)} + \frac{1}{(n+3)(n+2)(n+1)} + \ldots$$

Hence

$$\frac{1}{n+1} + \frac{1}{(n+2)(n+1)} + \frac{1}{(n+3)(n+2)(n+1)} + \ldots$$

must be equal to an integer; but this sum is less than $\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \ldots$, and therefore less than $\frac{1}{n+1} \Big/ \left(1 - \frac{1}{n+1}\right)$, that is less than $\frac{1}{n}$. But an integer cannot be less than $1/n$; it therefore follows that e cannot be equal to the commensurable number m/n.

304. The following proof of the Exponential Theorem is due to Prof. Hill*. It will be seen that it only assumes the truth of the Binomial Theorem for a *positive integral* exponent.

Let $f(m)$ denote the series $1 + m + \frac{m^2}{\underline{|2}} + \ldots\ldots + \frac{m^r}{\underline{|r}} + \ldots.$

Thus $f(m) \equiv 1 + m + \frac{m^2}{\underline{|2}} + \ldots\ldots + \frac{m^r}{\underline{|r}} + \ldots\ldots,$

* *Proceedings of the Cambridge Philosophical Society*, Vol. v. p. 415. Substantially the same proof is however given in Cauchy's *Analyse Algébrique.*

$$f(n) \equiv 1 + n + \frac{n^2}{\lfloor 2} + \ldots\ldots + \frac{n^s}{\lfloor s} + \ldots\ldots,$$

and $f(m+n) \equiv 1 + (m+n) + \dfrac{(m+n)^2}{\lfloor 2} + \ldots + \dfrac{(m+n)^p}{\lfloor p} + \ldots$

Now the coefficient of $m^r n^s$ in $f(m) \times f(n)$ is $\dfrac{1}{\lfloor r \, \lfloor s}$;

and in $f(m+n)$ the term $m^r n^s$ can only occur in $\dfrac{(m+n)^{r+s}}{\lfloor r+s}$,

and its coefficient will therefore be $\dfrac{\lfloor r+s}{\lfloor r \, \lfloor s} \dfrac{1}{\lfloor r+s}$, that is $\dfrac{1}{\lfloor r \, \lfloor s}$.

Hence, as the series $f(m)$, $f(n)$ and $f(m+n)$ are convergent for all values of m and n, and the coefficient of any term $m^r n^s$ is the same in $f(m) \times f(n)$ as in $f(m+n)$, it follows from Art. 280 that

$$f(m) \times f(n) = f(m+n) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

for all values of m and n.

Now let x be a positive integer; then from (i) we have

$$f(1) \times f(1) \times f(1) + \ldots\ldots \text{to } x \text{ factors,}$$
$$= f(1 + 1 + 1 + \ldots\ldots \text{to } x \text{ terms}),$$
$$\therefore \ \{f(1)\}^x = f(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Next let x be a positive fraction $\dfrac{p}{q}$, where p and q are positive integers. Then from (i)

$$\left\{f\left(\frac{p}{q}\right)\right\}^q = f\left(\frac{p}{q} + \frac{p}{q} + \frac{p}{q} + \ldots\ldots \text{to } q \text{ terms}\right) = f(p)$$
$$= \{f(1)\}^p, \text{ from (ii)};$$
$$\therefore f\left(\frac{p}{q}\right) = \{f(1)\}^{\frac{p}{q}}.$$

Hence, for all *positive* values of x, $\{f(1)\}^x = f(x)$.

Lastly, let x be negative, and equal to $-y$, so that y is positive; then $f(-y) \times f(y) = f(0)$ from (i); but $f(0) = 1$, therefore $f(-y) = 1/f(y)$.

Hence

$$f(x) = f(-y) = \frac{1}{f(y)} = \frac{1}{\{f(1)\}^y}, \text{ since } y \text{ is positive,}$$
$$= \{f(1)\}^{-y} = \{f(1)\}^x.$$

Hence, whatever x may be,

$$\{f(1)\}^x = f(x).$$

But $$f(1) = 1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \dots\dots\dots \equiv e,$$

therefore $$e^x = f(x) = 1 + \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \dots\dots + \frac{x^r}{\underline{|r}} + \dots\dots$$

305. To shew that

$$n^n - n(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - \dots = \underline{|n}.$$

We have from Art. 304

$$(e^x - 1)^n = \left(x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \dots\right)^n;$$

Also, by the binomial theorem,

$$(e^x - 1)^n = e^{nx} - n \,.\, e^{(n-1)x} + \frac{n(n-1)}{1.2} e_{(n-2)x} - \dots$$

Now the coefficient of x^r in $\left(x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \dots\right)^n$ is zero, if r is less than n, and is 1 if $r = n$.

Also the coefficient of x^r in $e^{nx} - ne^{(n-1)x} + \frac{n(n-1)}{1.2} e^{(n-2)x} - \dots$ is

$$\frac{1}{\underline{|r}}\left\{n_r - n(n-1)^r + \frac{n(n-1)}{1.2}(n-2)^r - \dots\right\}.$$

Hence, equating the coefficients of x^n in the expansions of the two expressions for $(e^x - 1)^n$, we have

$$\frac{1}{\underline{|n}}\left\{n^n - n(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - \dots\right\} = 1.$$

The above theorem may be generalised as follows:

We have

$$(e^{ax}-e^{bx})^n = e^{nax} - ne^{(\overline{n-1}\,.\,a+b)x} + \frac{n(n-1)}{1\,.\,2}e^{(\overline{n-2}\,.\,a+2b)x} - \ldots$$

and

$$(e^{ax}-e^{bx})^n = e^{nbx}(e^{(a-b)x}-1)^n = e^{nbx}\left\{(a-b)x + \frac{(a-b)^2x^2}{\lfloor 2} + \ldots\right\}^n.$$

Hence, equating coefficients of x^n in the two expressions for $(e^{ax}-e^{bx})^n$, we have

$$\frac{1}{\lfloor n}\left\{(na)^n - n(\overline{n-1}\,.\,a+b)^n + \frac{n(n-1)}{1\,.\,2}(\overline{n-2}\,.\,a+2b)^n - \ldots\right\} = (a-b)^n.$$

If we put $na=x$ and $b-a=y$, the last result becomes

$$x^n - n(x+y)^n + \frac{n(n-1)}{1\,.\,2}(x+2y) - \ldots = (-1)^n\,.\,y^n \lfloor n.$$

We have also, if k be any positive integer less than n,

$$x^k - n(x+y)^k + \frac{n(n-1)}{1\,.\,2}(x+2y)^k - \ldots \text{ to } n+1 \text{ terms} = 0.$$

The following particular cases are of importance, k being less than n.

$$1^k - n2^k + \frac{n(n-1)}{1\,.\,2}3^k - \ldots \text{ to } n+1 \text{ terms} = 0,$$

and $$m^k - n(m-1)^k + \frac{n(n-1)}{1\,.\,2}(m-2)^k - \ldots \text{ to } n+1 \text{ terms} = 0.$$

EXAMPLES XXX.

Ex. 1. Shew that the limit when n is infinite of $\left(1+\frac{x}{n}\right)^n$ is e^x.

Ex. 2. Shew that the limit when n is infinite of $\left(1+\frac{a}{n}\right)^{\frac{n}{b}}$ is $e^{\frac{a}{b}}$.

Ex. 3. Shew that

$$n^{n+1} - n(n-1)^{n+1} + \frac{n(n-1)}{1\,.\,2}(n-2)^{n+1} - \ldots = \frac{1}{2}n\lfloor n+1.$$

Ex. 4. Shew that

$$n^{n+2} - n(n-1)^{n+2} + \frac{n(n-1)}{1\,.\,2}(n-2)^{n+2} - \ldots = \frac{n}{24}(3n+1)\lfloor n+2).$$

Ex. 5. Shew that

$$\left(1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots\right)\left(1-\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}+\ldots\right)=1.$$

Ex. 6. Shew that $e^{-1}=\frac{2}{\underline{|3}}+\frac{4}{\underline{|5}}+\frac{6}{\underline{|7}}+\ldots$

Ex. 7. Shew that

$$\frac{3}{2}e=1+\frac{1+2}{\underline{|2}}+\frac{1+2+3}{\underline{|3}}+\frac{1+2+3+4}{\underline{|4}}+\ldots$$

Ex. 8. Shew that

$$\left(1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|6}}+\ldots\right)^2=1+\left(1+\frac{1}{\underline{|3}}+\frac{1}{\underline{|5}}+\frac{1}{\underline{|7}}+\ldots\right)^2.$$

Ex. 9. Shew that

$$e^{-1}=\frac{1}{1\,.\,3}+\frac{1}{1.2.3.5}+\ldots+\frac{1}{1.2.3\ldots(2n-1)(2n+1)}+\ldots$$

Ex. 10. Shew that

$$\frac{e-1}{e+1}=\left\{\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|6}}+\ldots\right\}\div\left\{\frac{1}{\underline{|1}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|5}}+\ldots\right\}.$$

Ex. 11. Shew that

$$\frac{e^2+1}{e^2-1}=\left\{1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|6}}+\ldots\right\}\div\left\{1+\frac{1}{\underline{|3}}+\frac{1}{\underline{|5}}+\frac{1}{\underline{|7}}+\ldots\right\}.$$

Ex. 12. Shew that the coefficient of x^n in the expansion of

$$1+\frac{(1+2x)}{\underline{|1}}+\frac{(1+2x)^2}{\underline{|2}}+\frac{(1+2x)^3}{\underline{|3}}+\ldots \text{ is } \frac{2^n e}{\underline{|n}}.$$

LOGARITHMS.

306. **Definition.** The index of the power to which one number must be raised to produce a second number is called the *logarithm* of the second number with respect to the first as base. Thus, if $a^x=y$, then x is called the logarithm of y to the base a, and this is expressed by the notation $x=\log_a y$.

We proceed to investigate the fundamental properties of logarithms, and to shew how logarithms can be found, and how they can be employed to shorten certain approximate calculations.

307. **Properties of Logarithms.** The following are the fundamental properties of logarithms.

I. Since $a^0=1$, for all values of a, it follows that $\log_a 1=0$.

Thus the logarithm of 1 *is* 0, *whatever the base may be.*

II. If $\log_a x=\alpha$, $\log_a y=\beta$, $\log_a z=\gamma, \ldots$

then $x=a^\alpha$, $y=a^\beta$, $z=a^\gamma, \ldots$;

$$\therefore xyz\ldots=a^\alpha . a^\beta . a^\gamma \ldots = a^{\alpha+\beta+\gamma+\ldots}$$

$$\therefore \log_a(xyz\ldots)=\alpha+\beta+\gamma+\ldots$$
$$=\log_a x+\log_a y+\log_a z+\ldots$$

Thus *the logarithm of a product is the sum of the logarithms of its factors.*

III. If $\log_a x=\alpha$, and $\log_a y=\beta$;

then $x=a^\alpha$, $y=a^\beta$, and $\therefore x\div y=a^{\alpha-\beta}$;

$$\therefore \log_a(x\div y)=\alpha-\beta=\log_a x-\log_a y.$$

Thus *the logarithm of a quotient is the algebraic difference of the logarithms of the dividend and the divisor.*

IV. If $x=a^\alpha$; then $x^m=a^{m\alpha}$, for all values of m.

Hence $\log_a x^m=m\alpha=m\log_a x$.

Thus *the logarithm of any power of a number is the product of the logarithm of that number by the index of the power.*

V. Let $\log_a x=\alpha$, and $\log_b x=\beta$; then $x=a^\alpha=b^\beta$; and hence $a=b^{\frac{\beta}{\alpha}}$, and $a^{\frac{\alpha}{\beta}}=b$.

Therefore $\dfrac{\beta}{\alpha} = \log_b a$, and $\dfrac{\alpha}{\beta} = \log_a b$.

Hence $\log_a b \times \log_b a = \dfrac{\alpha}{\beta} \times \dfrac{\beta}{\alpha} = 1$.

Also $\beta = \alpha \log_b a$, that is $\log_b x = \log_a x \, . \log_b a$.

Hence *the logarithm of any number to the base b will be found by multiplying the logarithm of that number to the base a by the constant multiplier $\log_b a$.*

308. **The logarithmic series.** Let $a = e^k$, so that $k = \log_e a$; then $a^x = e^{xk} = e^{x \log_e a}$. Hence from Art. 304, we have

$$a^x = e^{x \log_e a} = 1 + x \log_e a + \frac{(x \log_e a)^2}{\lfloor 2} + \ldots + \frac{(x \log_e a)^r}{\lfloor r} + \ldots$$

Put $a = 1 + y$; then we have

$$(1+y)^x = 1 + x \log_e (1+y) + \frac{1}{\lfloor 2} \{x \log_e (1+y)\}^2 + \ldots$$

Now, *provided y be numerically less than unity*, $(1+y)^x$ can be expanded by the binomial theorem; we then have

$$1 + xy + \frac{x(x-1)}{1 \, . \, 2} y^2 + \ldots + \frac{x(x-1)(x-2)\ldots(x-r+1)}{\lfloor r} y^r + \ldots$$

$$= 1 + x \log_e (1+y) + \frac{1}{\lfloor 2} \{x \log_e (1+y)\}^2 + \ldots$$

The series on the right is convergent for all values of x and y, and the series on the left is convergent for all values of x provided y is numerically less than unity. Hence, for such values of y, we may equate the coefficients of x on the two sides of the equation. We thus obtain

$$\log_e (1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \ldots\ldots + (-1)^{r-1} \frac{y^r}{r} + \ldots$$

This is called the *logarithmic series*.

Ex. 1. To express a^n+b^n in terms of powers of ab and $a+b$.

From the identity $(1-ax)(1-bx)\equiv 1-(a+b)x+abx^2$

$$\equiv 1-sx+px^2,$$

where s is put for $a+b$ and p for ab, we have

$$\log_e(1-ax)+\log_e(1-bx)=\log_e(1-sx+px^2).$$

Hence $\left(ax+\frac{a^2x^2}{2}+\frac{a^3x^3}{3}+\dots\dots\right)+\left(bx+\frac{b^2x^2}{2}+\frac{b^3x^3}{3}+\dots\dots\right)$

$$=\left\{x(s-px)+\frac{x^2(s-px)^2}{2}+\frac{x^3(s-px)^3}{3}+\dots\dots\right\}.$$

Equate the coefficients of x^n on the two sides of the last equation. [This is allowable since the series can clearly be made convergent by taking x sufficiently small.] Then the coefficient of x^n on the left is $\frac{1}{n}(a^n+b^n)$. On the right we have to pick out the coefficient of x^n from the terms (beginning at the highest in which it can appear)

$$\frac{x^n}{n}(s-px)^n+\frac{x^{n-1}}{n-1}(s-px)^{n-1}+\frac{x^{n-2}}{n-2}(s-px)^{n-2}+\dots\dots,$$

the coefficient of x^n is therefore

$$\frac{1}{n}s^n+\frac{1}{n-1}\{-(n-1)s^{n-2}p\}+\frac{1}{n-2}\left\{\frac{(n-2)(n-3)}{1.2}s^{n-4}p^2\right\}+\dots\dots$$

Hence we have

$$a^n+b^n=(a+b)^n-nab(a+b)^{n-2}+\frac{n(n-3)}{1.2}a^2b^2(a+b)^{n-4}$$

$$-\frac{n(n-4)(n-5)}{1.2.3}a^3b^3(a+b)^{n-6}+\dots$$

$$\dots+(-1)^r\frac{n(n-r-1)(n-r-2)\dots(n-2r+1)}{\underline{|r}}a^rb^r(a+b)^{n-2r}+\dots\dots$$

Ex. 2. Shew that, if $a+b+c=0$; then will

$$10(a^7+b^7+c^7)=7(a^2+b^2+c^2)(a^5+b^5+c^5).$$

Put $-p$ for $bc+ca+ab$, and q for abc; then we have the identity

$$(1-ax)(1-bx)(1-cx)=1-px^2-qx^3.$$

Now take logarithms, and equate the coefficients of the different powers of x in the two expansions. This gives $\frac{1}{r}(a^r+b^r+c^r)$ in terms of p and q, and the required result follows at once. [See also Art. 129.]

Ex. 3. To express $a^n+b^n+c^n$ in terms of abc and $bc+ca+ab$, when $a+b+c=0$.

Put $-p=bc+ca+ab$, and $q=abc$; then we have the identity $(1-ax)(1-bx)(1-cx) \equiv 1-px^2-qx^3$.

Hence, by taking logarithms, and equating the coefficients of like powers of x, we have

$$\frac{1}{n}(a^n+b^n+c^n)=\text{coefficient of } x^n \text{ in } \Sigma \frac{1}{r} x^{2r}(p+qx)^r,$$

which gives the required result.

The terms in $\Sigma \frac{1}{r} x^{2r}(p+qx)^r$ which contain $x^{6m\pm1}$ are

$$\frac{1}{2m} x^{4m}(p+qx)^{2m} + \frac{1}{2m+1} x^{4m+2}(p+qx)^{2m+1} + \frac{1}{2m+2} x^{4m+4}(p+qx)^{2m+2}$$
$$+ \dots + \frac{1}{3m-1} x^{6m-2}(p+qx)^{3m-1} + \frac{1}{3m} x^{6m}(p+qx)^{3m}.$$

Now by inspection we see that the coefficient of x^{6m-1} in each of the above terms in which it occurs contains pq as a factor; and also that the coefficient of x^{6m+1} in each of the terms in which it occurs contains p^2q as a factor.

Hence, when $a+b+c=0$, $a^n+b^n+c^n$ is algebraically divisible by $abc(bc+ca+ab)$ when n is of the form $6m-1$, and $a^n+b^n+c^n$ is algebraically divisible by $abc(bc+ca+ab)^2$ when n is of the form $6m+1$.

If we put $c=-(a+b)$, $bc+ca+ab$ becomes $-(a^2+ab+b^2)$, and we have Cauchy's Theorem, namely that $a^n+b^n-(a+b)^n$ is divisible by $ab(a+b)(a^2+ab+b^2)$ when n is of the form $6m-1$, and by $ab(a+b)(a^2+ab+b^2)^2$ when n is of the form $6m+1$.

[See papers on Cauchy's Theorem by Mr J. W. L. Glaisher and Mr T. Muir in the *Quarterly Journal*, Vol. XVI., and in the *Messenger of Mathematics*, Vol. VIII.]

309. In order to diminish the labour of finding the approximate value of the logarithm of any number, more rapidly converging series are obtained from the fundamental logarithmic series.

Changing the sign of y in the logarithmic series

$$\log_e(1+y)=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\dots\dots\dots\dots \text{(i)},$$

we have

$$\log_e(1-y)=-y-\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^4}{4}-\dots\dots\dots\dots \text{(ii)}.$$

Hence $\log_e \dfrac{1+y}{1-y} = \log_e(1+y) - \log_e(1-y)$

$$= 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \dots\dots\right) \qquad \dots\dots\text{(iii)}.$$

Put $\dfrac{m}{n}$ for $\dfrac{1+y}{1-y}$, and therefore $\dfrac{m-n}{m+n}$ for y; then

$$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \dots\right\}\dots\text{(iv)}.$$

We are now able to calculate logarithms to base e without much labour. For example:—

Put $m = 2$, $n = 1$, in formula (iv); then

$$\log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \dots\right\},$$

from which it is easy to obtain the value $\log_e 2 = {\cdot}693147\dots$

Having found $\log_e 2$, we have from (iv)

$$\log_e 3 - \log_e 2 = 2\left\{\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} + \dots\right\} = {\cdot}405465\dots.$$

Hence $\log_e 3 = {\cdot}693147 + {\cdot}405465 = 1{\cdot}09861$.

Proceeding in this way, the logarithm to base e of any number can be found to any requisite degree of approximation.

310. Logarithms to base e are called *Napierian* or *natural logarithms.*

The logarithms used in all theoretical investigations are Napierian logarithms; but when approximate numerical calculations are made by means of logarithms, the logarithms used are always those to base 10, for reasons which will shortly appear: on this account logarithms to base 10 are called *Common* logarithms.

We have shewn how logarithms to base e can be found; and having found logarithms to base e, the logarithms to base 10 are obtained by multiplying by the constant factor $\log_{10}e$, or by $1/\log_e 10$. [Art. 307, V.] This constant factor is called the *Modulus*: its value is ${\cdot}43429\dots$

EXAMPLES XXXI.

1. Shew that $\log(x+n) = \log x + \log\left(1+\frac{1}{x}\right) + \log\left(1+\frac{1}{1+x}\right) + \log\left(1+\frac{1}{2+x}\right) + \dots\dots + \log\left(1+\frac{1}{n-1+x}\right)$.

2. Shew that $\log_e \sqrt{12} = 1 + \left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4} + \left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2} + \left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3} + \left(\frac{1}{8}+\frac{1}{9}\right)\frac{1}{4^4} + \dots\dots$ to infinity.

3. Shew that $\log_e \sqrt{10} = \left\{1 + \frac{1}{3}\frac{1}{9} + \frac{1}{5}\frac{1}{9^2} + \frac{1}{7}\frac{1}{9^3} + \dots\dots \text{ to infinity}\right\} + \left\{\frac{1}{9} + \frac{1}{3}\frac{1}{9^3} + \frac{1}{5}\frac{1}{9^5} + \frac{1}{7}\frac{1}{9^7} + \dots\dots \text{ to infinity}\right\}$.

4. Shew that $\log_e 2 - \frac{1}{2} = \frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \dots\dots$ to infinity.

5. Shew that $\frac{5}{1.2.3} + \frac{7}{3.4.5} + \frac{9}{5.6.7} + \dots\dots$ to infinity $= 3\log_e 2 - 1$.

6. Shew that
$$\log_e \frac{x}{x-1} = 2\left\{\frac{1}{2x-1} + \frac{1}{3}\frac{1}{(2x-1)^3} + \frac{1}{5}\frac{1}{(2x-1)^5} + \dots\dots\right\}.$$

7. Shew that $\log_e x = \frac{x-1}{x+1} + \frac{1}{2}\frac{x^2-1}{(x+1)^2} + \frac{1}{3}\frac{x^3-1}{(x+1)^3} + \dots\dots$

8. Shew that
$$\log_e \frac{a+x}{a-x} = \frac{2ax}{a^2+x^2} + \frac{1}{3}\left(\frac{2ax}{a^2+x^2}\right)^3 + \frac{1}{5}\left(\frac{2ax}{a^2+x^2}\right)^5 + \dots\dots$$

9. Shew that

$$2\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\dots\dots\right)=\frac{2x}{1-x^2}-\frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3+\frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5-\dots\dots$$

10. Shew that

$$\{\log_e(1+x)\}^2=2\left\{\frac{1}{2}x^2-\frac{1}{3}\left(\frac{1}{1}+\frac{1}{2}\right)x^3+\frac{1}{4}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right)x^4-\dots\right\}.$$

11. Shew that, if $\log_e(1+x+x^2)$ be expanded in powers of x, the coefficient of x^n is either $\frac{1}{n}$ or $-\frac{2}{n}$, and distinguish the cases.

12. If $\log_e(1-x+x^2)$ be expanded in ascending powers of x in the form $a_1x+a_2x^2+a_3x^3+\dots\dots$, then will $a_3+a_6+a_9+\dots = \frac{2}{3}\log_e 2$.

13. Expand $\log_e \dfrac{1+x+x^2}{1-x+x^2}$ in ascending powers of x.

14. Shew that

$$\frac{1}{n}+\frac{x}{n(n+1)}+\frac{x^2}{n(n+1)(n+2)}+\frac{x^3}{n(n+1)(n+2)(n+3)}+\dots$$

$$=e^x\left\{\frac{1}{n}-\frac{x}{\lfloor 1\,(n+1)}+\frac{x^2}{\lfloor 2\,(n+2)}-\frac{x^3}{\lfloor 3\,(n+3)}+\dots\dots\right\}.$$

15. From the identity $2\log(1-x)\equiv\log(1-2x+x^2)$, prove that $2^n-n\,.\,2^{n-2}+\dfrac{n(n-3)}{1\,.\,2}2^{n-4}-\dfrac{n(n-4)(n-5)}{1\,.\,2\,.\,3}2^{n-6}+\dots=2.$

16. If $\log_e\dfrac{1}{1-x-x^2+x^3}$ be expanded in a series of positive integral powers of x, the coefficient of x^n will be $\frac{1}{n}$ or $\frac{3}{n}$ according as n is odd or even.

17. Shew that the coefficient of x^r in the expansion of $\dfrac{e^{nx}-1}{1-e^{-x}}$ is $\dfrac{1}{\lfloor r}\{1^r+2^r+3^r+\ldots\ldots+n^r\}$. Hence find the sum of n terms of the series $1^2+2^2+3^2+\ldots$, and also of $1^3+2^3+3^3+\ldots$

18. Shew that, if a_r be the coefficient of x^r in the expansion of e^{e^x}, then

$$a_r=\frac{1}{\lfloor r}\left\{\frac{1^r}{\lfloor 1}+\frac{2^r}{\lfloor 2}+\frac{3^r}{\lfloor 3}+\ldots\right\}.$$

Hence shew that

$$\frac{1^3}{\lfloor 1}+\frac{2^3}{\lfloor 2}+\frac{3^3}{\lfloor 3}+\ldots=5e,$$

and that

$$\frac{1^4}{\lfloor 1}+\frac{2^4}{\lfloor 2}+\frac{3^4}{\lfloor 3}+\ldots=15e.$$

19. Shew that

$$e\left\{1+\frac{n}{1^2}+\frac{n(n-1)}{1^2.2^2}+\frac{n(n-1)(n-2)}{1^2.2^2.3^2}+\ldots\right\}$$
$$=1+(n+1)+\frac{(n+1)(n+2)}{1^2.2^2}+\frac{(n+1)(n+2)(n+3)}{1^2.2^2.3^2}+\ldots$$

20. Shew that the sum of n terms of the series $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots$, beginning at the $(n+1)$th, becomes equal to $\log_e 2$ when n is increased without limit.

21. Shew that

$$\log_e(1+n)<\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}<1+\log_e(1+n).$$

22. Prove the following:—

(i) $(x+y)^7-x^7-y^7=7xy(x+y)(x^2+xy+y^2)^2$,

(ii) $(x+y)^{11}-x^{11}-y^{11}=11xy(x+y)(x^2+xy+y^2)\{(x^2+xy+y^2)^3+x^2y^2(x+y)^2\}$,

(iii) $(x+y)^{13}-x^{13}-y^{13}=13xy(x+y)(x^2+xy+y^2)^2\{(x^2+xy+y^2)^3+2x^2y^2(x+y)^2\}$.

23. Shew that $x^{2n} + y^{2n} + (x+y)^{2n}$

$$= 2p^n + n(n-2)p^{n-3}q^2 + \frac{n(n-3)(n-4)(n-5)}{3 \,.\, 4} p^{n-6}q^4 + \dots$$
$$+ \frac{n(n-r-1)\dots(n-3r+1)}{3 \,.\, 4 \dots 2r} p^{n-3r}q^{2r} + \dots,$$

where $p \equiv x^2 + xy + y^2$ and $q \equiv xy(x+y)$.

24. Shew that, (i) if n be any uneven integer, $(b-c)^n + (c-a)^n + (a-b)^n$ will be divisible by $(b-c)^3 + (c-a)^3 + (a-b)^3$; (ii) if n be of the form $6m \pm 1$, it will be also divisible by $(b-c)^2 + (c-a)^2 + (a-b)^2$; and (iii) if n be of the form $6m+1$ it will be divisible by $(b-c)^4 + (c-a)^4 + (a-b)^4$.

Common Logarithms.

311. In what follows the logarithms must always be supposed to be common logarithms, and the base, 10, need not be written.

If two numbers have the same figures, and therefore differ only in the position of the decimal point, the one must be the product of the other and some integral power of 10, and hence from Art. 307, II. the logarithms of the numbers will differ by an integer.

Thus $\log 421{\cdot}5 = \log 4{\cdot}215 + \log 100 = 2 + \log 4{\cdot}215$.

Again, knowing that $\log 2 = {\cdot}30103$, we have $\log {\cdot}02 = \log(2 \div 100) = \log 2 - \log 100 = {\cdot}30103 - 2$.

On account of the above property, common logarithms are always written with *the decimal part positive.* Thus $\log {\cdot}02$ is not written in the form $-1{\cdot}69897$ but $\bar{2}{\cdot}30103$, the minus sign referring only to the integral portion of the logarithm and being written above the figure to which it refers.

Definition. When a logarithm is so written that its decimal part is positive, the decimal part of the logarithm is called the *mantissa* and the integral part the *characteristic.*

312. *The characteristic of the logarithm of any number can be written down by inspection.* For, if the number be greater than 1, and n be the number of figures in its integral part, the number is clearly less than 10^n but not less than 10^{n-1}.

Hence its logarithm is between n and $n-1$: the logarithm is therefore equal to $n-1+$ a decimal.

Thus *the characteristic of the logarithm of any number greater than unity is one less than the number of figures in its integral part.*

Next, let the number be less than unity.

Express the number as a decimal, and let n be the number of ciphers before its first significant figure.

Then the number is greater than 10^{-n-1} and less than 10^{-n}.

Hence, as the decimal part of the logarithm must be positive, the logarithm of the number will be $-(n+1)+$ a decimal fraction, the characteristic being $-(n+1)$.

Thus, *if a number less than unity be expressed as a decimal, the characteristic of its logarithm is negative and one more than the number of ciphers before the first significant figure.*

For example, the characteristic of the logarithm of 3571·4 is 3, and that of ·00035714 is $\bar{4}$.

Conversely, if we know the characteristic of the logarithm of any number whose digits form a certain sequence of figures we know at once where to place the decimal point.

For example, knowing that the logarithm of a number whose digits form the sequence 35714 is 3·55283, we know that the number must be 3571·4.

313. Tables are published which give the logarithms of all numbers from 1 to 99999 calculated to seven places of decimals: these are called 'seven-figure' logarithms. For many purposes it is however sufficient to use five-figure logarithms.

In all Tables of logarithms the mantissae only are given, for the characteristics can always, as we have seen, be written down by inspection.

In making use of Tables of logarithms we have, I. to find the logarithm of a given number, and II. to find the number which has a given logarithm.

I. *To find the logarithm of a given number.*

If the number have no more than five significant figures, its logarithm will be given in the tables. But, if the number have more significant figures than are given in the tables, use must be made of the principle that when the difference of two numbers is small compared with either of them, the difference of the numbers is approximately proportional to the difference of their logarithms. This follows at once from Art. 308, for

$$\log_{10}(N+x) - \log_{10} N = \log_{10}\left(1+\frac{x}{N}\right) = \mu \log_e \left(1+\frac{x}{N}\right)$$

$$= \mu\left(\frac{x}{N} - \frac{1}{2}\frac{x^2}{N^2} + \dots\right) = \mu\frac{x}{N} \text{ approximately, when } \frac{x}{N} \text{ is}$$

small, μ being the modulus $1/\log_e 10$.

An example will shew how the above principle, called the *Principle of Proportional Differences,* is utilised.

Ex. To find the logarithm of 357·247.

We find from the tables that log 3·5724 = ·5529601, and log 3·5725 = ·5529722; and the difference of these logarithms is ·0000121. Now the difference between 3·57247 and 3·5724 is $\frac{7}{10}$*ths.* of the difference between 3·5724 and 3·5725; and hence if we add $\frac{7}{10}$*ths.* of ·0000121 to the logarithm of 3·5724 we shall obtain the approximate logarithm of 3·57247. Now $\frac{7}{10}$*ths.* of ·0000121 is ·00000847, which is nearer to ·0000085 than to ·0000084. Hence the nearest approximation we can find to the logarithm of 3·57247 is ·5529601 + ·0000085 = ·5529686.

The characteristic of the logarithm of 357·247 is obviously 2, and therefore the logarithm required is 2·5529686.

II. *To find the number which has a given logarithm.*

For example, let the given logarithm be $\bar{4}$·5529652.

We find from the tables that log 3·5724 = ·5529601 and that log 3·5725 = ·5529722, the mantissa of the given logarithm falling

between these two. Now the difference between ·5529601 and the given logarithm is $\frac{51}{121}$ of the difference between the logarithms of 3·5724 and 3·5725; and hence, by the principle of proportional differences, the number whose logarithm is ·5529652 is

$$3{\cdot}5724 + \frac{51}{121} \times {\cdot}0001 = 3{\cdot}5724 + {\cdot}00004 = 3{\cdot}57244.$$

[The approximation could only be relied upon for *one* figure.]

Thus $\cdot 5529652 = \log 3{\cdot}57244$, and therefore

$$\bar{4}{\cdot}5529652 = \log {\cdot}000357244.$$

Compound Interest and Annuities.

314. The approximate calculation of Compound Interest for a long period, and also of the value of an annuity, can be readily made by means of logarithms.

All problems of this kind depend upon the three following:—[The student is supposed to be acquainted with the arithmetical treatment of these subjects.]

I. *To find the amount of a given sum at compound interest, in a given number of years and at a given rate per cent. per annum.*

Let P denote the principal, n the number of years, $100r$ the rate per cent. per annum, and A the required amount.

Then the interest of P for one year will be Pr, and therefore the amount of principal and interest at the end of the first year will be $P(1+r)$. This last sum is the capital on which interest is to be paid for the second year; and therefore the amount at the end of the second year will be $\{P(1+r)\}(1+r) = P(1+r)^2$. Similarly the amount at the end of n years will be $P(1+r)^n$.

Thus $A = P(1+r)^n$; and hence

$$\log A = \log P + n \log (1+r).$$

If the interest is paid, and capitalised, half yearly, it can be easily seen that the amount will be $P\left(1+\frac{r}{2}\right)^{2n}$.

Ex. *Find the amount of £350 in 25 years at 5 per cent. per annum.*

Here $$P=350,\ r=\frac{5}{100}\ \text{and}\ n=25;$$

$$\therefore \log A=\log 350+25\log\left(1+\frac{5}{100}\right)$$

$$=\log 350+25\,(\log 105-\log 100).$$

From the tables we find that $\log 350=2{\cdot}5440680$ and $\log 105=2{\cdot}0211893$; hence $\log A=3{\cdot}0738005$. Whence it is found from the tables that $A=£1185{\cdot}22$.

II. *To find the present value of a sum of money which is to be paid at the end of a given time.*

Let A be the sum payable at the end of n years, and let P be its present worth, the interest on money being supposed to be $100r$ per cent. per annum. Then the amount of P in n years at $100r$ per cent. per annum must be just equal to A.

Hence from I. $\quad P=A\,(1+r)^{-n}$.

III. *To find the present value of an annuity of £A payable at the end of each of n successive years.*

If the interest on money be supposed to be $100r$ per cent. per annum; then from II.

The present value of the first payment is $A\,(1+r)^{-1}$

.... second .... $A\,(1+r)^{-2}$

....

..........................nth.............. $A\,(1+r)^{-n}$.

Hence the present value of the whole is

$$A\left\{\frac{1}{1+r}+\frac{1}{(1+r)^2}+\ldots+\frac{1}{(1+r)^n}\right\}=\frac{A}{r}\left\{1-\frac{1}{(1+r)^n}\right\}.$$

Ex. Find the present value of an annuity of £30 to be paid for 20 years, reckoning interest at 4 per cent.

Here $A=30,\ n=20,\ r=\dfrac{4}{100}=\dfrac{1}{25}$.

Hence the present value $= 30 \times 25 \left\{1 - \left(\frac{25}{26}\right)^{20}\right\}$.

Now $\log \left(\frac{25}{26}\right)^{20} = 20\{\log 25 - \log 26\}$

$= 20\{1\cdot3979400 - 1\cdot4149733\}$

$= 20(-\cdot0170333) = -\cdot340666 = \bar{1}\cdot659334$

$= \log \cdot456389$, from the Tables.

Hence the value required $= 30 \times 25 \times (1 - \cdot456389) = £407\cdot7$ ..

EXAMPLES XXXII.

The following logarithms are given

$\log 1\cdot02 = \cdot0086002$	$\log 1\cdot6386 = \cdot2144730$
$\log 1\cdot025 = \cdot0107239$	$\log 1\cdot6387 = \cdot2144995$
$\log 1\cdot033 = \cdot0141003$	$\log 1\cdot7292 = \cdot2378452$
$\log 1\cdot04 = \cdot0170333$	$\log 1\cdot7349 = \cdot2392744$
$\log 1\cdot05 = \cdot0211893$	$\log 2 = \cdot3010300$
$\log 1\cdot06 = \cdot0253059$	$\log 2\cdot0829 = \cdot3186684$
$\log 1\cdot1467 = \cdot0594498$	$\log 3 = \cdot4771213$
$\log 1\cdot1468 = \cdot0594877$	$\log 3\cdot0832 = \cdot4890017$
$\log 1\cdot2258 = \cdot0884196$	$\log 4\cdot4230 = \cdot6457169$
$\log 1\cdot2620 = \cdot1010594$	$\log 5\cdot1 = \cdot7075702$
$\log 1\cdot4816 = \cdot1707310$	$\log 5\cdot577 = \cdot7464006$
$\log 1\cdot4817 = \cdot1707603$	$\log 6\cdot3862 = \cdot8052425$
	$\log 7\cdot4297 = \cdot8709713$
	$\log 7\cdot4298 = \cdot8709771$

1. Find $\sqrt[20]{105}$.

2. Find $\sqrt[10]{51}$.

3. Find the amount of £100 in 50 years at 5 per cent. per annum.

4. Shew that money will more than double itself in 15 years at 5 per cent. per annum, and in 18 years at 4 per cent. per annum.

5. Find the amount of £500 in 10 years, interest at 4 per cent. being paid half yearly.

6. The number of births in a certain country every year is 85 per 1000 and the number of deaths 52 per 1000 of the population at the beginning of every year: shew that the population will be more than doubled in 22 years.

7. A man invests £30 a year in a Savings Bank which pays $2\frac{1}{2}$ per cent. per annum on all deposits. What will be the total amount at the end of 20 years?

8. What sum should be paid for an annuity of £100 a year to be paid for 40 years, money being supposed to be worth 4 per cent. per annum?

9. A corporation borrows £30000 which is to be repaid by 30 equal yearly payments. How much will have to be paid each year, money being supposed to be worth 4 per cent. per annum?

10. A house which is really worth £70 a year is let on a lease for 40 years at a rent of £10 a year, the lease being renewable at the end of every 14 years on payment of a fine. Calculate the amount of the fine, reckoning interest at 6 per cent.

CHAPTER XXV.

SUMMATION OF SERIES.

315. WE have already considered some important classes of series, namely the Progressions [Chapter XVII], Binomial series [Art. 288], and Exponential and Logarithmic series [Chapter XXIV]. In the present chapter some other important types of series will be considered.

316. The nth term of a series will be denoted by u_n, and the sum of n terms by S_n. When the series is convergent its sum to infinity will be denoted by S_∞.

317. No general method can be given by which the summation of series can be effected; but in a great number of cases the result can be obtained by expressing the general term of the series, u_n, as the difference of two expressions one of which involves $n-1$ in the same manner as the other involves n.

For example, in the series

$$\frac{a}{x(x+a)}+\frac{a}{(x+a)(x+2a)}+\frac{a}{(x+2a)(x+3a)}+\ldots,$$

the nth term, namely $\dfrac{a}{(x+\overline{n-1}.a)(x+na)}$, is equal to $\dfrac{1}{x+(n-1)a}-\dfrac{1}{x+na}$. Hence the series may be written

$$\left(\frac{1}{x}-\frac{1}{x+a}\right)+\left(\frac{1}{x+a}-\frac{1}{x+2a}\right)+\left(\frac{1}{x+2a}-\frac{1}{x+3a}\right)+\dots\dots$$

$+\left\{\frac{1}{x+(n-1)a}-\frac{1}{x+na}\right\}$; and it is now obvious that all the terms cancel except the first and last;

hence $$S_n=\frac{1}{x}-\frac{1}{x+na}=\frac{na}{x(x+na)}.$$

Ex. 1. Find the sum of n terms of the series

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots\dots+\frac{1}{n(n+1)}+\dots\dots$$

Ans. $1-\frac{1}{n+1}$.

Ex. 2. Find the sum of n terms of the series

$$\frac{1}{\underline{|2}}+\frac{2}{\underline{|3}}+\frac{3}{\underline{|4}}+\dots\dots+\frac{n}{\underline{|n+1}}+\dots\dots$$

Here $u_n=\frac{1}{\underline{|n}}-\frac{1}{\underline{|n+1}}$. *Ans.* $1-\frac{1}{\underline{|n+1}}$.

Ex. 3. Find the sum to infinity of the series

$$\frac{1}{3\,\underline{|1}}+\frac{1}{4\,\underline{|2}}+\frac{1}{5\,\underline{|3}}+\dots\dots+\frac{1}{(n+2)\,\underline{|n}}+\dots\dots$$

Here $u_n=\frac{1}{\underline{|n+1}}-\frac{1}{\underline{|n+2}}$. *Ans.* $\frac{1}{2}$.

Ex. 4. Find the sum to infinity of the series

$$\frac{3}{1^2.2^2}+\frac{5}{2^2.3^2}+\frac{7}{3^2.4^2}+\dots\dots+\frac{2n+1}{n^2(n+1)^2}+\dots\dots$$

Here $u_n=\frac{1}{n^2}-\frac{1}{(n+1)^2}$. *Ans.* 1.

Ex. 5. Find the sum of n terms of the series

$$\frac{1}{1.3}+\frac{2}{1.3.5}+\frac{3}{1.3.5.7}+\dots\dots+\frac{n}{1.3.5\dots(2n+1)}.$$

$$\left[2u_n=\frac{1}{1.3.5\dots(2n-1)}-\frac{1}{1.3.5\dots(2n-1)(2n+1)}\right].$$

Ans. $\frac{1}{2}\left\{1-\frac{1}{1.3.5\dots(2n+1)}\right\}$.

Ex. 6. Sum to infinity the series

$$\frac{2}{1.3}\frac{1}{3}+\frac{3}{3.5}\cdot\frac{1}{3^2}+\frac{4}{5.7}\cdot\frac{1}{3^3}+\ldots\ldots+\frac{n+1}{(2n-1)(2n+1)}\frac{1}{3^n}+\ldots\ldots$$

$$\left[\text{Since } \frac{n+1}{(2n-1)(2n+1)}=\frac{1}{4}\left(\frac{3}{2n-1}-\frac{1}{2n+1}\right),\right.$$

$$\left. 4u_n=\frac{1}{2n-1}\frac{1}{3^{n-1}}-\frac{1}{2n+1}\frac{1}{3^n}\right]. \qquad \textit{Ans. } \frac{1}{4}.$$

Ex. 7. Find the sum to infinity of the series

$$\frac{1}{2^2-1}+\frac{1}{4^2-1}+\frac{1}{6^2-1}+\frac{1}{8^2-1}+\ldots\ldots \qquad \textit{Ans. } \frac{1}{2}.$$

Ex. 8. Find the sum of n terms of the series

$$\frac{x}{(1-x)(1-x^2)}+\frac{x^2}{(1-x^2)(1-x^3)}+\frac{x^3}{(1-x^3)(1-x^4)}+\ldots\ldots$$

$$\textit{Ans. } \frac{1}{(1-x)^2}-\frac{1}{(1-x)(1-x^{n+1})}.$$

318. *To find the sum of n terms of the series*

$$\{a(a+b)\ldots(a+\overline{r-1}.b)\}+\{(a+b)(a+2b)\ldots(a+rb)\}$$
$$+\ldots+\{(a+\overline{n-1}.b)(a+nb)\ldots(a+\overline{n+r-2}.b)\}+\ldots$$

In the above series (i) each term contains r factors, (ii) the factors of any term are in arithmetical progression, and (iii) the first factors of the successive terms form the same A.P. as the successive factors of the first term.

Consider the series which is formed according to the same law but with one factor *added* at the *end* of every term, and let v_n be the nth term of this new series, so that $v_n=\{(a+\overline{n-1}.b)(a+nb)\ldots(a+\overline{n+r-1}.b)\}$.

Then

$$\begin{aligned} v_n-v_{n-1}&=\{(a+\overline{n-1}.b)(a+nb)\ldots(a+\overline{n+r-1}.b)\}\\ &\quad-\{(a+\overline{n-2}.b)(a+\overline{n-1}.b)\ldots(a+\overline{n+r-2}.b)\}\\ &=\{(a+\overline{n-1}.b)\ldots(a+\overline{n+r-2}.b)\}\{(a+\overline{n+r-1}.b)\\ &\qquad\qquad-(a+\overline{n-2}.b)\}\\ &=(r+1)b\{(a+\overline{n-1}.b)\ldots(a+\overline{n+r-2}.b)\}.\end{aligned}$$

Hence $$v_n - v_{n-1} = (r+1)b \times u_n.$$

Changing n into $n-1$ we have in succession

$$v_{n-1} - v_{n-2} = (r+1)b \times u_{n-1}$$

$$\dots\dots\dots\dots = \dots\dots\dots\dots\dots\dots$$

$$v_2 - v_1 = (r+1)b \times u_2.$$

Also $$v_1 - v_0 = (r+1)b \times u_1,$$

where v_0 is the term preceding v_1 which is formed according to the same law,

that is $v_0 = \{(a-b)a(a+b)\dots(a+\overline{r-1}\,b)\}$, so that v_0 is obtained by putting $n=0$ in the expression for v_n.

Hence by addition

$$v_n - v_0 = (r+1)bS_n;$$

$$\therefore S_n = (v_n - v_0)/(r+1)b.$$

Ex. 1. Sum the series $1.2+2.3+3.4+\dots\dots+n(n+1)$.

Here $u_n = n(n+1)$, $v_n = n(n+1)(n+2)$, $v_0 = 0.1.2$, $r=2$, and $b=1$.

Hence $S_n = \frac{1}{3}n(n+1)(n+2)$.

Or, by using the above method without quoting the result, which is preferable in very simple cases, we have

$$n(n+1) = \frac{1}{3}\{n(n+1)(n+2) - (n-1)n(n+1)\},$$

$$(n-1)n = \frac{1}{3}\{(n-1)n(n+1) - (n-2)(n-1)n\},$$

$$\dots\dots\dots = \dots\dots\dots$$

$$1.2 = \frac{1}{3}\{1.2.3 - 0.1.2\}.$$

Hence $$S_n = \frac{1}{3}n(n+1)(n+2).$$

Ex. 2. Sum the series $1.2.3+2.3.4+\dots\dots+n(n+1)(n+2)$.

Ans. $\frac{1}{4}n(n+1)(n+2)(n+3)$.

Ex. 3. Sum the series

$$1.2.3.4+2.3.4.5+\ldots\ldots+n(n+1)(n+2)(n+3).$$

Ans. $\frac{1}{5}n(n+1)(n+2)(n+3)(n+4)$.

Ex. 4. Find the sum of n terms of the series

$$3.5.7+5.7.9+7.9.11+\ldots\ldots$$

Here

$$u_n=(2n+1)(2n+3)(2n+5),\ \ v_n=(2n+1)(2n+3)(2n+5)(2n+7),$$

$$v_0=1.3.5.7,\ \ r=3,\ \text{and}\ b=2.$$

Hence $S_n=\frac{1}{4.2}\{(2n+1)(2n+3)(2n+5)(2n+7)-1.3.5.7\}$.

Many series which are not of the requisite form can be expressed as the algebraic sum of a number of series which are all of the required form; and the sum of the given series can then be written down. The following are examples.

Ex. 5. Find the sum of n terms of the series $1.3+2.4+3.5+\ldots\ldots$

Here $u_n=n(n+2)=n(n+1)+n$.

The sum of the series $1.2+2.3+\ldots+n(n+1)$ is

$$\frac{1}{3}\{n(n+1)(n+2)-0.1.2\},$$

and the sum of the series $1+2+\ldots+n$ is $\frac{1}{2}\{n(n+1)-0.1\}$.

Hence the required sum is $\frac{1}{3}n(n+1)(n+2)+\frac{1}{2}n(n+1)$.

Ex. 6. Find the sum of the series

$$2.3.1+3.4.4+4.5.7+\ldots\ldots+(n+1)(n+2)(3n-2).$$

Here $u_n=(n+1)(n+2)(3n-2)=3n(n+1)(n+2)-2(n+1)(n+2)$.

$$\therefore\ S_n=\frac{3}{4}\{n(n+1)(n+2)(n+3)-0.1.2.3\}$$

$$-\frac{2}{3}\{(n+1)(n+2)(n+3)-1.2.3\}$$

$$=\frac{1}{12}(9n-8)(n+1)(n+2)(n+3)+4.$$

319. *To find the sum of n terms of the series whose general term is*

$$1/\{(a+\overline{n-1}\,.\,b)(a+nb)(a+\overline{n+1}\,.\,b)\ldots(a+\overline{n+r-2}\,.\,b)\}.$$

Consider the series which is formed according to the same law but with one factor *taken away* from the *beginning* of each term, and let v_n be the nth term of this second series, so that $v_n = 1 / \{(a+nb)\ldots(a+\overline{n+r-2}\,.\,b)\}$.

Then

$$v_n - v_{n-1} = \frac{1}{\{(a+nb)\ldots(a+\overline{n+r-2}\,.\,b)\}} - \frac{1}{\{(a+\overline{n-1}\,.\,b)(a+nb)\ldots(a+\overline{n+r-3}\,.\,b)\}}$$

$$= \frac{1}{\{(a+\overline{n-1}\,.\,b)\ldots(a+\overline{n+r-2}\,.\,b)\}}\{(a+\overline{n-1}\,.\,b) - (a+\overline{n+r-2}\,.\,b)\};$$

$$\therefore\ v_n - v_{n-1} = -(r-1)\,b \times u_n.$$

Changing n into $n-1$ we have in succession

$$v_{n-1} - v_{n-2} = -(r-1)\,b \times u_{n-1},$$

$$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$v_2 - v_1 = -(r-1)\,b \times u_2.$$

Also $$v_1 - v_0 = -(r-1)\,b \times u_1,$$

where v_0 is the term which precedes v_1 and which is formed according to the same law, that is

$$v_0 = 1 / \{a(a+b)\ldots(a+\overline{r-2}\,.\,b)\}.$$

Hence, by addition,

$$v_n - v_0 = -(r-1)\,b \times S_n;$$

$$\therefore\ S_n = (v_0 - v_n) / (r-1)\,b.$$

Ex. 1. Sum the series $\frac{1}{2\,.\,3} + \frac{1}{3\,.\,4} + \ldots + \frac{1}{(n+1)(n+2)}$.

Here $u_n=\dfrac{1}{(n+1)(n+2)}$, $v_n=\dfrac{1}{n+2}$, $v_0=\dfrac{1}{2}$, $r=2$, $b=1$.

Hence $S_n=\dfrac{1}{1\,.\,1}\left\{\dfrac{1}{2}-\dfrac{1}{n+2}\right\}=\dfrac{1}{2}-\dfrac{1}{n+2}$.

Ex. 2. Sum the series $\dfrac{1}{1.2.3.4}+\dfrac{1}{2.3.4.5}+\ldots+\dfrac{1}{n(n+1)(n+2)(n+3)}$ to n terms and to infinity.

Here $u_n=\dfrac{1}{n(n+1)(n+2)(n+3)}$, $v_n=\dfrac{1}{(n+1)(n+2)(n+3)}$, $v_0=\dfrac{1}{1.2.3}$, $r=4$, and $b=1$.

Hence $S_n=\dfrac{1}{3\,.\,1}\left\{\dfrac{1}{1\,.\,2\,.\,3}-\dfrac{1}{(n+1)(n+2)(n+3)}\right\}$,

and $S_\infty=\dfrac{1}{3}\cdot\dfrac{1}{1\,.\,2\,.\,3}=\dfrac{1}{18}$.

Ex. 3. Sum the series $\dfrac{1}{3.7.11}+\dfrac{1}{7.11.15}+\ldots+\dfrac{1}{(4n-1)(4n+3)(4n+7)}$.

Ans. $S_n=\dfrac{1}{8}\left\{\dfrac{1}{3\,.\,7}-\dfrac{1}{(4n+3)(4n+7)}\right\}$.

Many series which are not of the above form can be expressed as the algebraic sum of a number of series which are all of the required form; and the sum of the series can then be written down. The following are examples.

Ex. 4. Sum the series $\dfrac{1}{1\,.\,3}+\dfrac{1}{2\,.\,4}+\dfrac{1}{3\,.\,5}+\ldots$

Here

$$u_n=\frac{1}{n(n+2)}=\frac{n+1}{n(n+1)(n+2)}=\frac{1}{(n+1)(n+2)}+\frac{1}{n(n+1)(n+2)}.$$

The series whose general terms are $\dfrac{1}{(n+1)(n+2)}$ and $\dfrac{1}{n(n+1)(n+2)}$ are of the required form. Hence the sum of the given series is given by

$$S_n=\left(\frac{1}{2}-\frac{1}{n+2}\right)+\frac{1}{2}\left(\frac{1}{1\,.\,2}-\frac{1}{(n+1)(n+2)}\right)=\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}.$$

Ex. 5. Sum the series $\frac{1}{1.3.5}+\frac{1}{2.4.6}+\ldots+\frac{1}{n(n+2)(n+4)}$.

$$u_n=\frac{1}{n(n+2)(n+4)}=\frac{(n+1)(n+3)}{n(n+1)(n+2)(n+3)(n+4)}$$

$$=\frac{n(n+4)+3}{n(n+1)(n+2)(n+3)(n+4)}$$

$$=\frac{1}{(n+1)(n+2)(n+3)}+\frac{3}{n(n+1)(n+2)(n+3)(n+4)}.$$

Hence

$$S_n=\frac{1}{2}\left\{\frac{1}{2.3}-\frac{1}{(n+2)(n+3)}\right\}+\frac{3}{4}\left\{\frac{1}{1.2.3.4}-\frac{1}{(n+1)(n+2)(n+3)(n+4)}\right\}.$$

320. The sum of series of the kind just considered may be obtained by means of partial fractions.

The method will be seen from the following example.

To find the sum of the series $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\ldots+\frac{1}{n(n+2)}$.

Let $\frac{1}{n(n+2)}=\frac{A}{n}+\frac{B}{n+2}$; then, as in Chapter XXIII, we find that $A=\frac{1}{2}$ and $B=-\frac{1}{2}$.

Hence $$2u_n=\frac{1}{n}-\frac{1}{n+2}.$$

We have therefore the following series of equations:

$$2u_1=\frac{1}{1}-\frac{1}{3},\qquad 2u_2=\frac{1}{2}-\frac{1}{4},\qquad 2u_3=\frac{1}{3}-\frac{1}{5},\qquad \ldots\ldots=\ldots\ldots,$$

$$2u_{n-2}=\frac{1}{n-2}-\frac{1}{n},\qquad 2u_{n-1}=\frac{1}{n-1}-\frac{1}{n+1},\qquad \text{and } 2u_n=\frac{1}{n}-\frac{1}{n+2}.$$

Hence, by addition,

$$2S_n=\frac{1}{1}+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2},$$

the other terms all cancelling.

Hence $$S_n=\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}.$$

321. *To find the sum of the rth powers of the first n whole numbers.*

We will first consider the two simplest cases.

Case I. To find the sum of $1^2 + 2^2 + 3^2 + \ldots + n^2$.

Here $$u_n \equiv n^2 = n(n+1) - n.$$

Hence, by Art. 318,

$$S_n = \frac{1}{3} n(n+1)(n+2) - \frac{1}{2} n(n+1)$$

$$= \frac{1}{6} n(n+1)(2n+1).$$

Case II. To find the sum of $1^3 + 2^3 + 3^3 + \ldots + n^3$.

Here $$u_n \equiv n^3 = n(n+1)(n+2) - 3n^2 - 2n$$

$$= n(n+1)(n+2) - 3n(n+1) + n.$$

Hence, by Art. 318,

$$S_n = \frac{1}{4} n(n+1)(n+2)(n+3) - \frac{3}{3} n(n+1)(n+2)$$

$$+ \frac{1}{2} n(n+1)$$

$$= \frac{1}{4} n(n+1)\{(n+2)(n+3) - 4(n+2) + 2\}$$

$$= \frac{1}{4} n^2 (n+1)^2.$$

Since $$1 + 2 + \ldots + n = \frac{1}{2} n(n+1),$$

the above result shews that

$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2,$$

so that *the sum of the cubes of the first n whole numbers is equal to the square of the sum of the numbers.*

The sum of the cubes of the first n integers can also be easily found by means of the identity $4n^3 \equiv \{n(n+1)\}^2 - \{(n-1)n\}^2$.

For we have in succession

$$4n^3 = \{n(n+1)\}^2 - \{(n-1)n\}^2,$$

$$4(n-1)^3 = \{(n-1)n\}^2 - \{(n-2)(n-1)\}^2,$$

$$\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$4 \cdot 2^3 = (2 \cdot 3)^2 - (1 \cdot 2)^2,$$

and $$4 \cdot 1^3 = (1 \cdot 2)^2 - (0 \cdot 1)^2.$$

Hence, by addition, $4S_n = n^2(n+1)^2$.

Case III. To find the sum of $1^r + 2^r + 3^r + \ldots + n^r$.

The sum for any particular value of r can be found by the same method as that adopted for the values 2 and 3.

For example, the sum of the fourth powers can be written down as soon as n^4 is expressed in the form

$$n^4 \equiv n(n+1)(n+2)(n+3) - 6n(n+1)(n+2) + 7n(n+1) - n.$$

By means of the Binomial Theorem a formula can be found which gives the sum of the rth powers in terms of the sum of powers lower than the rth; and this formula can be used for finding the sum of the 2nd, 3rd, 4th, &c. powers in succession. The formula has however the great disadvantage that in order to find by means of it the sum of the rth powers, it is necessary to know the sums of all the powers lower than the rth.

By the Binomial Theorem, we have in succession

$$(n+1)^{r+1} = n^{r+1} + (r+1)n^r + \frac{(r+1)r}{1.2}n^{r-1} + \ldots + 1,$$

$$(n)^{r+1} = (n-1)^{r+1} + (r+1)(n-1)^r + \frac{(r+1)r}{1.2}(n-1)^{r-1} + \ldots + 1,$$

$$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$3^{r+1} = 2^{r+1} + (r+1)2^r + \frac{(r+1)r}{1.2}2^{r-1} + \ldots + 1,$$

$$2^{r+1} = 1^{r+1} + (r+1)1^r + \frac{(r+1)r}{1.2}1^{r-1} + \ldots + 1.$$

Hence, by addition, we have $(n+1)^{r+1} - (n+1)$

$$= (r+1)S_n^r + \frac{(r+1)r}{1.2}S_n^{r-1} + \ldots + (r+1)S_n^1,$$

where S_n^r is written for the sum of n terms of the series

$$1^r + 2^r + 3^r + \ldots$$

We can in a similar manner find a formula for summing the rth powers of any series of quantities a, $a+b$, $a+2b$, ... in arithmetical progression. The result is

$$(a+nb)^{r+1}-a^{r+1}-nb^{r+1}=(r+1)\,bS_n^{\,r}+\frac{(r+1)\,r}{1\,.\,2}\,b^2S_n^{\,r-1}+\ldots+(r+1)\,b^rS_n^{\,1},$$

where $$S_n^{\,r}\equiv a^r+(a+b)^r+\ldots+(a+\overline{n-1}\,b)^r.$$

322. **Piles of Shot.** *To find the number of spherical balls in a pyramidal heap, when the base is* (I) *an equilateral triangle,* (II) *a square, and* (III) *a rectangle.*

I. In a pile of this kind the balls which rest on the ground form an equilateral triangle, and upon this first layer a number of balls are placed forming another equilateral triangle having one ball fewer in each side than in the side of the base; and so on; a single ball being at the top.

If n be the number of balls in each side of the base, the total number in the base will be

$$n+(n-1)+(n-2)+\ldots+2+1,$$

that is $\frac{1}{2}n\,(n+1)$. The whole number of the balls in the pile will therefore be

$$\tfrac{1}{2}\,\{n\,(n+1)+(n-1)\,n+\ldots+1\,.\,2\},$$

that is $\frac{1}{6}n\,(n+1)\,(n+2)$.

II. In this case the balls in any layer form a square with one ball fewer in each side than in the layer next below. Hence if n be the number of balls in each side of the lowest layer, n^2 will be the number of balls in the base, and therefore the whole number of the balls will be $n^2+(n-1)^2+(n-2)^2+\ldots+1^2$, that is $\frac{1}{6}n\,(n+1)\,(2n+1)$.

III. In this case the balls in any layer form a rectangle with one ball fewer in each side than in the layer next below. Hence if n and m be the number of balls in the sides of the lowest layer, nm will be the number of balls in the base and therefore the whole number of the balls will be, n being greater than m,

$$nm+(\overline{n-1})\,(m-1)+(n-2)\,(m-2)+\ldots\,(n-m+1)\,1$$
$$=(n-m+m)\,m+(\overline{n-m}+m-1)(m-1)+\ldots(\overline{n-m}+1)\,1$$

$$=(n-m)\{m+(m-1)+\ldots+1\}+m^2+(m-1)^2+\ldots+1^2$$
$$=\tfrac{1}{2}(n-m)\,m\,(m+1)+\tfrac{1}{6}m\,(m+1)\,(2m+1)$$
$$=\tfrac{1}{6}m\,(m+1)\,(3n-m+1).$$

Ex. 1. How many balls are contained in 8 layers of an unfinished triangular pile, the number in one side of the base being 12?

If the pile were completed it would contain $\frac{1}{6}\,.\,12\,.\,13\,.\,14$ balls; and there are $\frac{1}{6}\,.\,4\,.\,5\,.\,6$ missing from the complete pile; hence the required number is $\frac{1}{6}\,(12\,.\,13\,.\,14-4\,.\,5\,.\,6)$.

Ex. 2. How many balls are contained in 10 layers of an incomplete pile of balls whose base is a rectangle with 20 and 25 balls in its sides?

The number $=\Sigma n\,(n+5)$ from $n=11$ to $n=20$.

Ans. 3260.

323. **Figurate numbers.** Series of numbers which are such that the nth term of any series is the sum of the first n terms of the preceding series, all the numbers of the first series being unity, are called *orders* of *figurate numbers.*

Thus the different orders of figurate numbers are :—

First order, 1, 1, 1, 1, 1,......

Second order, 1, 2, 3, 4, 5,......

Third order, 1, 3, 6, 10, 15,......

....

It follows from the definition that the nth term of the *second* order of figurate numbers is n; the nth term of the *third* order will therefore be $(1+2+3+\ldots+n)$, that is $\tfrac{1}{2}n\,(n+1)$; the nth term of the *fourth* order will therefore be $\tfrac{1}{2}\{n\,(n+1)+(n-1)\,n+\ldots+1\,.\,2\}$, that is $\dfrac{n\,(n+1)\,(n+2)}{2\,.\,3}$; the nth term of the *fifth* order will therefore be $\dfrac{1}{2\,.\,3}\{n\,(n+1)\,(n+2)+(n-1)\,n\,(n+1)+\ldots+1\,.\,2\,.\,3\}$, that

is $\frac{1}{2.3.4}n(n+1)(n+2)(n+3)$; and so on, the nth term of the rth order being

$$\frac{n(n+1)(n+2)\ldots(n+r-2)}{\lfloor r-1}.$$

324. Polygonal numbers. Consider the arithmetical progressions whose first two terms are respectively 1, 1; 1, 2; 1, 3; 1, 4; and so on. Then the series formed by taking 1, 2, 3,..., n of the terms of these different arithmetical progressions, namely the series

$$\begin{array}{llll}
1, & 2, & 3,\ldots\ldots\ldots, & n, \ldots\ldots\ldots \\
1, & 3, & 6,\ldots\ldots\ldots, & \tfrac{1}{2}n(n+1), \ldots\ldots \\
1, & 4, & 9,\ldots\ldots\ldots, & n^2, \ldots\ldots. \\
1, & 5, & 12,\ldots\ldots\ldots, & n+\tfrac{3}{2}n(n-1), \ldots\ldots \\
\ldots & \ldots & \ldots & \ldots \\
1, & r, & 3r-3, \ldots, & n+\tfrac{1}{2}n(n-1)(r-2), \ldots
\end{array}$$

are called series of *linear, triangular, square, pentagonal,... r-gonal* numbers.

The sum of n terms of a series of r-gonal numbers can be written down at once, for the sum of n terms of the series whose general term is $n+\frac{1}{2}n(n-1)(r-2)$ is $\frac{1}{2}n(n+1)+\frac{1}{6}(n-1)n(n+1)(r-2)$ [Art. 318].

EXAMPLES XXXIII.

Find the sum of n terms of each of the following series, and find also the sum to infinity when the series is convergent.

1. $4.7.10+7.10.13+10.13.16+\ldots$

2. $\frac{1}{3.7.11}+\frac{1}{7.11.15}+\frac{1}{11.15.19}+\ldots$

3. $1.3.4+2.4.5+3.5.6+\ldots$

4. $1.5+3.7+5.9+7.11+\ldots$

5. $1 \,.\, 2 \,.\, 3 + 2 \,.\, 3 \,.\, 5 + 3 \,.\, 4 \,.\, 7 + 4 \,.\, 5 \,.\, 9 + \ldots$

6. $1 \,.\, 2^2 + 2 \,.\, 3^2 + 3 \,.\, 4^2 + 4 \,.\, 5^2 + \ldots$

7. $1 \,.\, 3^2 + 3 \,.\, 5^2 + 5 \,.\, 7^2 + 7 \,.\, 9^2 + \ldots$

8. $\frac{1}{1\,.\,3\,.\,7} + \frac{1}{3\,.\,5\,.\,9} + \frac{1}{5\,.\,7\,.\,11} + \frac{1}{7\,.\,9\,.\,13} + \ldots$

9. $\frac{1}{1\,.\,3\,.\,4} + \frac{1}{2\,.\,4\,.\,5} + \frac{1}{3\,.\,5\,.\,6} + \frac{1}{4\,.\,6\,.\,7} + \ldots$

10. $\frac{4}{1\,.\,2\,.\,3} + \frac{5}{2\,.\,3\,.\,4} + \frac{6}{3\,.\,4\,.\,5} + \frac{7}{4\,.\,5\,.\,6} + \ldots$

11. $\frac{1}{1\,.\,3\,.\,5} + \frac{2}{3\,.\,5\,.\,7} + \frac{3}{5\,.\,7\,.\,9} + \frac{4}{7\,.\,9\,.\,11} + \ldots$

12. $\frac{3}{1\,.\,2\,.\,4} + \frac{4}{2\,.\,3\,.\,5} + \frac{5}{3\,.\,4\,.\,6} + \frac{6}{4\,.\,5\,.\,7} + \ldots$

13. $\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \ldots$

14. $\frac{1^2}{1} + \frac{1^2+2^2}{2} + \frac{1^2+2^2+3^2}{3} + \frac{1^2+2^2+3^2+4^2}{4} + \ldots$

15. $1 \,.\, 1^2 + 2\,(1^2+2^2) + 3\,(1^2+2^2+3^2) + 4\,(1^2+2^2+3^2+4^2) + \ldots$

16. $a^2 + (a+b)^2 + (a+2b)^2 + \ldots$

17. $a^3 + (a+b)^3 + (a+2b)^3 + \ldots$

18. $1^2 + 3^2 + 5^2 + 7^2 + \ldots$

19. $1^2 + 5^2 + 9^2 + 13^2 + \ldots$

20. Shew that
$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (2n+1)^2 = (n+1)\,(2n+1).$$

21. Shew that $1^2 - 2^2 + 3^2 - 4^2 + \ldots - (2n)^2 = -\,n\,(2n+1)$.

22. Shew that
$$1^3 - 2^3 + 3^3 - 4^3 + \ldots + (2n+1)^3 = 4n^3 + 9n^2 + 6n + 1.$$

23. Find the sum of the series
$$1 \,.\, n + 2\,(n-1) + 3\,(n-2) + \ldots + n \,.\, 1.$$

24. Find the sum of the series $n . n+(n-1)(n+1)+(n-2)(n+2)+\ldots+2(2n-2)+1 . (2n-1)$.

25. Find the sum of n terms of the series

$$ab+(a-1)(b-1)+(a-2)(b-2)+\ldots$$

26. Prove that, if $S_n^{\,r} \equiv 1^r+2^r+\ldots n^r$; then will

(i) $5S_n^{\,4}=6S_n^{\,1}\times S_n^{\,3}-S_n^{\,2}$.

(ii) $S_n^{\,7}+S_n^{\,5}=2\,(S_n^{\,3})^2$.

27. Find the sum of the following series to n terms:

(i) $\dfrac{1}{2.3}\,2+\dfrac{2}{3.4}\,2^2+\dfrac{3}{4.5}\,2^3+\ldots$

(ii) $\dfrac{3}{1.2}\dfrac{1}{2}+\dfrac{4}{2.3}\dfrac{1}{2^2}+\dfrac{5}{3.4}\dfrac{1}{2^3}+\ldots$

(iii) $\dfrac{4}{1.2}\left(\dfrac{2}{3}\right)+\dfrac{5}{2.3}\left(\dfrac{2}{3}\right)^2+\dfrac{6}{3.4}\left(\dfrac{2}{3}\right)^3+\ldots$

(iv) $\dfrac{8}{1.2.3}\left(\dfrac{5}{7}\right)+\dfrac{9}{2.3.4}\left(\dfrac{5}{7}\right)^2+\dfrac{10}{3.4.5}\left(\dfrac{5}{7}\right)^3+\ldots$

(v) $\dfrac{9}{1.2.3}\left(\dfrac{3}{4}\right)+\dfrac{10}{2.3.4}\left(\dfrac{3}{4}\right)^2+\dfrac{11}{3.4.5}\left(\dfrac{3}{4}\right)^3+\ldots$

(vi) $\dfrac{15}{1.2.3}\left(\dfrac{6}{7}\right)+\dfrac{16}{2.3.4}\left(\dfrac{6}{7}\right)^2+\dfrac{17}{3.4.5}\left(\dfrac{6}{7}\right)^3+\ldots$

28. Shew that the sum of all the products of the first n natural numbers two together is $\dfrac{1}{24}(n-1)\,n\,(n+1)(3n+2)$.

29. Shew that the sum of all the products of the first n natural numbers three together is $\dfrac{1}{48}(n-2)(n-1)\,n^2\,(n+1)^2$.

30. Shew that the sum of the products of every pair of the squares of the first n whole numbers is

$$\frac{1}{360}\,n\,(n^2-1)(4n^2-1)(5n+6).$$

325. To find the sum of n terms of the series

$$\frac{a}{b}+\frac{a(a+x)}{b(b+x)}+\frac{a(a+x)(a+2x)}{b(b+x)(b+2x)}+\dots$$
$$+\frac{a(a+x)\dots(a+\overline{n-1}\,x)}{b(b+x)\dots(b+n-1\,x)}+\dots$$

In the above series there is an additional factor both in the numerator and in the denominator for every successive term, and the successive factors of the numerator and denominator form two arithmetical progressions with *the same common difference.*

Consider the series formed according to the same law but with an additional factor in the numerator, and let v_n be the general term of this second series, so that

$$v_n=\frac{a(a+x)\dots(a+\overline{n-1}\,x)(a+nx)}{b(b+x)\dots(b+\overline{n-1}\,x)}.$$

Then

$$v_n-v_{n-1}=\frac{a(a+x)\dots(a+\overline{n-1}\,x)(a+nx)}{b(b+x)\dots(b+\overline{n-1}\,x)}$$
$$-\frac{a(a+x)\dots(a+\overline{n-1}\,x)}{b(b+x)\dots(b+\overline{n-2}\,x)}$$
$$=\frac{a(a+x)\dots(a+\overline{n-1}\,x)}{b(b+x)\dots(b+\overline{n-1}\,x)}\left\{(a+nx)-(b+\overline{n-1}\,x)\right\};$$
$$\therefore\ v_n-v_{n-1}=u_n\times(a+x-b).$$

So also $v_{n-1}-v_{n-2}=u_{n-1}\times(a+x-b)$

$$\dots\dots\dots\dots=\dots\ \dots\dots\dots\dots$$
$$v_2-v_1=u_2\times(a+x-b).$$

Also $\quad v_1=a\dfrac{(a+x)}{b}=(a+x)\,u_1$

$$=u_1\times(a+x-b)+bu_1.$$

Hence $S_n \times (a+x-b) = v_n - a$;

$$\therefore\ S_n = \frac{a}{a+x-b}\left\{\frac{(a+x)\ldots(a+nx)}{b(b+x)\ldots(b+\overline{n-1}\,x)} - 1\right\}.$$

The sum of n terms of the series

$$\frac{a}{b} - \frac{a(a-x)}{b(b+x)} + \frac{a(a-x)(a-2x)}{b(b+x)(b+2x)} - \ldots$$

in which the successive factors of the numerator and denominator form two arithmetical progressions whose common differences are equal in magnitude but of opposite sign, can be found by changing the sign of a in the previous result: the sum can, however, be obtained independently by the same method. Thus

$$\frac{a}{b} = \frac{1}{a+b-x}\left[\frac{a}{1} + \frac{a(a-x)}{b}\right]$$

$$-\frac{a(a-x)}{b(b+x)} = -\frac{1}{a+b-x}\left[\frac{a(a-x)}{b} + \frac{a(a-x)(a-2x)}{b(b+x)}\right]$$

$$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$$

$$(-1)^{n-1}\frac{a(a-x)\ldots(a-\overline{n-1}\,x)}{b(b+x)\ldots(b+\overline{n-1}\,x)}$$

$$= (-1)^{n-1}\left[\frac{a(a-x)\ldots(a-\overline{n-1}\,x)}{b(b+x)\ldots(b+\overline{n-2}\,x)} + \frac{a(a-x)\ldots(a-nx)}{b(b+x)\ldots(b+\overline{n-1}\,x)}\right].$$

Hence

$$S_n = \frac{a}{a+b-x}\left[1 - (-1)^n\frac{(a-x)(a-2x)\ldots(a-nx)}{b(b+x)\ldots(b+\overline{n-1}\,x)}\right].$$

Ex. 1. To find the sum of n terms of the series $\frac{2}{3} + \frac{2.5}{3.6} + \frac{2.5.8}{3.6.9} + \ldots$

We have

$$\frac{2}{3}=\frac{1}{2}\left(\frac{2.5}{3}-\frac{2}{1}\right), \quad \frac{2.5}{3.6}=\frac{1}{2}\left(\frac{2.5.8}{3.6}-\frac{2.5}{3}\right), \ldots\ldots$$

$$\frac{2.5.8\ldots(3n-1)}{3.6.9\ldots 3n}=\frac{1}{2}\left\{\frac{2.5.8\ldots(3n+2)}{3.6.9\ldots 3n}-\frac{2.5.8\ldots(3n-1)}{3.6.9\ldots(3n-3)}\right\}.$$

Hence $$S_n=\frac{1}{2}\left\{\frac{2.5.8\ldots(3n+2)}{3.6.9\ldots 3n}-\frac{2}{1}\right\}.$$

[This particular series is a binomial series, the successive terms being the coefficients of x, x^2, &c., in the expansion of $(1-x)^{-\frac{2}{3}}$. Hence [Art. 287] $1+S_n=$ sum of the first $(n+1)$ coefficients in the expansion of $(1-x)^{-\frac{2}{3}}=$ coefficient of x^n in $(1-x)^{-\frac{2}{3}}\times(1-x)^{-1}$, that is in $(1-x)^{-\frac{5}{3}}$].

Ex. 2. Find the sum of n terms of the series $\frac{2}{3}+\frac{2.6}{3.7}+\frac{2.6.10}{3.7.11}+\ldots$

Ans. $\frac{2}{3}\left\{\frac{6.10\ldots(4n+2)}{3.7\ldots(4n-1)}-1\right\}$.

Ex. 3. Find the sum of n terms of the series

$$1-\frac{m}{1}+\frac{m(m-1)}{1.2}-\frac{m(m-1)(m-2)}{1.2.3}+\ldots$$

Ans. $(-1)^{n-1}\frac{(m-1)(m-2)\ldots(m-n+1)}{1.2\ldots(n-1)}$.

326. The sum of $n+1$ terms of the series

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n,$$

where a_n is any integral expression of the rth degree in n, can be found in the following manner.

$$S_n=a_0+a_1x+a_2x^2+\ldots+a_nx^n,$$

$$(1-x)^{r+1}=1-(r+1)x+\frac{(r+1)r}{1.2}x^2-\ldots+(-1)^{r+1}x^{r+1}.$$

Hence $$S_n\times(1-x)^{r+1}=a_0+\{a_1-(r+1)a_0\}x+\ldots$$

$$+\left\{a_p-(r+1)a_{p-1}+\frac{(r+1)r}{1.2}a_{p-2}-\ldots\right\}x^p+\ldots$$

$$+(-1)^{r+1}a_nx^{n+r+1}.$$

Now a_p is by supposition an integral expression of the rth degree in p; hence

$$a_p = A_r p^r + A_{r-1} p^{r-1} + A_{r-2} p^{r-2} + \ldots + A_0,$$

where $A_r, A_{r-1}, \ldots, A_0$ do not contain p.

Also, by Art. 305, the sum of the series

$$p^k - (r+1)(p-1)^k + \frac{(r+1)r}{1.2}(p-2)^k - \ldots \text{ to } (r+2) \text{ terms},$$

is zero for all integral values of k less than $r+1$. Hence

$$a_p - (r+1)\,a_{p-1} + \frac{(r+1)r}{1.2}\,a_{p-2} - \ldots \text{ to } (r+2) \text{ terms}$$

is zero for all values of p.

All the terms of the product $S_n \times (1-x)^{r+1}$ will therefore vanish except those near the beginning, or the end, for which the series $a_p - (r+1)\,a_{p-1} + \ldots$ is not continued for $(r+2)$ terms, that is all the terms of the product will vanish except the first $r+1$ terms and the last $r+1$ terms.

Hence

$$\begin{aligned} S_n \times (1-x)^{r+1} = a_0 &+ \{a_1 - (r+1)\,a_0\}\,x + \ldots \\ &+ \{a_{r-1} - (r+1)\,a_{r-2} + \ldots + (-1)^r (r+1)\,a_0\}\,x^r \\ &+ \left\{-(r+1)\,a_n + \frac{(r+1)r}{1.2}\,a_{n-1} - \ldots\right\} x^{n+1} \\ &+ \ldots + (-1)^{r+1}\,a_n\,x^{n+r+1}, \end{aligned}$$

whence the value of S_n is found.

Ex. 1. Find the sum of the series

$$1 + 2x + 3x^2 + 4x^3 + \ldots\ldots + (n+1)\,x^n.$$

$$S_{n+1} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots + (n+1)\,x^n,$$

$$(1-x)^2 = 1 - 2x + x^2;$$

$$\therefore\ (1-x)^2 S_{n+1} \equiv 1 + x^{n+1}\{n - 2(n+1)\} + (n+1)\,x^{n+2},$$

[all the other terms vanishing on account of the identity

$$k - 2(k-1) + (k-2) \equiv 0]$$

$$= 1 - (n+2)\,x^{n+1} + (n+1)\,x^{n+2};$$

$$\therefore\ S_{n+1} = \frac{1}{(1-x)^2} - \frac{(n+2)\,x^{n+1} - (n+1)\,x^{n+2}}{(1-x)^2}.$$

Ex. 2. **Find the sum of $n+1$ terms of the series**

$$1^3+2^3x+3^3x^2+\ldots\ldots+(n+1)^3x^n.$$

$$S_{n+1}=1^3+2^3x+3^3x^2+\ldots\ldots+(n+1)^3x^n,$$

$$(1-x)^4=1-4x+6x^2-4x^3+x^4;$$

$$\therefore\ S_{n+1}\times(1-x)^4=1+(2^3-4)\,x+(3^3-4\,.\,2^3+6\,.\,1^3)\,x^2$$
$$+\,(4^3-4\,.\,3^3+6\,.\,2^3-4\,.\,1^3)\,x^3$$
$$+\{-4\,(n+1)^3+6n^3-4\,(n-1)^3+(n-2)^3\}\,x^{n+1}$$
$$+\{6\,(n+1)^3-4n^3+(n-1)^3\}\,x^{n+2}$$
$$+\{-4\,(n+1)^3+n^3\}\,x^{n+3}$$
$$+(n+1)^3\,x^{n+4}.$$

[The other terms all vanishing, since

$$k^3-4\,(k-1)^3+6\,(k-2)^3-4\,(k-3)^3+(k-4)^3\equiv 0 \text{ identically.}]$$

Hence $S_{n+1}=[1+4x+x^2-(n^3+6n^2+12n+8)\,x^{n+1}$
$$+\,(3n^3+15n^2+21n+5)\,x^{n+2}$$
$$-\,(3n^3+12n^2+12n+4)\,x^{n+3}$$
$$+\,(n+1)^3\,x^{n+4}]/(1-x)^4.$$

When x is numerically less than 1, the series is convergent, and the sum of the series continued to infinity is $(1+4x+x^2)/(1-x)^4$.

327. **Series whose law is not given.** We have hitherto considered series in which the general term was given, or in which the law of the series was obvious on inspection. We proceed to consider cases in which the law of the series is not given. With reference to series in which the law is not given, but only a certain number of the terms of the series, it is of importance to remark that in *no case can the actual law of the series be really determined:* all that can be done is to find the *simplest law* the few terms which are given will obey.

There are for instance an indefinite number of series whose first few terms are given by $x+x^2+x^3+\ldots$, the simplest of all the series being the geometrical progression whose nth term is x^n: another series which has the given terms is that formed by the expansion of $\dfrac{x+x^2+x^3+\ldots+x^9}{1-x^{10}}$,

which agrees with the geometrical progression except at every 10th term.

Note. In what follows it must be understood that by *the* law of a series is meant the simplest law which satisfies the given conditions.

Method of Differences.

328. If in any arithmetical series

$$a_1 + a_2 + a_3 + \ldots + a_n,$$

each term be taken from the succeeding term, a new series is formed, namely the series

$$(a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1}) + \ldots,$$

which is called the *first order of differences.*

If the new series be operated upon in the same way, the series obtained is called the *second order of differences.* And so forth.

Thus, for the series 2, 7, 15, 26, 40, ...,
the first order of differences is 5, 8, 11, 14, ...,
and the second order of differences is 3, 3, 3, ...

329. When the law of a series is not given, it can often be found by forming the series of successive orders of differences; if the law of one of these orders of differences can be seen by inspection, the law of the preceding order of differences can often be found, and then the law of the next preceding order of differences, and so on until the law of the series itself is obtained. The method will be seen from the following examples.

Ex. 1. Find the nth term of the series

$$1 + 6 + 23 + 58 + 117 + 206 + \ldots\ldots$$

The first order of differences is $5 + 17 + 35 + 59 + 89 + \ldots\ldots$
„ second „ „ „ $12 + 18 + 24 + 30 + \ldots\ldots$
„ third „ „ „ $6 + 6 + 6 + \ldots\ldots$

The second order of differences is clearly an arithmetical progression whose nth term is $6(n+1)$.

Hence, if v_n be the nth term of the first order of differences, we have in succession

$$v_n - v_{n-1} = 6n\ ;\quad v_{n-1} - v_{n-2} = 6(n-1)\ ;\ \ldots\ldots;\ v_2 - v_1 = 6 \cdot 2.$$

Also $v_1 = 6 \cdot 1 - 1$. Hence, by addition,

$$v_n = 6(1 + 2 + \ldots\ldots + n) - 1 = 3n(n+1) - 1.$$

Then again, we have in succession $u_n - u_{n-1} = v_{n-1} = 3(n-1)n - 1$; $u_{n-1} - u_{n-2} = 3(n-2)(n-1) - 1$; ...; $u_2 - u_1 = 3 \cdot 1 \cdot 2 - 1$. Also $u_1 = 1$. Hence $u_n = 3\{(n-1)n + \ldots\ldots + 1 \cdot 2\} - n + 2 = (n-1)n(n+1) - n + 2$.

Ex. 2. Find the nth term and the sum of n terms of the series

$$6 + 9 + 14 + 23 + 40 + \ldots\ldots$$

The first order of differences is $3 + 5 + 9 + 17 + \ldots\ldots$

„ second „ „ „ $2 + 4 + 8 + \ldots\ldots$

Hence the second order of differences is a geometrical progression, the $(n-1)$th term being 2^{n-1}. Hence, if v_n be the nth term of the first order of differences, we have in succession

$$v_n - v_{n-1} = 2^{n-1},\quad v_{n-1} - v_{n-2} = 2^{n-2},\ \ldots\ldots,\ v_2 - v_1 = 2^1.$$

Also $v_1 = 3$. Hence, by addition, $v_n = (2 + 2^2 + \ldots\ldots + 2^{n-1}) + 3 = 2^n + 1$.

Then again, we have in succession $u_n - u_{n-1} = v_{n-1} = 2^{n-1} + 1$,

$u_{n-1} - u_{n-2} = 2^{n-2} + 1$, ..., $u_2 - u_1 = 2^1 + 1$. Also $u_1 = 6$.

Hence $u_n = (2^{n-1} + \ldots + 2) + n + 5 = 2^n + n + 3$.

The sum of n terms of the series can now be written down: for the sum of n terms of the series whose general term is $2^n + n + 3$ is

$$(2 + 2^2 + \ldots + 2^n) + \{n + (n-1) + \ldots + 1\} + 3n = 2^{n+1} - 2 + \frac{1}{2}n(n+1) + 3n.$$

Note. By the method adopted in the preceding examples the nth term of a series can always be found *provided the terms of one of its orders of differences are all the same, or are in geometrical progression.*

330. It is of importance to notice that when the nth term of a series is an integral expression of the rth degree in n, all the terms of the rth order of differences will be the same.

For, if $u_n \equiv A_r n^r + A_{r-1} n^{r-1} + \ldots + A_0$, where $A_r, A_{r-1}, \ldots$ do not contain n, the nth term of the first order of differences will be

$$\{A_r(n+1)^r + A_{r-1}(n+1)^{r-1} + \ldots\} - \{A_r n^r + A_{r-1} n^{r-1} + \ldots\},$$

which only contains n to the $(r-1)$th degree.

Similarly the nth term of the second order of differences will be of the $(r-2)$th degree in n; and so on, the nth term of the rth order of differences being of the $(r-r)$th degree in n, so that the nth term of the rth order of differences will not contain n, and therefore all the terms of that order of differences will be the same.

When therefore it is found that all the terms of the rth order of differences are the same, we may at once assume that $u_n \equiv A_r n^r + A_{r-1} n^{r-1} + \ldots + A_0$, and find the values of $A_r, A_{r-1}, \ldots, A_0$ by comparing the actual terms of the series with the values obtained by putting $n = 1$, $n = 2$, &c. in the assumed value of u_n. This method will not however give the value of u_n in a convenient form for finding the sum of the series; for, if r be greater than 3, the sum of n terms of the series whose general term is $A_r n^r + A_{r-1} n^{r-1} + \ldots$ cannot be found [see Art. 321] without a troublesome transformation which will in fact reduce u_n to the form in which it is obtained by the method of the preceding Article. A much better method would be to assume that $u_n = A_r(n)_r + A_{r-1}(n)_{r-1} + \ldots$, and then to find $A_r, A_{r-1}, \ldots, A_0$ as above.

Recurring Series.

331. **Definitions.** When $r+1$ successive terms of the series $a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n + \ldots$ are connected by a relation of the form $a_n x^n + px(a_{n-1} x^{n-1}) + qx^2(a_{n-2} x^{n-2}) + \ldots = 0$, the series is called a *recurring series* of the rth *order*, and $1 + px + qx^2 + \ldots$ is called its *scale of relation*. The relation does not hold good unless there are r terms before the nth, so that the relation only holds good after the first r terms of the series.

For example, the series $1+2x+4x^2+8x^3+\ldots\ldots$ is a recurring series of the first order, the scale of relation being $1-2x$. Again, it will be found that the series $1+3x+5x^2+7x^3+9x^4+\ldots\ldots$ is a recurring series of the second order, the scale of relation being $1-2x+x^2$.

332. *To find the sum of n terms of a given recurring series.*

Let the series be $a_0+a_1x+\ldots\ldots+a_nx^n+\ldots$, and let the scale of relation be $1+px+qx^2$. [This assumes that the recurring series is of the second order, but the method is perfectly general]. Then

$$S_n = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n;$$

$$\therefore S_n(1+px+qx^2) = a_0 + (a_1+pa_0)x + (a_2+pa_1+qa_0)x^2 + \ldots + (a_n+pa_{n-1}+qa_{n-2})x^n + (pa_n+qa_{n-1})x^{n+1} + qa_n x^{n+2}$$
$$= a_0 + (a_1+pa_0)x + (pa_n+qa_{n-1})x^{n+1} + qa_n x^{n+2},$$

since all the other terms vanish in virtue of the relation $a_k x^k + px(a_{k-1}x^{k-1}) + qx^2(a_{k-2}x^{k-2}) = 0$, which is by supposition true for all values of k greater than 1.

Hence

$$S_n = \frac{a_0 + (a_1+pa_0)x + (pa_n+qa_{n-1})x^{n+1} + qa_nx^{n+2}}{1+px+qx^2}.$$

If the given series be a convergent series, the nth term will be indefinitely small when n is increased without limit; and the sum of the series continued to infinity will in this case be given by

$$S_\infty = \frac{a_0+(a_1+pa_0)x}{1+px+qx^2}.$$

The expression $\dfrac{a_0+(a_1+pa_0)x}{1+px+qx^2}$ is therefore such that if it can be expanded in a convergent series proceeding according to ascending powers of x, the coefficient of x^n in its expansion will be the same as in the recurring series.

On this account the expression $\dfrac{a_0+(a_1+pa_0)x}{1+px+qx^2}$ is called the *generating function* of the series.

333. *A recurring series of the* r*th order is determined when the first* 2r *terms are given.*

For let the series be

$$a_0+a_1x+a_2x^2+\ldots\ldots+a_nx^n+\ldots\ldots$$

Then, the series being a recurring series of the rth order, if we assume that the unknown scale of relation is $1+p_1x+p_2x^2+\ldots+p_rx^r$, we have by definition the following equations

$$\begin{aligned}
a_{r+1}+p_1a_r+p_2a_{r-1}+\ldots+p_ra_0&=0,\\
a_{r+2}+p_1a_{r+1}+p_2a_r+\ldots+p_ra_1&=0,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&=0,\\
a_{2r}+p_1a_{2r-1}+p_2a_{2r-2}+\ldots+p_ra_{r-1}&=0.
\end{aligned}$$

We have therefore r equations which are sufficient to determine the r unknown quantities $p_1, p_2, \ldots, p_r$ in the scale of relation; and when the scale of relation is determined the series can be continued term by term, for a_{2r+1} is given by the equation $a_{2r+1}+p_1a_{2r}+\ldots+p_ra_r=0$; and when a_{2r+1} is found, a_{2r+2} can be found in a similar manner; and so on.

The series is similarly determined when *any* $2r$ consecutive terms are given.

334. From Art. 305 we know that if $p<r+1$,

$$k^r-(r+1)(k-1)^p+\frac{(r+1)r}{1.2}(k-2)^p-\ldots$$
$$\text{to } r+2 \text{ terms}=0,$$

for all values of k.

This shews that the series

$$1^r + 2^r x + 3^r x^2 + \ldots + (n+1)^r x^n + \ldots$$

is a recurring series whose scale of relation is $(1-x)^{r+1}$.

It also shews that the series

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

is a recurring series whose scale of relation is $(1-x)^{r+1}$ whenever a_n is a rational and integral expression of the rth degree in n.

335. In order to find the sum of any number of terms of a recurring series by the method of Art. 332, it is necessary to know the *general term* of the series; we must therefore shew how to obtain the general term of a recurring series when the first few terms are given.

By Art. 333 the scale of relation of a recurring series of the rth order can be found when the $2r$ first terms are given; and, having found the scale of relation, the generating function is at once given by the formula of Art. 332.

Now, provided the scale of relation can be expressed in factors of the first degree, the generating function can be expressed as a series of partial fractions of the form $\dfrac{A}{1-ax}$ or of the form $\dfrac{A}{(1-ax)^k}$, and the coefficient of any power of x in the expansion of the generating function can be at once written down by the binomial theorem; and thus the general term of the series is found.

When the value of x is such that the given recurring series is not convergent, the generating function will not be equal to the given series continued to infinity nor can it be expanded in a series of ascending powers of x; but, taking as an example the generating function in Art. 332, the expression $\dfrac{a_0 + (a_1 + pa_0)y}{1 + py + qy^2}$ can always be expanded in ascending powers of y, if y be taken sufficiently small, and

the coefficients of y^0 and y^1 in this expansion will clearly be a_0 and a_1 respectively and all succeeding terms will obey the law $a_k + pa_{k-1} + qa_{k-2} = 0$, and hence all the coefficients of the expansion will be the same as the corresponding coefficients in the given series. We may therefore in all cases, whether the series is convergent or not, find the general term of a recurring series by writing down the expansion of its generating function in ascending powers of x on the supposition that x is sufficiently small.

Ex. 1. Find the nth term of the recurring series $3+4x+6x^2+10x^3+\ldots$

In an example of this kind, in which the *order* of the recurring series is not given, it must always be understood that what is wanted is the recurring series of the *lowest possible order* whose first few terms agree with the given series. In the present example there is a sufficient number of terms given to determine a recurring series of the *second* order, but an indefinite number of recurring series of the *third*, or of any higher order than the second, could be found whose first four terms were the same as those of the given series. [See Art. 327.]

Assuming then that the scale of relation is $1+px+qx^2$, we have the equations $6+4p+3q=0$, and $10+6p+4q=0$, whence $p=-3$ and $q=2$. Hence the scale of relation is $1-3x+2x^2$.

The generating function is therefore

$$\frac{3+(4-9)x}{1-3x+2x^2}=\frac{3-5x}{1-3x+2x^2}=\frac{2}{1-x}+\frac{1}{1-2x}$$

$$=2\{1+x+\ldots+x^{n-1}\}+\{1+2x+\ldots+2^{n-1}x^{n-1}+\ldots\}.$$

Hence the general term of the series is $(2+2^{n-1})\,x^{n-1}$.

The sum of n terms can now be found by the method of Art. 332; the sum can however be written down at once, for the sum of n terms of the series $2(1+x+x^2+\ldots)$ is $2(1-x^n)/(1-x)$ and the sum of n terms of the series $1+2x+4x^2+\ldots$ is $(1-2^nx^n)/(1-2x)$.

We may remark that the given series is convergent provided $x<\frac{1}{2}$.

Ex. 2. Find the nth term and the sum of n terms of the series $1+3+7+13+21+31+\ldots$.

Consider the series $1+3x+7x^2+13x^3+21x^4+31x^5+\ldots$

Then, *assuming* that the series is a recurring series, and also that a sufficient number of terms are given to determine the recurring series completely, it follows that the series is of the third order.

Let then the scale of relation be $1+px+qx^2+rx^3$; we then have the following equations to find p, q, r:

$$13+7p+3q+r=0,$$
$$21+13p+7q+3r=0,$$
and $$31+21p+13q+7r=0,$$
whence $$p=-3, \quad q=3 \text{ and } r=-1,$$
so that the scale of relation is $1-3x+3x^2-x^3$.

The generating function is now found to be

$$\frac{1+x^2}{(1-x)^3}=\frac{2}{(1-x)^3}-\frac{2}{(1-x)^2}+\frac{1}{1-x}.$$

Hence the general term of the series

$1+3x+7x^2+\ldots$ is $x^{n-1}\{n(n+1)-2n+1\}=(n^2-n+1)\,x^{n-1}$.

Thus the general term of the given series is n^2-n+1.

Having found the general term of the series the sum of the first n terms can be written down, for the sum of n terms of the series whose nth term is $n(n-1)+1$ is $\frac{1}{3}(n-1)\,n\,(n+1)+n$.

Ex. 3. Find the nth term of the series $2+2+8+20+\ldots\ldots$

Considered as a recurring series of the lowest possible order, the generating function of $2+2x+8x^2+20x^3+\ldots$ will be found to be

$$\frac{2-2x}{1-2x-2x^2}.$$

Now the factors of $1-2x-2x^2$ are irrational, and therefore the nth term of the series, considered as a recurring series of the *second* order, will be a complicated expression containing radicals.

On the other hand, by the method of Art. 329, we should be led to conclude that the nth term of the series was $(3n^2-9n+8)\,x^{n-1}$, which by Art. 334 is a recurring series of the *third* order.

As we have already remarked, the actual law of a series cannot be determined from any finite number of its terms, and the above is a case in which it would be difficult to decide as to what is the *simplest* law that the few terms given obey, for the recurring series of the lowest order which has the given terms for its first four terms is not the recurring series which gives the simplest expression for the nth term.

Convergency and Divergency.

336. We shall now investigate certain theorems in convergency which were not considered in Chapter XXI.

337. **Convergency of infinite products.** A product composed of an infinite number of factors cannot be convergent unless the factors tend to unity as their limit; for otherwise the addition of a factor would always make a finite change in the continued product, and there could be no definite quantity to which the product approached without limit as the number of factors was indefinitely increased.

It is therefore only necessary to consider infinite products of the form

$$\Pi(1+u_r) \equiv (1+u_1)(1+u_2)(1+u_3)\ldots(1+u_n)\ldots,$$

where u_n becomes indefinitely small as n is indefinitely increased; and the convergency or divergency of such products is determined by the following theorem.

Theorem. *The infinite product* $\Pi(1+u_r)$, *in which all the factors are greater than unity, is convergent or divergent according as the infinite series* Σu_r *is convergent or divergent.*

Since $e^x > 1 + x$, for all positive values of x, it follows that

$$(1+u_1)(1+u_2)(1+u_3)\ldots < e^{u_1}.e^{u_2}.e^{u_3}\ldots < e^{u_1+u_2+u_3+\ldots}$$

Hence, if Σu_r be convergent, $\Pi(1+u_r)$ will also be convergent.

Again, $$(1+u_1)(1+u_2) > 1+u_1+u_2,$$

$$(1+u_1)(1+u_2)(1+u_3) > (1+u_1+u_2)(1+u_3) > 1+u_1+u_2+u_3,$$

and so on, so that

$$\Pi(1+u_r) > 1 + \Sigma u_r.$$

Hence, if Σu_r be divergent, $\Pi(1+u_r)$ will also be divergent.

Ex. 1. To shew that $\dfrac{a(a+1)(a+2)\ldots(a+n-1)}{b(b+1)(b+2)\ldots(b+n-1)}$, is infinite or zero, when n is indefinitely increased, according as a is greater or less than b.

For, if $a > b$, the expression may be written in the form

$$\left(1+\frac{a-b}{b}\right)\left(1+\frac{a-b}{b+1}\right)\ldots\ldots\left(1+\frac{a-b}{b+n-1}\right)\ldots\ldots,$$

which is greater than

$$1+(a-b)\left\{\frac{1}{b}+\frac{1}{b+1}+\frac{1}{b+2}+\ldots\ldots\right\}.$$

But $\frac{1}{b}+\frac{1}{b+1}+\frac{1}{b+2}+\ldots$ is a divergent series [Art. 274]: the given expression is therefore infinite when n is infinite, a being greater than b.

If $b > a$; then as before, $\frac{b(b+1)(b+2)\ldots\ldots}{a(a+1)(a+2)\ldots\ldots}$ is infinite; and therefore $\frac{a(a+1)(a+2)\ldots\ldots}{b(b+1)(b+2)\ldots\ldots}$ must be zero.

Ex. 2. Determine whether the series

$$\frac{a}{b}+\frac{a(a+x)}{b(b+x)}+\frac{a(a+x)(a+2x)}{b(b+x)(b+2x)}+\ldots\ldots$$

is convergent or divergent.

From Art. 325, we have

$$S_n=\frac{a}{a+x-b}\left\{\frac{(a+x)(a+2x)\ldots(a+nx)}{b(b+x)(b+2x)\ldots(b+\overline{n-1}\,.\,x)}-1\right\}.$$

Now by Ex. 1, $\frac{(a+x)(a+2x)\ldots(a+nx)}{b(b+x)\ldots(b+\overline{n-1}x)}$ is infinite or zero according as $a+x \gtrless b$.

Hence the given series is convergent, and its sum is then $\frac{a}{b-a-x}$, if $b > a+x$. Also the series is divergent if $b < a+x$.

Also if $b=a+x$, the series becomes $\frac{a}{b}+\frac{a}{b+x}+\frac{a}{b+2x}$ which is known to be divergent [Art. 274].

338. The Binomial Series. We have already proved that the binomial series, namely

$$1+mx+\frac{m(m-1)}{1\,.\,2}x^2+\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}x^3+\ldots$$

is convergent or divergent, for all values of m, according as x is numerically less or greater than unity.

If $x=1$, the series becomes

$$1+m+\frac{m(m-1)}{1\,.\,2}+\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}+\ldots$$

Now we know that the terms of this series are *alternately positive and negative* after the rth term, where r is the first positive integer greater than $m+1$. Moreover the ratio u_{n+1}/u_n is numerically less or greater than unity according as $m+1$ is positive or negative. The series will therefore, from theorem V. Chapter XXI. be convergent when $m+1$ is positive provided the nth term decreases without limit as n is increased without limit.

Now $$\pm\frac{1}{u_n} = \frac{1\,.\,2\,......\,n}{(-m)(1-m)...(n-1-m)}$$

$$\therefore\ \pm\frac{1}{u_n} = \frac{1}{m}\left(1+\frac{1+m}{1-m}\right)\left(1+\frac{1+m}{2-m}\right)...\left(1+\frac{1+m}{n-1-m}\right).$$

Now, if $m+1$ be positive and less than r, the product of the factors from the rth onwards is greater than

$$(1+m)\left\{\frac{1}{r-m}+\frac{1}{r+1-m}+...\right\};$$

and the product of the preceding factors is finite.

Hence, when n is increased without limit, $1/u_n$ is infinitely great, and therefore u_n indefinitely small, provided $1+m$ be positive.

Thus *the binomial series is convergent if* $x=1$, *provided* $m > -1$.

If $x=-1$, the series becomes

$$1-m+\frac{m(m-1)}{1\,.\,2}-\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}+...$$

The sum of n terms of the above series is easily found to be [see Art. 287 or Art. 325]

$$\frac{(1-m)(2-m)(3-m)...(n-1-m)}{1\,.\,2\,.\,3...(n-1)}.$$

The sum of n terms of the series is therefore [Ex. 1, Art. 337], zero or infinite, when n is infinite, according as m is positive or negative.

Thus *the binomial series is convergent when* $x=-1$, *provided m is positive.*

339. **Cauchy's Theorem.** *If the series $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ have all its terms positive, and if each term be less than the preceding, then the series will be convergent or divergent according as the series $u_1 + au_a + a^2u_{a^2} + \ldots + a^nu_{a^n} + \ldots$ is convergent or divergent, a being any positive integer.*

For, since each term is less than the preceding, we have the following series of relations

$$u_1 + u_2 + \ldots + u_a < au_1 < (a-1)\,u_1 + u_1,$$

$$u_{a+1} + u_{a+2} \ldots + u_{a^2} < (a^2 - a)\,u_a < (a-1)\,au_a,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u_{a^n+1} + u_{a^n+2} + \ldots + u_{a^{n+1}} < (a^{n+1} - a^n)\,u_{a^n} < (a-1)\,a^nu_{a^n}.$$

Hence, by addition, $S < (a-1)\,\Sigma + u_1$ .... (I),

where S and Σ stand for the sum of the first and second series respectively.

Again, we have since a is $\nless 2$,

$$a\,(u_1 + u_2 + u_3 + \ldots + u_a) > au_a$$

$$a\,(u_{a+1} + u_{a+2} + \ldots + u_{a^2}) > a\,(a^2 - a)\,u_{a^2} > a^2u_{a^2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a\,(u_{a^{n-1}+1} + u_{a^{n-1}+2} + \ldots + u_{a^n}) > a\,(a^n - a^{n-1})\,u_{a^n} > a^nu_{a^n}.$$

Hence $\qquad aS > \Sigma - u_1$ ....(II).

From I and II it follows that if S is finite so also is Σ, and that if S is infinite so also is Σ.

Ex. To shew that the series $\dfrac{1}{n\,(\log n)^k}$ is convergent if k be greater than unity, and divergent if k be equal or less than unity.

By Cauchy's theorem the series will be convergent or divergent according as the series whose general term is $\dfrac{a^n}{a^n\,(\log a^n)^k}$ is convergent or divergent.

Now $\Sigma \frac{a^n}{a^n (\log a^n)^k} = \Sigma \frac{1}{n^k} \frac{1}{(\log a)^k} = \frac{1}{(\log a)^k} \Sigma \frac{1}{n^k}$;

it therefore follows from Art. 274 that the given series is convergent if $k > 1$ and divergent if $k \ngtr 1$.

340. We shall conclude with the two following tests of convergency which are sometimes of use, referring the student to Boole's Finite Differences and Bertrand's Differential Calculus for further information on the subject.

341. **Theorem.** *A series is convergent when, from and after any particular term, the ratio of each term to the preceding is less than the corresponding ratio in a known convergent series whose terms are all positive.*

For let the series, beginning at the term in question, be

$$U = u_1 + u_2 + u_3 + \ldots + u_n + \ldots,$$

and the known convergent series, beginning at the same term, be

$$V = v_1 + v_2 + v_3 + \ldots + v_n + \ldots.$$

Then, since $\frac{u_{r+1}}{u_r} < \frac{v_{r+1}}{v_r}$ for all values of r, we have

$$V = v_1 + v_1 \cdot \frac{v_2}{v_1} + v_1 \frac{v_3}{v_2} \frac{v_2}{v_1} + v_1 \frac{v_4}{v_3} \frac{v_3}{v_2} \frac{v_2}{v_1} + \ldots$$

$$> v_1 + v_1 \frac{u_2}{u_1} + v_1 \frac{u_3}{u_2} \frac{u_2}{u_1} + v_1 \frac{u_4}{u_3} \frac{u_3}{u_2} \frac{u_2}{u_1} + \ldots$$

$$> \frac{v_1}{u_1} (u_1 + u_2 + u_3 + u_4 + \ldots) > \frac{v_1}{u_1} U.$$

Hence as V is convergent, U must also be convergent.

The given series is therefore convergent, for the sum of the finite number of terms preceding the first term of U must be finite.

We can prove similarly that if, from and after any particular term, $u_{r+1} : u_r > v_{r+1} : v_r$, and all the terms of Σu_r have the same sign; then Σu_r will be divergent if Σv_r be divergent.

342. Theorem. *A series, all of whose terms are positive, is convergent or divergent according as the limit of* $n\left(1-\frac{u_{n+1}}{u_n}\right)$ *is greater or less than unity.*

For let the limit of $n\left(1-\frac{u_{n+1}}{u_n}\right)$ be α.

Consider the series $\Sigma\frac{1}{n^\beta}\equiv\Sigma v_n$; then

$$n\left(1-\frac{v_{n+1}}{v_n}\right)=n\left\{\frac{(n+1)^\beta-n^\beta}{(n+1)^\beta}\right\}=\frac{\beta n^\beta+\text{lower powers of } n}{n^\beta+\text{lower powers of } n}.$$

Hence the limit of $n\left(1-\frac{v_{n+1}}{v_n}\right)$, when n is infinitely great, is β.

First suppose $\alpha>1$, and let β be chosen between α and 1.

Then since the *limit* of $n\left(1-\frac{u_{n+1}}{u_n}\right)$ is greater than the *limit* of $n\left(1-\frac{v_{n+1}}{v_n}\right)$, there must be some finite value of n from and after which the former is constantly greater than the latter.

But when $\quad n\left(1-\frac{u_{n+1}}{u_n}\right)>n\left(1-\frac{v_{n+1}}{v_n}\right)$,

we have $\qquad \frac{v_{n+1}}{v_n}>\frac{u_{n+1}}{u_n}$.

Hence, by the previous theorem, Σu_n will be convergent if Σv_n be convergent; but Σv_n is convergent since $\beta>1$.

Similarly, if α be <1, and β be taken between α and 1, we can prove that Σu_n is divergent if Σv_n is divergent, and the latter series is known to be divergent when $\beta<1$.

If the limit of $n\left(1-\frac{u_{n+1}}{u_n}\right)$ be unity the test fails.

Ex. 1. Is the series $\frac{a}{b}+\frac{a(a+1)}{b(b+1)}x+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^2+\dots$ convergent or divergent?

Here $\frac{u_{n+1}}{u_n}=\frac{a+n}{b+n}x$, the limit of which is x. Hence, either from the beginning or after a finite number of terms,

$$\frac{u_{n+1}}{u_n} \gtrless 1 \text{ according as } x \gtrless 1.$$

Hence the series is divergent if $x>1$, and convergent if $x<1$.

If $x=1$, the limit of $\frac{u_{n+1}}{u_n}$ is unity. But

$$n\left(1-\frac{u_{n+1}}{u_n}\right)=n\left(1-\frac{a+n}{b+n}\right),$$

the limit of which is $b-a$.

Thus, if $x=1$, the series is convergent when $b-a>1$ and divergent when $b-a<1$. When $b=a+1$, the series becomes

$$\frac{a}{b}+\frac{a}{b+1}+\frac{a}{b+2}+\dots\dots,$$

which is divergent.

[These are the results arrived at in Ex. 2, Art. 337.]

EXAMPLES XXXIV

1. Find the sum of each of the following series to n terms, and when possible to infinity:—

(i) $\frac{4}{5}+\frac{4.7}{5.8}+\frac{4.7.10}{5.8.11}+\dots.$

(ii) $\frac{2}{4}+\frac{2.5}{4.7}+\frac{2.5.8}{4.7.10}+\dots.$

(iii) $\frac{3}{8}+\frac{3.5}{8.10}+\frac{3.5.7}{8.10.12}+\dots.$

(iv) $\frac{11}{14}+\frac{11.13}{14.16}+\frac{11.13.15}{14.16.18}+\dots.$

2. Find, by the method of differences, the nth term and the sum of n terms of the following series:—

(i) $2+2+8+20+38+\dots.$

(ii) $7+14+19+22+23+22+\dots.$

(iii) $1+4+11+26+57+120+\dots.$

(iv) $1 + 0 + 1 + 8 + 29 + 80 + 193 + \dots.$

(v) $1 + 5 + 15 + 35 + 70 + 126 + \dots.$

(vi) $1 + 2 + 29 + 130 + 377 + 866 + 1717 + \dots.$

3. Find the generating function of each of the following series on the supposition that it is a determinate recurring series:—

(i) $2 + 4x + 14x^2 + 52x^3 + \dots.$

(ii) $1 + 3x + 11x^2 + 43x^3 + \dots.$

(iii) $1 + 6x + 40x^2 + 288x^3 + \dots.$

(iv) $1 + x + 2x^2 + 7x^3 + 14x^4 + 35x^5 + \dots.$

(v) $1^2 + 2^2x + 3^2x^2 + 4^2x^3 + 5^2x^4 + 6^2x^5 + \dots.$

4. Find the nth term, and the sum of n terms of the following recurring series:—

(i) $2 + 6 + 14 + 30 + \dots.$

(ii) $2 - 5 + 29 - 89 + \dots.$

(iii) $1 + 2 + 7 + 20 + \dots.$

5. Find the nth term of the series 1, 3, 4, 7, &c.; where, after the second, each term is formed by adding the two preceding terms.

6. Determine a, b, c, d so that the coefficient of x^n in the expansion of $\dfrac{a + bx + cx^2 + dx^3}{(1-x)^4}$ may be $(n+1)^3$.

7. Shew that the series $1^r + 2^r x + 3^r x^2 + 4^r x^3 + \dots$ is the expansion of an expression of the form $\dfrac{a_0 + a_1 x + \dots + a_r x^r}{(1-x)^{r+1}}$; shew also that $a_r = 0$; and that $a_{r-s} = a_{s-1}$.

8. Find the sum to infinity of the recurring series

$$2 + 5x + 9x^2 + 15x^3 + 25x^4 + 43x^5 + \dots$$

supposed convergent, it being given that the scale of relation is of the form $1 + px + qx^2 + rx^3$. Shew that the $(n+1)$th term of the series is $(2^n + 2n + 1)x^n$.

9. Find the sum to infinity of the series

$$1+4x+11x^2+26x^3+57x^4+120x^5+\ldots,$$

x being less than $\frac{1}{2}$.

10. Find the sum of n terms of the series

$$1+\frac{x}{a}+\frac{x(x+a)}{ab}+\frac{x(x+a)(x+b)}{abc}+\ldots.$$

11. Shew that

$$\frac{1}{x+a}+\frac{a}{(x+a)(x+b)}+\frac{ab}{(x+a)(x+b)(x+c)}+\ldots$$
$$+\frac{abc\ldots k}{(x+a)(x+b)\ldots(x+k)(x+l)}=\frac{1}{x}-\frac{abc\ldots kl}{x(x+a)(x+b)\ldots(x+l)}.$$

12. Shew that

$$\frac{1}{p+n}+\frac{1+n}{(p+n)(p+2n)}+\frac{(1+n)(1+2n)}{(p+n)(p+2n)(p+3n)}$$
$$+\ldots \text{ to infinity}=\frac{1}{p-1},$$

provided that $p>1$ and $p+n>0$.

13. Shew that, if m be greater than 1,

$$1+\frac{1}{m+1}+\frac{1.2}{(m+1)(m+2)}+\frac{1.2.3}{(m+1)(m+2)(m+3)}$$
$$+\ldots \text{ to infinity}=\frac{m}{m-1}.$$

14. Shew that

$$\frac{1}{m+1}-\frac{n-1}{(m+1)(m+2)}+\frac{(n-1)(n-2)}{(m+1)(m+2)(m+3)}-\ldots=\frac{1}{m+n},$$

if $m+n$ be positive, or if n be a positive integer.

15. Shew that, if n be any positive integer,

$$\frac{n}{n+1}-\frac{n(n-1)}{(n+1)(n+2)}+\frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)}+\ldots$$
$$\mp\frac{n(n-1)(n-2)\ldots 2.1}{(n+1)(n+2)\ldots 2n}=\frac{1}{2}.$$

16. Shew that, if m be a positive integer,

$$1-m\frac{2n+1}{2n+2}+\frac{m(m-1)}{1.2}\frac{(2n+1)(2n+3)}{(2n+2)(2n+4)}-\dots$$
$$=\frac{1.3.5\dots(2m-1)}{(2n+2)(2n+4)\dots(2n+2m)}.$$

17. Shew that, if m, n and $m-n+1$ are positive integers; then

$$1+n\frac{m}{m-n+1}+\frac{n(n-1)}{1.2}\frac{m(m-1)}{(m-n+1)(m-n+2)}$$
$$+\frac{n(n-1)(n-2)}{1.2.3}\frac{m(m-1)(m-2)}{(m-n+1)(m-n+2)(m-n+3)}$$
$$+\dots \text{ to } (n+1) \text{ terms}=\frac{(m+1)(m+2)\dots(m+m)}{(m-n+1)(m-n+2)\dots(m-n+m)}.$$

18. Shew that, if $m+1>0$, then

$$\frac{1}{2}-\frac{1}{3}m+\frac{1}{4}\frac{m(m-1)}{1.2}-\frac{1}{5}\frac{m(m-1)(m-2)}{1.2.3}+\dots$$
$$=\frac{1}{(m+1)(m+2)}.$$

19. Shew that, if P_r be the sum of the products r together of the first n even numbers, and Q_r be the sum of the products r together of the first n odd numbers; then will

$$1+P_1+P_2+\dots\dots+P_n=1.3.5\dots(2n+1),$$

and $$1+Q_1+Q_2+\dots\dots+Q_n=2.4.6\dots 2n.$$

20. Prove that

$$\{a+(a+1)+(a+2)+\dots+(a+n)\}\{a^2+(a+1)+(a+2)+\dots$$
$$\dots+(a+n)\}=a^3+(a+1)^3+\dots+(a+n)^3.$$

21. Shew that the series

$$1-\frac{1-a^n}{1-a}+\frac{(1-a^n)(1-a^{n-1})}{(1-a)(1-a^2)}-\frac{(1-a^n)(1-a^{n-1})(1-a^{n-2})}{(1-a)(1-a^2)(1-a^3)}+\dots$$

is zero when n is an odd integer, and is equal to $(1-a)(1-a^3)\dots(1-a^{n-1})$ when n is an even integer [Gauss].

22. Find the sum of the series

$$\frac{n}{1.2.3}+\frac{n-1}{2.3.4}+\frac{n-2}{3.4.5}+\ldots+\frac{1}{n(n+1)(n+2)}.$$

23. Sum to infinity $\frac{2x^2}{1.3}-\frac{3x^3}{2.4}+\frac{4x^4}{3.5}-\ldots.$

24. Sum, when convergent, the series

$$\frac{x}{1.2}+\frac{x^2}{2.3}+\ldots+\frac{x^n}{n(n+1)}+\ldots.$$

25. Sum to infinity the series

$$1.2.3+3.4.5x+5.6.7x^2+7.8.9x^3+\ldots,$$

x being less than unity.

26. Shew that, if n is a positive integer

$$1-3n+\frac{3n(3n-3)}{1.2}-\frac{3n(3n-4)(3n-5)}{1.2.3}+\ldots=2(-1)^n.$$

27. Shew that, if a_1, a_2, a_3,... be all positive, and if $a_1+a_2+a_3+\ldots$ be divergent, then

$$\frac{a_1}{a_1+1}+\frac{a_2}{(a_1+1)(a_2+1)}+\frac{a_3}{(a_1+1)(a_2+1)(a_3+1)}+\ldots$$

is convergent and equal to unity.

28. Shew that the series

$$\frac{1}{2^{m+x}}+\frac{2^m}{3^{m+x}}+\frac{3^m}{4^{m+x}}+\ldots+\frac{n^m}{(n+1)^{m+x}}+\ldots$$

is convergent if $x>1$, and is divergent if $x \ngtr 1$.

29. Shew that, if the series $u_1+u_2+u_3+\ldots+u_n+\ldots$ be divergent, the series

$$\frac{u_2}{u_1}+\frac{u_3}{u_1+u_2}+\ldots+\frac{u_n}{u_1+u_2+\ldots+u_{n-1}}+\ldots$$

will also be divergent.

30. For what values of x has the infinite product $(1+a)(1+ax)(1+ax^2)(1+ax^3)\ldots$ a finite value?

31. Prove that, if v_n is always finite and greater than unity but approaches unity without limit as n increases indefinitely, the two infinite products $v_1v_2v_3v_4\ldots$, $v_1v_2{}^2v_4{}^4v_8{}^8\ldots$ are either both finite or both infinite.

32. Test the convergency of the following series:—

(i) $$\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\ldots+\frac{n^n}{(n+1)^{n+1}}+\ldots.$$

(ii) $$\frac{1}{1}+\frac{1}{\sqrt[2]{2^3}}+\frac{1}{\sqrt[3]{3^4}}+\ldots+\frac{1}{\sqrt[n]{n^{n+1}}}+\ldots.$$

(iii) $$\frac{2}{3.4}+\frac{2.4}{3.5.6}+\frac{2.4.6}{3.5.7.8}+\ldots+\frac{2.4.6\ldots2n}{3.5.7\ldots(2n+1)(2n+2)}+\ldots.$$

(iv) $$\frac{1}{2.3}+\frac{1.3}{2.4.5}+\frac{1.3.5}{2.4.6.7}+\ldots+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n(2n+1)}+\ldots.$$

(v) $$\frac{\alpha\beta}{1.\gamma}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}+\ldots+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}+\ldots.$$

CHAPTER XXVI.

INEQUALITIES.

343. We have already proved [Art. 232] the theorem that the arithmetic mean of any two positive quantities is greater than their geometric mean. We now proceed to consider other theorems of this nature, which are called *Inequalities.*

Note. Throughout the present chapter every letter is supposed to denote a real positive quantity.

344. The following elementary principles of inequalities can be easily demonstrated:

I. If $a > b$; then $a + x > b + x$, and $a - x > b - x$.

II. If $a > b$; then $-a < -b$.

III. If $a > b$; then $ma > mb$, and $-ma < -mb$.

IV. If $a > b$, $a' > b'$, $a'' > b''$, &c.;
then $a + a' + a'' + \dots > b + b' + b'' + \dots$,
and $aa'a'' \dots > bb'b'' \dots$.

V. If $a > b$; then $a^m > b^m$, and $a^{-m} < b^{-m}$.

Ex. 1. Prove that $a^3 + b^3 > a^2b + ab^2$.

We have to prove that

$$a^3 - a^2b - ab^2 + b^3 > 0, \text{ or that } (a^2 - b^2)(a - b) > 0,$$

which must be true since *both* factors are positive or *both* negative according as a is greater or less than b.

Ex. 2. Prove that $a^m + a^{-m} > a^n + a^{-n}$, if $m > n$.

We have to prove that $(a^m - a^n)(1 - a^{-m-n}) > 0$, which must be the case since *both* factors are positive or *both* negative according as a is greater or less than 1.

Ex. 3. Prove that $(l^2+m^2+n^2)(l'^2+m'^2+n'^2) > (ll'+mm'+nn')^2$.

It is easily seen that

$$(l^2+m^2+n^2)(l'^2+m'^2+n'^2) - (ll'+mm'+nn')^2$$
$$= (mn'-m'n)^2 + (nl'-n'l)^2 + (lm'-l'm)^2.$$

Now the last expression can never be negative, and can only be zero when $mn'-m'n$, $nl'-n'l$ and $lm'-l'm$ are all separately zero, the conditions for which are $\frac{l}{l'} = \frac{m}{m'} = \frac{n}{n'}$.

Hence $(l^2+m^2+n^2)(l'^2+m'^2+n'^2) > (ll'+mm'+nn')^2$, except when $l/l' = m/m' = n/n'$, in which case the inequality becomes an equality.

345. **Theorem I.** *The product of two positive quantities, whose sum is given, is greatest when the two factors are equal to one another.*

For let $2a$ be the given sum, and let $a + x$ and $a - x$ be the two factors. Then the product of the two quantities is $a^2 - x^2$, which is clearly greatest when x is zero, in which case each factor is half the given sum.

The above theorem is really the same as that of Art. 232; for from Art. 232 we have $\left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}\right) > ab$.

346. **Theorem II.** *The product of any number of positive quantities, whose sum is given, is greatest when the quantities are all equal.*

For, suppose that any two of the factors, a and b, are unequal.

Then, keeping all the other factors unchanged, take $\frac{1}{2}(a+b)$ and $\frac{1}{2}(a+b)$ instead of a and b: we thus, without altering the sum of all the factors, increase their continued product since $\frac{1}{2}(a+b) \times \frac{1}{2}(a+b) > ab$, except when $a = b$.

Hence, so long as any two of the factors are unequal, the continued product can be increased without altering the sum; and therefore all the factors must be equal to one another when their continued product has its greatest possible value.

Thus, unless the n quantities $a, b, c, \dots$ are all equal,

$$abcd \dots < \left(\frac{a+b+c+d+\dots}{n}\right)^n,$$

and therefore

$$\frac{a+b+c+d+\dots}{n} > \sqrt[n]{(abcd \dots)}.$$

By extending the meaning of the terms *arithmetic mean* and *geometric mean*, the last result may be enunciated as follows :—

Theorem III. *The arithmetic mean of any number of positive quantities is greater than their geometric mean.*

Ex. 1. Shew that $a^3+b^3+c^3 > 3abc$.

We have $\dfrac{a^3+b^3+c^3}{3} > \sqrt[3]{(a^3 . b^3 . c^3)} > abc.$

Ex. 2. Shew that $\dfrac{a_1}{a_2}+\dfrac{a_2}{a_3}+\dfrac{a_3}{a_4}+\dots\dots+\dfrac{a_n}{a_1} > n$.

We have $\dfrac{1}{n}\left(\dfrac{a_1}{a_2}+\dfrac{a_2}{a_3}+\dots\dots+\dfrac{a_n}{a_1}\right) > \sqrt[n]{\left(\dfrac{a_1}{a_2}.\dfrac{a_2}{a_3}\dots\dfrac{a_n}{a_1}\right)} > \sqrt[n]{1}.$

Ex. 3. Find the greatest value of $(a-x)(b-y)(cx+dy)$, where a, b, c are known positive quantities and $a-x$, $b-y$ are also positive.

The expression is greatest when $(ac-cx)(bd-dy)(cx+dy)$ is greatest, and this is the case, since the sum of the factors is now constant, when $ac-cx=bd-dy=cx+dy$. Whence the greatest value is found to be $(ac+bd)^3/27cd$.

Ex. 4. Find when $x^\alpha y^\beta z^\gamma$ has its greatest value, for different values of x, y and z subject to the condition that $x+y+z$ is constant.

Let $P=x^\alpha y^\beta z^\gamma$; then

$$\frac{P}{\alpha^\alpha \beta^\beta \gamma^\gamma} = \left(\frac{x}{\alpha}\right)^\alpha . \left(\frac{y}{\beta}\right)^\beta . \left(\frac{z}{\gamma}\right)^\gamma$$

$$= \frac{x}{\alpha}.\frac{x}{\alpha}.\frac{x}{\alpha}\dots\dots\frac{y}{\beta}.\frac{y}{\beta}.\frac{y}{\beta}\dots\dots\frac{z}{\gamma}.\frac{z}{\gamma}.\frac{z}{\gamma}\dots\dots$$

The sum of the factors in the last product is constant, since there are α factors each $\dfrac{x}{\alpha}$, β factors each $\dfrac{y}{\beta}$, and γ factors each $\dfrac{z}{\gamma}$, and therefore the sum of all the factors is $x+y+z$.

Hence, from Theorem II, $\left(\frac{x}{\alpha}\right)^{\alpha}\left(\frac{y}{\beta}\right)^{\beta}\left(\frac{z}{\gamma}\right)^{\gamma}$ has its greatest value when *all* the factors are equal, that is when $\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}$.

It is clear that P is greatest when $P/\alpha^{\alpha}\beta^{\beta}\gamma^{\gamma}$ is greatest, since α, β, γ are constant; hence P is greatest when $x/\alpha=y/\beta=z/\gamma$.

In the above it was assumed that α, β, γ were integers; if this be not the case, let n be the least common multiple of the denominators of α, β, γ. Then $x^{\alpha}y^{\beta}z^{\gamma}$ will have its greatest value when $x^{n\alpha}y^{n\beta}z^{n\gamma}$ has its greatest value, which by the above, since $n\alpha$, $n\beta$ and $n\gamma$ are all integers, will be when $\frac{x}{n\alpha}=\frac{y}{n\beta}=\frac{z}{n\gamma}$, that is when $\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}$.

Thus, whether α, β, γ are integral or not, $x^{\alpha}y^{\beta}z^{\gamma}$ is greatest for values of x, y and z such that $x+y+z$ is constant, when $x/\alpha=y/\beta=z/\gamma$.

347. **Theorem IV.** *The sum of any number of positive quantities, whose product is given, is least when the quantities are all equal.*

First suppose that there are two quantities denoted by a and b.

Then, if a and b are unequal, $(\sqrt{a}\sim\sqrt{b})^2>0$, and therefore $a+b>\sqrt{ab}+\sqrt{ab}$. Hence the sum of any two unequal quantities a, b is greater than the sum of the two equal quantities $\sqrt{ab}, \sqrt{ab}$ which have the same product.

Next suppose that there are more than two quantities.

Let a, b, any two of the quantities, be unequal. Then, keeping all the others unchanged, take $\sqrt{ab}$ and $\sqrt{ab}$ instead of a and b: we thus, without altering the product of all the quantities, diminish their sum since $\sqrt{ab}+\sqrt{ab}<a+b$. Hence, so long as any two of the quantities are unequal, their sum can be diminished without altering their product; and therefore all the quantities must be equal to one another when their sum has its least possible value.

348. **Theorem V.** *If m and r be positive, and $m > r$; then*

$$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} \text{ will be greater than}$$

$$\frac{a_1^r + a_2^r + \ldots + a_n^r}{n} \times \frac{a_1^{m-r} + a_2^{m-r} + \ldots + a_n^{m-r}}{n}.$$

We have to prove that

$$n(a_1^m + a_2^m + \ldots) > (a_1^r + a_2^r + \ldots)(a_1^{m-r} + a_2^{m-r} + \ldots),$$

or that

$$(n-1)(a_1^m + a_2^m + \ldots) > \Sigma(a_1^r a_2^{m-r} + a_1^{m-r} a_2^r),$$

or that

$$\Sigma(a_1^m + a_2^m - a_1^r a_2^{m-r} - a_1^{m-r} a_2^r) > 0,$$

every letter being taken with each of the $(n-1)$ other letters.

Now

$$a_1^m + a_2^m - a_1^r a_2^{m-r} - a_1^{m-r} a_2^r = (a_1^r - a_2^r)(a_1^{m-r} - a_2^{m-r}),$$

which is positive since $a_1^r - a_2^r$ and $a_1^{m-r} - a_2^{m-r}$ are *both* positive or *both* negative according as a_1 is greater or less than a_2.

Hence $\Sigma(a_1^m + a_2^m - a_1^r a_2^{m-r} - a_1^{m-r} a_2^r) > 0$, which proves the proposition.

By repeated application of the above we have

$$\frac{\Sigma a_1^m}{n} > \frac{\Sigma a_1^\alpha}{n} \cdot \frac{\Sigma a_1^\beta}{n} \cdot \frac{\Sigma a_1^\gamma}{n} \ldots,$$

where $\alpha, \beta, \gamma, \ldots$ are positive quantities such that

$$\alpha + \beta + \gamma + \ldots\ldots = m.$$

Ex. 1. Shew that $3(a^3 + b^3 + c^3) > (a+b+c)(a^2+b^2+c^2)$.

Ex. 2. Shew that $a^5 + b^5 + c^5 > abc(a^2+b^2+c^2)$.

From Theorem V, $\dfrac{a^5+b^5+c^5}{3} > \dfrac{a^2+b^2+c^2}{3} \cdot \left(\dfrac{a+b+c}{3}\right)^3$,

$$> \frac{a^2+b^2+c^2}{3} \cdot abc, \text{ from Theorem III.}$$

349. **Theorem VI.*** *To prove that, if a, b, c, ... and $\alpha, \beta, \gamma, \ldots$ be all positive, then*

$$\left(\frac{a\alpha + b\beta + c\gamma + \ldots}{a+b+c+\ldots}\right)^{a+b+c+\ldots} > \alpha^a\beta^b\gamma^c\ldots.$$

First, let $a, b, c, \ldots$ be integers. Take a things each α, b things each β, and so on. Then, by Theorem III,

$$\frac{(\alpha+\alpha+\ldots \text{ to } a \text{ terms}) + (\beta+\beta+\ldots \text{ to } b \text{ terms}) + \ldots}{a+b+\ldots}$$

$$> \sqrt[a+b+\ldots]{\{\alpha^a\beta^b\ldots\ldots\}},$$

that is $$\frac{a\alpha + b\beta + \ldots}{a+b+\ldots} > \sqrt[a+b+\ldots]{\{\alpha^a\beta^b\ldots\}}.$$

If $a, b, c, \ldots$ be not integral, let m be the least common multiple of the denominators of $a, b, c, \ldots$; then $ma, mb, mc, \ldots$ are all integers, and we have

$$\frac{ma\alpha + mb\beta + \ldots}{ma+mb+\ldots} > \sqrt[ma+mb+\ldots]{\{\alpha^{ma}\beta^{mb}\ldots\}}.$$

Hence $$\left\{\frac{a\alpha + b\beta + \ldots}{a+b+\ldots}\right\}^{a+b+\ldots} > \alpha^a\beta^b \ldots\ldots\ldots\ldots\ldots [A].$$

COR. I. Put $\alpha = \frac{1}{a}$, $\beta = \frac{1}{b}$, ..., and let there be n of the letters $a, b, \ldots$; then

$$\left\{\frac{n}{a+b+\ldots}\right\}^{a+b+\ldots} > \frac{1}{a^a b^b\ldots};$$

$$\therefore \left\{\frac{a+b+\ldots}{n}\right\}^{a+b+\ldots} < a^a b^b\ldots$$

COR. II. Substitute in (A) a^r for a, b^r for $b, \ldots$; also substitute a^{m-r} for α, b^{m-r} for β, ..., where $m > r$.

$$\left\{\frac{a^m + b^m + \ldots}{a^r + b^r + \ldots}\right\}^{a^r+b^r+\ldots} > \{a^{a^r}b^{b^r}\ldots\}^{m-r} \ldots\ldots [B].$$

* See a paper by Mr L. J. Rogers in the *Messenger of Mathematics*, Vol. XVII.

Again, substitute $a^r, b^r, \ldots$ for $a, b, \ldots$ respectively, and $a^{t-r}, b^{t-r}, \ldots$ for α, β respectively, where $t<r$.

Then

$$\left\{\frac{a^t+b^t+\ldots}{a^r+b^r+\ldots}\right\}^{a^r+b^r+\ldots} > \{a^{a^r}b^{b^r}\ldots\}^{t-r}$$

$$\therefore \left\{\frac{a^r+b^r+\ldots}{a^t+b^t+\ldots}\right\}^{a^r+b^r+\ldots} < \{a^{a^r}b^{b^r}\ldots\}^{r-t} \ldots\ldots[\mathrm{C}].$$

Hence, as $m-r$ and $r-t$ are both positive, we have from [B] and [C]

$$\left\{\frac{a^m+b^m+\ldots}{a^r+b^r+\ldots}\right\}^{\frac{1}{m-r}} > \left\{\frac{a^r+b^r+\ldots}{a^t+b^t+\ldots}\right\}^{\frac{1}{r-t}}.$$

Hence, *provided* $m>r>t$,

$$\{a^m+b^m+\ldots\}^{r-t}\times\{a^r+b^r+\ldots\}^{t-m}\times\{a^t+b^t+\ldots\}^{m-r}>1\ldots.[\mathrm{D}].$$

The following are particular cases of [D].

Put $t=0$; then, since $a^0+b^0+\ldots=n$, we have *provided* $m>r$

$$\left\{\frac{a^m+b^m+\ldots}{n}\right\}^r > \left\{\frac{a^r+b^r+\ldots}{n}\right\}^m \ldots\ldots\ldots[\mathrm{E}].$$

Again, put $t=0$, $m=1$; then since $m>r>t$, r must be a proper fraction.

Hence, *if r be a proper fraction,*

$$\left\{\frac{a+b+\ldots}{n}\right\}^r > \frac{a^r+b^r+\ldots}{n} \ldots\ldots\ldots\ldots[\mathrm{F}].$$

Again, put $t=0$, $r=1$; then $m>1$.

Hence, *if $m>1$ we have*

$$\frac{a^m+b^m+\ldots}{n} > \left\{\frac{a+b+\ldots}{n}\right\}^m \ldots\ldots\ldots\ldots[\mathrm{G}].$$

Now put $m=1$, $r=0$, then t is negative. Hence, *provided t be negative,*

$$(a+b+\ldots)^{-t}\times n^{t-1}\times(a^t+b^t+\ldots)>0;$$

$$\therefore \frac{a^t+b^t+\ldots}{n} > \left(\frac{a+b+\ldots}{n}\right)^t \ldots\ldots\ldots\ldots[\mathrm{H}].$$

From [F], [G] and [H] we see that

$$\frac{a^x+b^x+\dots}{n} \gtrless \left[\frac{a+b+\dots}{n}\right]^x$$

according as x is not or is a proper fraction.

350. We shall conclude this chapter by solving the following examples. [See also Art. 133.]

Ex. 1. Shew that, if $s=a_1+a_2+\dots+a_n$,

$$\frac{s}{s-a_1}+\frac{s}{s-a_2}+\dots+\frac{s}{s-a_n}>\frac{n^2}{n-1}, \text{ unless } a_1=a_2=\dots=a_n.$$

Unless $a_1=a_2=\dots=a_n$, we have

$$\frac{1}{n}\left(\frac{s}{s-a_1}+\frac{s}{s-a_2}+\dots+\frac{s}{s-a_n}\right)>\sqrt[n]{\frac{s^n}{(s-a_1)(s-a_2)\dots(s-a_n)}},$$

and $\dfrac{(s-a_1)+(s-a_2)+\dots+(s-a_n)}{n}>\sqrt[n]{\{(s-a_1)(s-a_2)\dots(s-a_n)\}}$.

By multiplication, since $(s-a_1)+(s-a_2)+\dots+(s-a_n)=ns-s$, we have

$$\frac{n-1}{n^2}\left(\frac{s}{s-a_1}+\frac{s}{s-a_2}+\dots+\frac{s}{s-a_n}\right)>1.$$

Ex. 2. Shew that, if $a+b+c+d=3s$; then will

$$abcd>81(s-a)(s-b)(s-c)(s-d).$$

For $3\sqrt[3]{\{(s-b)(s-c)(s-d)\}}<\{(s-b)+(s-c)+(s-d)\}<a$.

So also $3\sqrt[3]{\{(s-c)(s-d)(s-a)\}}<b$, $3\sqrt[3]{\{(s-d)(s-a)(s-b)\}}<c$,

and $3\sqrt[3]{\{(s-a)(s-b)(s-c)\}}<d$.

Hence $81(s-a)(s-b)(s-c)(s-d)<abcd$.

Ex. 3. Shew that $\left(\dfrac{x^2+y^2+z^2}{x+y+z}\right)^{x+y+z}>x^xy^yz^z$, unless $x=y=z$.

First suppose that x, y and z are integral; then by Theorem III.

$$\frac{(x+x+\dots\text{ to } x \text{ terms})+(y+y+\dots\text{ to } y\text{ terms})+(z+z+\dots\text{ to } z\text{ terms})}{x+y+z}$$

$$>\sqrt[x+y+z]{(x^xy^yz^z)};$$

$$\therefore \left(\frac{x^2+y^2+z^2}{x+y+z}\right)^{x+y+z}>x^xy^yz^z.$$

If x, y, z be not integral let m be the least common multiple of their denominators; then mx, my and mz are integral, and we have by the first case

$$\left(\frac{m^2x^2+m^2y^2+m^2z^2}{mx+my+mz}\right)^{mx+my+mz}>(mx)^{mx}(my)^{my}(mz)^{mz};$$

that is $\left(\frac{x^2+y^2+z^2}{x+y+z}\right)^{m(x+y+z)} \times m^{m(x+y+z)} > (x^x y^y z^z)^m \times m^{m(x+y+z)}$;

$$\therefore \left(\frac{x^2+y^2+z^2}{x+y+z}\right)^{x+y+z} > x^x y^y z^z.$$

The Theorem can in a similar manner be proved to be true for any number of quantities.

EXAMPLES XXXV.

Prove the following inequalities, all the letters being supposed to represent positive quantities:—

1. $y^2z^2 + z^2x^2 + x^2y^2 \nless xyz(x+y+z)$.

2. $(a_1^2 + a_2^2 + a_3^2 + \ldots\ldots)(b_1^2 + b_2^2 + b_3^2 + \ldots\ldots)$
$$\nless (a_1b_1 + a_2b_2 + a_3b_3 + \ldots\ldots)^2.$$

3. $(a_1b_1 + a_2b_2 + a_3b_3 + \ldots\ldots)\left(\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \ldots\ldots\right)$
$$\nless (a_1 + a_2 + a_3 + \ldots\ldots)^2.$$

4. $a^6 + b^6 \nless a^5b + ab^5$.

5. $\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right)\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right) \nless 9$.

6. $(a+b+c)(a^2+b^2+c^2) \nless 9abc$.

7. $a^2cd + b^2ad + c^2ab + d^2bc \nless 4abcd$.

8. $(bc+ca+ab)^2 \nless 3abc(a+b+c)$.

9. $a^4 + b^4 + c^4 \nless abc(a+b+c)$.

10. $a^5 + b^5 + c^5 + d^5 \nless abcd(a+b+c+d)$.

11. $\frac{a-x}{a+x} < \frac{a^2-x^2}{a^2+x^2}$, if $x < a$.

12. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \nless \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} + \frac{1}{\sqrt{ab}}$.

13. Shew that, if $x_1^2 + x_2^2 + \ldots\ldots + x_n^2 = a$,

then
$$na > (x_1 + x_2 + \ldots\ldots + x_n)^2 > a.$$

14. Prove that, if $x_1, x_2, x_3, \ldots\ldots, x_n$ be each greater than a, and be such that $(x_1-a)(x_2-a)\ldots\ldots(x_n-a)=b^n$, the least value of $x_1x_2x_3\ldots x_n$ will be $(a+b)^n$, a and b being positive.

15. Shew that $$\frac{(a+b)xy}{ay+bx} \ngtr \frac{ax+by}{a+b}.$$

16. $$\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b} \nless \frac{9}{a+b+c}.$$

17. $$\frac{3}{b+c+d}+\frac{3}{c+d+a}+\frac{3}{d+a+b}+\frac{3}{a+b+c} \nless \frac{16}{a+b+c+d}.$$

18. Shew that if $a>b>c$; then
$$\left(\frac{a+c}{a-c}\right)^a < \left(\frac{b+c}{b-c}\right)^b.$$

19. If $x^2=y^2+z^2$, then will $x^n \gtrless y^n+z^n$ according as $n \gtrless 2$.

20. Shew that $(abcd)^{\frac{1}{p+q+r+s}}$ lies between the greatest and least of $a^{\frac{1}{p}}, b^{\frac{1}{q}}, c^{\frac{1}{r}}, d^{\frac{1}{s}}$.

21. Shew that $1+x+x^2+\ldots\ldots+x^{2n} \nless (2n+1)x^n$.

22. If n be a positive integer, and $a>1$; then
$$n\frac{a^{2n+1}+1}{a^{2n}-1} > \frac{a}{a-1}.$$

23. $(m+1)(m+2)(m+3)\ldots\ldots(m+2n-1)<(m+n)^{2n-1}$.

24. $abc \nless (b+c-a)(c+a-b)(a+b-c)$.

25. $abcd \nless (b+c+d-2a)(c+d+a-2b)(d+a+b-2c)$
$(a+b+c-2d)$.

26. $a_1a_2a_3\ldots a_n \nless (n-1)^n(s-a_1)(s-a_2)\ldots(s-a_n)$,
where $(n-1)s=a_1+a_2+\ldots\ldots+a_n$.

27. If a, b, c be unequal positive quantities and such that the sum of any two is greater than the third, then
$$\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c} > \frac{9}{a+b+c}.$$

28. Shew that, unless $a=b=c$,

$$(b-c)^2(b+c-a)+(c-a)^2(c+a-b)+(a-b)^2(a+b-c)>0.$$

29. Shew that, if a, b, c be unequal positive quantities, then

$$a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b)>0.$$

30. Shew that $px^{q-r}+qx^{r-p}+rx^{p-q}>p+q+r$, unless $x=1$, or $p=q=r$.

31. Shew that $\dfrac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}<\sqrt{\dfrac{1}{2n+1}}$.

32. Shew that $\dfrac{3.7.11\ldots(4n-1)}{5.9.13\ldots(4n+1)}<\sqrt{\dfrac{3}{4n+3}}$.

33. Find the greatest value of $x^a y^\beta z^\gamma$, for different values of x, y, and z subject to the condition that $ax^a+by^\beta+cz^\gamma=d$.

34. Prove that, if $n>2$, $(\lfloor n)^2>n^n$.

35. Shew that, if n be positive,

$$(1+x)^n(1+x^n)>2^{n+1}x^n.$$

36. In a geometrical progression of an odd number of terms, the arithmetic mean of the odd terms is greater than the arithmetic mean of the even terms.

37. Prove that, if an arithmetical and a geometrical progression have the same first term, the same last term, and the same number of terms; then the sum of the series in A. P. will be greater than the sum of the series in G. P.

38. Shew that, if P_r denote the arithmetic mean of all those quantities each of which is the geometric mean of r out of n given positive quantities; then P_1, P_2, ..., P_n are in descending order of magnitude.

39. Shew that, if $s=a+b+c+\ldots$,

$$\left(\frac{s-a}{n-1}\right)^a\left(\frac{s-b}{n-1}\right)^b\left(\frac{s-c}{n-1}\right)^c\ldots<\left(\frac{s}{n}\right)^s,$$

n being the number of the unequal positive quantities a, b, c, ...

40. Shew that, if n be any positive integer,

$$2^{n(n+1)}>(n+1)^{n+1}\left(\frac{n}{1}\right)^n\left(\frac{n-1}{2}\right)^{n-1}\ldots\left(\frac{2}{n-1}\right)^2\left(\frac{1}{n}\right)^1.$$

CHAPTER XXVII.

Continued Fractions.

351. Any expression of the form $a \pm \cfrac{b}{c \pm \cfrac{d}{e \pm \text{\&c.}}}$

is called a *continued fraction.*

Continued fractions are generally written for convenience in the form

$$a \pm \frac{b}{c\pm}\;\frac{d}{e\pm}\;\frac{f}{g\pm}\;\ldots\ldots$$

352. The fraction obtained by stopping at any stage is called a *convergent* of the continued fraction. Thus a and $a \pm \frac{b}{c}$, that is $\frac{a}{1}$ and $\frac{ac \pm b}{c}$, are respectively the first and second convergents of the continued fraction $a \pm \frac{b}{c\pm}\;\frac{d}{e\pm}\ldots$

The rth convergent of any continued fraction will be denoted by $\frac{p_r}{q_r}$.

The fractions a, $\frac{b}{c}$, $\frac{d}{e}$, &c. will be called the first, second, third, &c. *elements* of the continued fraction.

353. *In a continued fraction of the form* $a + \frac{b}{c+} \frac{d}{e+} \dots$, *where a, b, c, &c. are all positive, the convergents are alternately less and greater than the fraction itself.*

For the first convergent is too small because the part $\frac{b}{c+} \dots$ is omitted; the second convergent, $a + \frac{b}{c}$, is too great because the denominator is really greater than c; then again, the third is too small, because $c + \frac{d}{e}$ is greater than $c + \frac{d}{e+} \dots$; and so on.

354. In order to find any convergent to a continued fraction, the most natural method is to begin at the *bottom*, as in Arithmetic: thus

$$\frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3} = \frac{a_1}{b_1 + \dfrac{a_2 b_3}{b_2 b_3 + a_3}} = \frac{a_1 b_2 b_3 + a_1 a_3}{b_1 b_2 b_3 + b_1 a_3 + a_2 b_3}.$$

If only *one* convergent has to be found, this method answers the purpose; but there would be a great waste of labour in so finding a succession of convergents, for in finding any one convergent no use could be made of the previous results: the successive convergents to a continued fraction are, however, connected by a simple law which we proceed to prove.

355. *To prove the law of formation of the successive convergents to the continued fraction*

$$a + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \dots\dots$$

The first three convergents will be found to be

$$\frac{a}{1}, \; \frac{ab_1 + a_1}{b_1} \text{ and } \frac{ab_1 b_2 + aa_2 + a_1 b_2}{b_1 b_2 + a_2}.$$

Now the third convergent can be written in the form

$$\frac{(ab_1+a_1)\,b_2+(a)\,a_2}{b_1\,.\,b_2+1\,.\,a_2},$$

from which it appears that its numerator is the *sum of the numerators of the two preceding convergents multiplied respectively by the denominator and numerator of the last element which is taken into account;* and a similar law holds for the denominator.

We will now shew by induction that all the convergents after the second are formed according to the above law *provided there is no cancelling at any stage.*

For, assume that the law holds up to the nth convergent, for which the last element is a_{n-1}/b_{n-1}, and let p_r/q_r denote the rth convergent; then by supposition

$$p_n=b_{n-1}p_{n-1}+a_{n-1}p_{n-2} \text{ and } q_n=b_{n-1}q_{n-1}+a_{n-1}q_{n-2}\ldots\text{(i)}.$$

Then the $(n+1)$th convergent will be obtained by changing $\dfrac{a_{n-1}}{b_{n-1}}$ into $\dfrac{a_{n-1}}{b_{n-1}+}\dfrac{a_n}{b_n}$, that is into $\dfrac{a_{n-1}b_n}{b_{n-1}b_n+a_n}$. Hence in (i) we must put $a_{n-1}b_n$ for a_{n-1} and $b_{n-1}b_n+a_n$ for b_{n-1}; we then have

$$\begin{aligned} p_{n+1} &= (b_{n-1}b_n+a_n)\,p_{n-1}+a_{n-1}b_n\,p_{n-2} \\ &= b_n\,(b_{n-1}p_{n-1}+a_{n-1}p_{n-2})+a_n p_{n-1} \\ &= b_n\,p_n+a_n\,p_{n-1} \qquad \text{[from i.].} \end{aligned}$$

Similarly $\qquad q_{n+1}=b_nq_n+a_nq_{n-1}.$

Thus the law will hold good for the $(n+1)$th convergent if it holds good for the nth convergent. But we know that the law holds good for the third convergent; it must therefore hold good for all subsequent ones.

COR. I. In the fraction $a_1+\dfrac{1}{a_2+}\dfrac{1}{a_3+}\cdots$,

$$p_n=a_n\,p_{n-1}+p_{n-2} \text{ and } q_n=a_nq_{n-1}+q_{n-2}.$$

COR. II. In the fraction $\dfrac{a_1}{b_1 -}\,\dfrac{a_2}{b_2 -}\,\dfrac{a_3}{b_3 -}\,\dots$,

$$p_n = b_n p_{n-1} - a_n p_{n-2} \text{ and } q_n = b_n q_{n-1} - a_n q_{n-2}.$$

Ex. By means of the law connecting successive convergents to a continued fraction, find the fifth convergent of each of the following fractions:

(i) $1 + \dfrac{1}{1+}\,\dfrac{1}{2+}\,\dfrac{1}{3+}\,\dfrac{1}{4}$. (ii) $\dfrac{1}{4+}\,\dfrac{1}{1+}\,\dfrac{1}{4+}\,\dfrac{1}{1+}\,\dfrac{1}{4+}\,\dots$

(iii) $\dfrac{1}{2+}\,\dfrac{2}{3+}\,\dfrac{3}{4+}\,\dfrac{4}{5+}\,\dfrac{5}{6}$. (iv) $3 + \dfrac{2}{5+}\,\dfrac{2}{5+}\,\dfrac{2}{5+}\,\dfrac{2}{5+}\,\dots$

(v) $\dfrac{1}{1+}\,\dfrac{2}{1+}\,\dfrac{3}{1+}\,\dfrac{4}{1+}\,\dfrac{5}{1}$. (vi) $\dfrac{4}{4+}\,\dfrac{3}{3+}\,\dfrac{2}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{2+}\,\dots$

(vii) $\dfrac{2}{3-}\,\dfrac{2}{3-}\,\dfrac{2}{3-}\,\dfrac{2}{3-}\,\dfrac{2}{3-}\,\dots$ (viii) $\dfrac{1}{1-}\,\dfrac{1}{4-}\,\dfrac{1}{1-}\,\dfrac{1}{4-}\,\dfrac{1}{1-}\,\dots$

Ans. $\frac{73}{43}$, $\frac{29}{140}$, $\frac{597}{1522}$, $\frac{2627}{779}$, $\frac{28}{48}$, $\frac{120}{144}$, $\frac{62}{63}$, $\frac{5}{3}$.

356. The convergents to continued fractions of the form $a + \dfrac{1}{b+}\,\dfrac{1}{c+}\,\dfrac{1}{d+}\,\dots$, where $a, b, c, d, \dots$ are all positive integers, have certain properties on account of which such fractions have special utility: these properties we proceed to consider. We first however shew that any rational fraction can be reduced to a continued fraction of this type with a finite number of elements.

For let $\dfrac{m}{n}$ be the given fraction; then, if m be greater than n, divide m by n and let a be the quotient and p the remainder, so that $\dfrac{m}{n} = a + \dfrac{p}{n}$. Now divide n by p and let b be the quotient and q the remainder; then $\dfrac{p}{n} = \dfrac{1}{\dfrac{n}{p}} = \dfrac{1}{b + \dfrac{q}{p}}$. Now divide p by q and let c be the quotient and r the remainder; then $\dfrac{q}{p} = \dfrac{1}{\dfrac{p}{q}} = \dfrac{1}{c + \dfrac{r}{q}}$. By

proceeding in this way, we find $\frac{m}{n}$ in the required form,

namely $\frac{m}{n} = a + \frac{p}{n} = a + \cfrac{1}{b + \frac{q}{p}} = a + \frac{1}{b+}\,\frac{1}{c+} \dots$.

Since the numbers p, q, r, ... become necessarily smaller at every stage, it is obvious that one of them will sooner or later become unity, unless there is an exact division at some earlier stage, so that the process must terminate after a finite number of divisions.

It should be noticed that the process above described is exactly the same as that for finding the G.C.M. of m and n, the numbers a, b, c, ... being the successive quotients. On this account the numbers a, b, c &c. in the continued fraction $a + \frac{1}{b+}\,\frac{1}{c+} \dots$ are often called the first, second, third, &c. *partial quotients.*

It is easy to see that the continued fractions, found as above, for $\frac{m}{n}$ and $\frac{mk}{nk}$, where k is any integer, will be the *same.*

Ex. Convert $\frac{491}{1224}$ and $3 \cdot 14159$ into continued fractions, and find in each case the fourth convergent. *Ans.* $\frac{71}{177}$, $\frac{355}{113}$.

357. **Properties of Convergents.** Let the continued fraction be $a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+} \dots$, and let $\frac{p_n}{q_n}$ denote the nth convergent.

I. From Art. 355 we have

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} - \frac{p_{n-1}}{q_{n-1}} = \frac{p_{n-2} q_{n-1} - p_{n-1} q_{n-2}}{q_n q_{n-1}};$$

$$\therefore \quad p_n q_{n-1} - p_{n-1} q_n = -(p_{n-1} q_{n-2} - p_{n-2} q_{n-1}).$$

So also in succession

$$p_{n-1}q_{n-2}-p_{n-2}q_{n-1}=-(p_{n-2}q_{n-3}-p_{n-3}q_{n-2}),$$

$$\dots\dots\dots\dots\dots\dots = \dots\dots\dots\dots\dots\dots$$

$$p_3q_2-p_2q_3=-(p_2q_1-p_1q_2).$$

But $\quad p_2q_1-p_1q_2=(a_1a_2+1)-a_1a_2=1.$

Hence $p_nq_{n-1}-p_{n-1}q_n=(-1)^n$ ....(i).

Hence also $\quad \dfrac{p_n}{q_n}-\dfrac{p_{n-1}}{q_{n-1}}=\dfrac{(-1)^n}{q_nq_{n-1}}$ ....(ii).

COR. In the continued fraction $\dfrac{1}{a_1+}\dfrac{1}{a_2}+\dots$, which is less than unity, we have

$$p_nq_{n-1}-p_{n-1}q_n=(-1)^{n-1} \text{ and } \frac{p_n}{q_n}-\frac{p_{n-1}}{q_{n-1}}=\frac{(-1)^{n-1}}{q_nq_{n-1}}.$$

II. Every common measure of p_n and q_n must also be a measure of $p_nq_{n-1}-p_{n-1}q_n$, that is, from I., a measure of ± 1. Hence p_n and q_n can have no common measure.

Thus *all convergents are in their lowest terms.*

III. If $F\equiv a_1+\dfrac{1}{a_2+}\dfrac{1}{a_3+}\dots\dfrac{1}{+a_n+}\dots$; then F will be obtained from the nth convergent by putting $\dfrac{1}{a_n+}\dfrac{1}{a_{n+1}+}\dots$ in the place of $\dfrac{1}{a_n}$.

Hence $\quad F=\dfrac{\left(a_n+\dfrac{1}{a_{n+1}+}\dots\right)p_{n-1}+p_{n-2}}{\left(a_n+\dfrac{1}{a_{n+1}+}\dots\right)q_{n+1}+q_{n-2}}=\dfrac{p_n+\lambda p_{n-1}}{q_n+\lambda q_{n-1}},$

where λ is written instead of $\dfrac{1}{a_{n+1}+}\dots$, so that λ is some *positive quantity less than unity.*

Hence $F - \dfrac{p_n}{q_n} = \dfrac{p_n + \lambda p_{n-1}}{q_n + \lambda q_{n-1}} - \dfrac{p_n}{q_n} = \dfrac{\lambda(p_{n-1}q_n - p_n q_{n-1})}{q_n(q_n + \lambda q_{n-1})}$

$$= \frac{(-1)^{n-1}\lambda}{q_n(q_n + \lambda q_{n-1})}.$$

Also $F - \dfrac{p_{n-1}}{q_{n-1}} = \dfrac{p_n + \lambda p_{n-1}}{q_n + \lambda q_{n-1}} - \dfrac{p_{n-1}}{q_{n-1}} = \dfrac{(p_n q_{n-1} - p_{n-1} q_n)}{q_{n-1}(q_n + \lambda q_{n-1})}$

$$= \frac{(-1)^n}{q_{n-1}(q_n + \lambda q_{n-1})}.$$

Now λ is less than 1, and q_n is greater than q_{n-1}; hence $F \sim \dfrac{p_n}{q_n}$ is less than $F \sim \dfrac{p_{n-1}}{q_{n-1}}$.

Thus *any convergent is nearer to the continued fraction than the immediately preceding convergent, and therefore nearer than any preceding convergent.*

IV. If any fraction, $\dfrac{x}{y}$ suppose, be nearer to a continued fraction than the nth convergent, then $\dfrac{x}{y}$ must from III. be also nearer than the $(n-1)$th convergent; and, as the continued fraction itself lies between the nth and the $(n-1)$th convergents [Art. 353], it follows that $\dfrac{x}{y}$ must also lie between these convergents.

Hence $\qquad \dfrac{p_{n-1}}{q_{n-1}} \sim \dfrac{x}{y}$ must be $< \dfrac{p_{n-1}}{q_{n-1}} \sim \dfrac{p_n}{q_n}$;

i.e. $\qquad \dfrac{p_{n-1}y \sim q_{n-1}x}{q_{n-1}y} < \dfrac{1}{q_n q_{n-1}}$;

$$\therefore\ y \text{ must be } > q_n(p_{n-1}y \sim q_{n-1}x).$$

Hence, as all the quantities are integral, y must be greater than q_n.

Thus *every fraction which is nearer to a continued fraction than any particular convergent must have a greater denominator than that convergent.*

V. We have seen in III. that

$$F \sim \frac{p_{n-1}}{q_{n-1}} = \frac{1}{q_{n-1}(q_n + \lambda q_{n-1})},$$

where λ is a positive quantity less than unity.

Hence $$F \sim \frac{p_{n-1}}{q_{n-1}} > \frac{1}{q_{n-1}(q_n + q_{n-1})};$$

also $$F \sim \frac{p_{n-1}}{q_{n-1}} < \frac{1}{q_{n-1}q_n}.$$

Thus *any convergent to a continued fraction differs from the fraction itself by a quantity which lies between* $\frac{1}{d_1 d_2}$ *and* $\frac{1}{d_1(d_1 + d_2)}$, *where* d_1 *and* d_2 *are respectively the denominators of the convergent in question and the next succeeding convergent.*

Ex. 1. Shew that, if p_r/q_r be the rth convergent to the continued fraction $a_1 + \frac{1}{a_2 +}\frac{1}{a_3 +} \cdots\cdots \frac{1}{+ a_n}$, then will

$$\frac{p_n}{p_{n-1}} = a_n + \frac{1}{a_{n-1} +} \cdots\cdots \frac{1}{+ a_2 +} \frac{1}{a_1}.$$

For we have $p_n = a_n p_{n-1} + p_{n-2}$,

$$p_{n-1} = a_{n-1} p_{n-2} + p_{n-3},$$

$$\cdots\cdots\cdots = \cdots\cdots\cdots$$

$$p_2 = a_2 p_1 + 1, \text{ and } p_1 = a_1.$$

Hence $$\frac{p_n}{p_{n-1}} = a_n + \frac{p_{n-2}}{p_{n-1}} = a_n + \frac{1}{\frac{p_{n-1}}{p_{n-2}}} = a_n + \frac{1}{a_{n-1} + \frac{p_{n-3}}{p_{n-2}}}$$

$$= a_n + \frac{1}{a_{n-1} +} \frac{1}{a_{n-2} +} \cdots\cdots \frac{1}{+ a_3 +} \frac{p_1}{p_2}$$

$$= a_n + \frac{1}{a_{n-1} +} \cdots\cdots \frac{1}{+ a_3 +} \frac{1}{a_2 +} \frac{1}{a_1}.$$

It can be proved in a similar manner that

$$\frac{q_n}{q_{n-1}} = a_n + \frac{1}{a_{n-1}+} \cdots \frac{1}{+a_3+} \frac{1}{a_2}.$$

Ex. 2. To shew that $\frac{n}{n+1} = \frac{1}{2-}\frac{1}{2-}\frac{1}{2-}$ ... to n quotients, where n is a positive integer.

We have

$$\frac{n}{n+1} = \frac{1}{2-\frac{n-1}{n}}, \quad \frac{n-1}{n} = \frac{1}{2-\frac{n-2}{n-1}}, \quad \ldots\ldots, \quad \text{and } \frac{2}{3} = \frac{1}{2-\frac{1}{2}}.$$

Hence $\frac{n}{n+1} = \frac{1}{2-}\frac{1}{2-}\frac{1}{2-}$ ...... to n quotients.

Ex. 3. Shew that, if p_r/q_r be the rth convergent of $\frac{a}{b+}\frac{a}{b+}\frac{a}{b+}$ ......; then will $p_{n+1} = aq_n$.

$$\frac{p_{n+1}}{q_{n+1}} = \frac{a}{b+}\frac{p_n}{q_n} = \frac{aq_n}{bq_n + p_n}. \quad \text{Hence } p_{n+1} = aq_n.$$

EXAMPLES XXXVI.

1. Shew that, if $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$ be three successive convergents to any continued fraction with unit numerators, then will $p_3 - p_1 : q_3 - q_1 = p_2 : q_2$.

2. Shew that, if

$\frac{p_n}{q_n}$ be the nth convergent of $\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\ldots$;

then will $\quad p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} a_1 a_2 \ldots\ldots a_n$.

3. Two graduated rulers have their zero points coincident, and the 100th graduation of one coincides exactly with the 63rd of the other: shew that the 27th and the 17th more nearly coincide than any other two graduations.

4. Shew that, if $a_1, a_2, \ldots\ldots, a_n$ be in harmonical progression; then will $\dfrac{a_n}{a_{n-1}} = \dfrac{1}{2-}\,\dfrac{1}{2-}\cdots\dfrac{1}{-2-}\,\dfrac{a_2}{a_1}$.

5. Shew that

$$na_1 + \frac{1}{na_2+}\,\frac{1}{na_3+}\,\frac{1}{na_4+}\cdots \equiv n\left\{a_1 + \frac{1}{n^2a_2+}\,\frac{1}{a_3+}\,\frac{1}{n^2a_4+}\cdots\right\}.$$

6. Shew that, if $P \equiv \dfrac{a}{a+}\,\dfrac{b}{b+}\,\dfrac{c}{c+}\cdots\dfrac{k}{+k+1}$,

and $$Q \equiv \frac{a}{b+}\,\frac{b}{c+}\,\frac{c}{d+}\cdots\frac{h}{+k};$$

then will $$P(a+Q+1) = a+Q.$$

7. Find the value of

$$\frac{n}{n+}\,\frac{n-1}{n-1+}\,\frac{n-2}{n-2+}\cdots\frac{2}{+2+}\,\frac{1}{1+}\,\frac{1}{2}.$$

8. Shew that, whether n be even or odd, $\dfrac{1}{1-}\,\dfrac{1}{4-}\,\dfrac{1}{1-}\,\dfrac{1}{4-}\cdots$ to n quotients $= \dfrac{2n}{n+1}$.

9. Prove that the ascending continued fraction

$$\frac{b_1 + \dfrac{b_2 + \dfrac{b_3 + \cdots}{a_3}}{a_2}}{a_1} \text{ is equal to } \frac{b_1}{a_1} + \frac{b_2}{a_1a_2} + \frac{b_3}{a_1a_2a_3} + \ldots$$

10. If p_n be the numerator of the nth convergent to the fraction $\dfrac{a_1}{b_1+}\,\dfrac{a_2}{b_2+}\,\dfrac{a_3}{b_3+}\ldots$, shew that a linear relation connects every successive four of the series $p_1^2, p_2^2, p_3^2, \ldots$; and find what the relation is.

11. If p_r/q_r be the rth convergent of $\dfrac{1}{a+}\,\dfrac{1}{b+}\,\dfrac{1}{a+}\,\dfrac{1}{b+}\cdots$; shew that $p_{2n+2} = p_{2n} + bq_{2n}$, and that $q_{2n+2} = ap_{2n} + (ab+1)q_{2n}$.

12. If p_r/q_r be the rth convergent of the continued fraction $\frac{1}{a}\,\frac{1}{+b}\,\frac{1}{+c}\,\frac{1}{+a}\,\frac{1}{+b}\,\frac{1}{+c}+\ldots$, shew that $p_{3n+3}=bp_{3n}+(bc+1)q_{3n}$.

13. If p_r/q_r be the rth convergent of $\frac{a}{1}\,\frac{b}{+1}\,\frac{a}{+1}\,\frac{b}{+1}+\ldots$, shew that $p_{2n}\,q_{2n-1}-q_{2n}\,p_{2n-1}=-a^nb^n$.

14. Shew that, if $\frac{p_n}{q_n}$ be the nth convergent to the continued fraction

$\frac{1}{a}\,\frac{1}{+b}\,\frac{1}{+a}\,\frac{1}{+b}+\ldots$, then $q_{2n}=p_{2n+1}$ and $bq_{2n+1}=ap_{2n}+abp_{2n+1}$.

15. Shew that, if $\frac{p}{q}=a+\frac{1}{b}\,\frac{1}{+c}+\ldots\frac{1}{k}\,\frac{1}{+l}$; then will

$$\frac{1}{l}\,\frac{1}{+k}+\ldots\frac{1}{+c}\,\frac{1}{+b}\,\frac{1}{+a}\sim\frac{1}{l}\,\frac{1}{+k}+\ldots\frac{1}{+c}\,\frac{1}{+b}=\frac{1}{pq}.$$

16. Shew that, if $\frac{P}{Q}$ be converted into a continued fraction, the first quotient being a, and the convergent preceding $\frac{P}{Q}$ being $\frac{p}{q}$; then, if $\frac{Q}{q}$ be converted into a continued fraction, the last convergent will be $(P-aQ)/(p-aq)$.

17. Shew that, if $\frac{p}{q}$ and $\frac{p'}{q'}$ be any two consecutive convergents of a continued fraction x, then will $\frac{pp'}{qq'}\gtrless x^2$ according as $\frac{p}{q}\gtrless\frac{p'}{q'}$.

358. *To find the nth convergent of a continued fraction.*

We have in Art. 355 found a law connecting three successive convergents to a continued fraction, so that the

convergents can always be determined in succession. In some cases an expression can be found for any convergent which does not involve the preceding convergents: the method of procedure will be seen from the following examples.

Ex. 1. To find the nth convergent of the continued fraction

$$\frac{1}{3+}\,\frac{1.3}{4+}\,\frac{3.5}{4+}\,\frac{5.7}{4+}\cdots\cdot$$

Here the nth element is $\dfrac{(2n-3)(2n-1)}{4}$, and therefore

$$p_n = 4p_{n-1} + (2n-3)(2n-1)\,p_{n-2}.$$

The above relation may be written

$$p_n - (2n+1)\,p_{n-1} = -(2n-3)\,\{p_{n-1} - (2n-1)\,p_{n-2}\}.$$

Changing n into $n-1$ we have in succession

$$p_{n-1} - (2n-1)\,p_{n-2} = -(2n-5)\,\{p_{n-2} - (2n-3)\,p_{n-3}\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$p_3 - 7p_2 = -3\,\{p_2 - 5p_1\}.$$

But, by inspection, $p=1$, $p_2=4$; $\therefore\ p_2 - 5p_1 = -1$.

Hence $p_n - (2n+1)\,p_{n-1} = (-1)^{n-1}\,(2n-3)(2n-5)\ldots 3.1.$

Then again

$$\frac{p_n}{1.3\ldots(2n+1)} - \frac{p_{n-1}}{1.3\ldots(2n-1)} = \frac{(-1)^{n-1}}{(2n+1)(2n-1)},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$$

$$\frac{p_2}{1.3.5} - \frac{p_1}{1.3} = \frac{(-1)^1}{3.5},$$

and
$$\frac{p_1}{1.3} = \frac{1}{1.3}.$$

Hence $$\frac{p_n}{1.3.5\ldots(2n+1)} = \frac{1}{1.3} - \frac{1}{3.5} + \ldots\ldots + \frac{(-1)^{n-1}}{(2n+1)(2n-1)}.$$

Since the denominators of convergents are formed according to the same law as the numerators, we have from the above

$$q_n - (2n+1)\,q_{n-1} = (-1)^{n-2}\,3.5\ldots(2n-3)\,\{q_2 - 5q_1\} = 0,$$

since $q_1 = 3$ and $q_2 = 15$.

Hence

$$\frac{q_n}{(2n+1)(2n-1)\dots 3.1} = \frac{q_{n-1}}{(2n-1)\dots 3.1} = \dots\dots = \frac{q_2}{5.3.1} = \frac{q_1}{3.1} = 1;$$

$$\therefore\ q_n = 1.3\dots(2n-1)(2n+1).$$

Hence p_n/q_n, the nth convergent required, is

$$\frac{1}{1.3} - \frac{1}{3.5} + \dots\dots + \frac{(-1)^{n-1}}{(2n-1)(2n+1)}.$$

Ex. 2. To find the nth convergent of the continued fraction

$$\frac{1}{1+}\,\frac{2}{2+}\,\frac{3}{3+}\,\frac{4}{4+}\dots.$$

The necessary transformations are given in Ex. 5, Art. 251.

It will be found that $p_n = \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \dots\dots + \frac{(-1)^{n-1}}{\lfloor n+1}$,

and that $$q_n = 1 - \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} - \dots\dots + \frac{(-1)^n}{\lfloor n+1}.$$

If n be supposed infinitely great

$$\frac{p_n}{q_n} = \frac{e^{-1}}{1-e^{-1}} = \frac{1}{e-1}.$$

359. Periodic continued fractions. When the elements of a continued fraction continually recur in the same order, the fraction is said to be a *periodic continued fraction*; and a periodic continued fraction is said to be *simple* or *mixed* according as the recurrence begins at the beginning or not.

Thus $$a + \frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{a+}\dots$$

is a simple, and $$\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{b+}\,\frac{1}{b+}\dots$$

is a mixed periodic continued fraction.

360. *To find the nth convergent of a periodic continued fraction with one recurring element.*

Let the fraction be $a+\frac{b}{c+}\frac{b}{c+}\frac{b}{c+}\dots$ Then, for all convergents after the second, we have $p_n = cp_{n-1} + bp_{n-2}$ where b and c are constants, that is, are the same for all values of n.

Now, if $u_1 + u_2x + u_3x^2 + \dots + u_nx^{n-1} + \dots$ be the recurring series formed by the expansion of $\frac{A+Bx}{1-cx-bx^2}$, the successive coefficients after the second are connected by the law $u_n = cu_{n-1} + bu_{n-2}$. Hence, if A and B are so chosen that $u_1 = p_1$ and $u_2 = p_2$, then will $u_n = p_n$ for all values of n. The necessary values of A and B are respectively p_1 and $p_2 - cp_1$, that is a and b.

Hence the numerator of the nth convergent to the continued fraction $a+\frac{b}{c+}\frac{b}{c+}\dots$ is the coefficient of x^{n-1} in the expansion of $\frac{a+bx}{1-cx-bx^2}$.

Similarly the denominator of the nth convergent is the coefficient of x^{n-1} in the expansion of $\frac{q_1+(q_2-cq_1)x}{1-cx-bx^2}$, that is of $\frac{1}{1-cx-bx^2}$.

Ex. 1. Find the nth convergent of the continued fraction

$$1+\frac{3}{2+}\frac{3}{2+}\frac{3}{2+}\dots.$$

The numerator of the nth convergent is the coefficient of x^{n-1} in the expansion of $\frac{1+3x}{1-2x-3x^2}$, that is of $\frac{3}{2(1-3x)}-\frac{1}{2+2x}$.

Hence $$p_n = \frac{1}{2}\{3^n + (-1)^n\}.$$

Also q_n = coefficient of x^{n-1} in the expansion of

$$\frac{1}{1-2x-3x^2} = \frac{3}{4(1-3x)} + \frac{1}{4(1+x)}.$$

Hence $$q_n = \frac{1}{4}\{3^n - (-1)^n\}.$$

Thus the nth convergent is $2\dfrac{3^n + (-1)^n}{3^n - (-1)^n}$.

Ex. 2. Find the nth convergent of the continued fraction

$$\frac{a}{b+}\,\frac{c}{d+}\,\frac{a}{b+}\,\frac{c}{d+}\cdots.$$

We have
$$p_{2n} = dp_{2n-1} + cp_{2n-2},$$
$$p_{2n-1} = bp_{2n-2} + ap_{2n-3},$$
$$p_{2n-2} = dp_{2n-3} + cp_{2n-4}.$$

Hence, eliminating p_{2n-1} and p_{2n-3}, we have

$$p_{2n} - (a+c+bd)\,p_{2n-2} + acp_{2n-4} = 0.$$

Since the last result is symmetrical in a and c, and also in b and d, it follows that

$$p_{2n-1} - (a+c+bd)\,p_{2n-3} + acp_{2n-5} = 0.$$

Hence the relation

$$p_n - (a+c+bd)\,p_{n-2} + acp_{n-4} = 0$$

holds good for all values of n.

Hence p_n will be the coefficient of x^{n-1} in the expansion of

$$\frac{A + Bx + Cx^2 + Dx^3}{1 - (a+c+bd)\,x^2 + acx^4},$$

provided the values of A, B, C, D are so chosen that the result holds good for the first four convergents. It will thus be found that p_n is the coefficient of x^{n-1} in the expansion of

$$\frac{a + adx - acx^2}{1 - (a+c+bd)\,x^2 + acx^4}.$$

It will similarly be found that q_n is the coefficient of x^{n-1} in the expansion of

$$\frac{b + (bd+c)\,x - acx^3}{1 - (a+c+bd)\,x^2 + acx^4}.$$

361. **Convergency of continued fractions.** When a continued fraction has an infinite number of elements it is of importance to determine whether it is convergent or not. When an expression can be found for the nth con-

vergent, the rules already investigated can be employed; the nth convergent cannot however be often found.

In the continued fraction $\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\dots$ it is easy to shew, as in Art. 357, that

$$\frac{p_n}{q_n}-\frac{p_{n-1}}{q_{n-1}}=(-1)^{n-1}\frac{a_1a_2\dots a_n}{q_{n-1}q_n},$$

and hence that

$$\frac{p_n}{q_n}=\frac{a_1}{q_1}-\frac{a_1a_2}{q_1q_2}+\dots+(-1)^n\frac{a_1a_2\dots a_n}{q_{n-1}q_n}.$$

Now, if all the letters are supposed to denote positive quantities, the terms of the series on the right are alternately positive and negative; also each term is less than the preceding, for the ratio of the rth term to the preceding term is $\frac{a_r q_{r-2}}{q_r}$, which is less than unity since $q_r=b_rq_{r-1}+a_rq_{r-2}$. Hence the series, and therefore the continued fraction, is convergent provided the nth term diminishes indefinitely when n is indefinitely increased.

It can be shewn that the condition of convergency is satisfied whenever the ratio $b_n.b_{n-1} : a_n$ is finite for all values of n*.

For let b_nb_{n-1} be always greater than $k.a_n$, where k is some finite quantity.

$$\text{Then } u_n=\frac{a_1a_2\dots a_{n-1}a_n}{q_{n-1}q_n}=\frac{a_1a_2\dots a_{n-1}}{q_{n-1}\left(q_{n-2}+\frac{b_n}{a_n}q_{n-1}\right)}.$$

$$\text{But } \frac{b_n}{a_n}q_{n-1}>\frac{k}{b_{n-1}}q_{n-1}>\frac{k}{b_{n-1}}(b_{n-1}q_{n-2}+a_{n-1}q_{n-3})>kq_{n-2}.$$

$$\text{Hence } u_n<\frac{a_1a_2\dots a_{n-1}}{q_{n-1}q_{n-2}(1+k)}.$$

$$\text{Whence } u_n<\frac{a_1a_2}{q_1q_2(1+k)^{n-2}}.$$

* Todhunter's *Algebra*, Art. 783.

But $(1+k)^{n-2}$ increases indefinitely with n, since k is finite; hence u_n decreases without limit as n is increased without limit.

We have therefore the following **Theorem.** *The infinite continued fraction* $\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\ldots$, *in which all the letters represent positive quantities, is convergent if the ratio* $b_n b_{n-1} : a_n$ *is always greater than some fixed finite quantity.*

It should be remarked that any infinite continued fraction of the form $a+\frac{1}{b+}\frac{1}{c+}\ldots$, in which $a, b, c,\ldots$ are positive integers, is convergent.

362. In the following five Articles the continued fractions will all be supposed to be of the form $a+\frac{1}{b+}\frac{1}{c+}\ldots$, where $a, b, c,\ldots$ are *positive integers.*

This form of continued fraction possesses two great advantages, for we know that every convergent is in its lowest terms, and we can also see by inspection, within narrow limits, the difference between any convergent and the true value.

*363. **Theorem.** *Every simple periodic continued fraction is a root of a quadratic equation with rational coefficients whose roots are of contrary signs, one root being greater and the other less than unity. Also the reciprocal of the negative root is equal in magnitude to the continued fraction which has the same quotients in inverse order.*

Let the fraction be

$$x=a+\frac{1}{b+}\frac{1}{c+}\cdots\frac{1}{+k+}\frac{1}{l+}\frac{1}{a+}\frac{1}{b+}\cdots.$$

Let $\frac{P'}{Q'}$ and $\frac{P}{Q}$ be the last two convergents of the first

* Articles 363, 364, and 368 are taken from a paper by Gerono, *Nouvelles Annales de Mathematiques*, t. I.

period. Then

$$x = \frac{xP + P'}{xQ + Q'};$$

$$\therefore x^2Q + x(Q' - P) - P' = 0 \text{.............(i).}$$

The roots of (i) are obviously of different signs, and the positive root is the value of the continued fraction.

Now, from Art. 357, Ex. 1,

$$\frac{P}{P'} = l + \frac{1}{k+} \cdots \frac{1}{+b} \frac{1}{+a},$$

and
$$\frac{Q}{Q'} = l + \frac{1}{k+} \cdots \frac{1}{+b}.$$

Hence, if $y = l + \frac{1}{k+} \cdots \frac{1}{+a} \frac{1}{+l} \frac{1}{+k+} \cdots$,

we have
$$y = \frac{Py + Q}{P'y + Q'};$$

$$\therefore y^2P' + y(Q' - P) - Q = 0 \text{.............(ii).}$$

The roots of (ii) are obviously of different signs, and the positive root is the value of the continued fraction

$$l + \frac{1}{k+} \cdots \frac{1}{+a} \frac{1}{+l+} \cdots.$$

From (i) and (ii) we see that the positive root of (ii) is equal in magnitude to the reciprocal of the negative root of (i); and therefore the reciprocal of the negative root of (i) is $-\left(l + \frac{1}{k+} \cdots \frac{1}{+a} \frac{1}{+l+} \cdots\right)$.

The positive roots of (i) and (ii) are both greater than unity, as may be seen by inspection; the negative root of (i) must therefore be less than unity.

The fraction $\frac{1}{a+} \frac{1}{b+} \cdots \frac{1}{+l} \frac{1}{+a+} \cdots$ requires no special

examination, for we have only to change x into $\frac{1}{x}$, and y into $\frac{1}{y}$; thus $\frac{1}{a+}\frac{1}{b+}\cdots\frac{1}{+k+}\frac{1}{l+}\frac{1}{a+}\cdots$ is equal to the positive root of $P'x^2-(Q'-P)x-Q=0$, and the negative root is $-\left\{l+\frac{1}{k+}\cdots\frac{1}{+b+}\frac{1}{a}\right\}$.

Hence, as before, one root of the quadratic equation in x is greater and the other is less than unity.

364. **Theorem.** *Every mixed periodic continued fraction, which has more than one non-periodic element, is a root of a quadratic equation with rational coefficients whose roots are both of the same sign.*

Let the fraction be

$$x=a+\frac{1}{b+}\cdots\frac{1}{+k+}\frac{1}{\alpha+}\frac{1}{\beta+}\cdots\frac{1}{+\mu+}\frac{1}{\nu+}\frac{1}{\alpha+}\frac{1}{\beta+}\cdots,$$

and let

$$y=\alpha+\frac{1}{\beta+}\cdots\frac{1}{+\mu+}\frac{1}{\nu+}\frac{1}{\alpha+}\frac{1}{\beta+}\cdots.$$

Let $\frac{A'}{B'}$ and $\frac{A}{B}$ be the two last convergents of the non-periodic part; then

$$x=\frac{yA+A'}{yB+B'}\quad\text{.......................(i).}$$

Let $\frac{P'}{Q'}$ and $\frac{P}{Q}$ be the last two convergents of the first period of y; then

$$y=\frac{yP+P'}{yQ+Q'}\quad\text{.......................(ii).}$$

The elimination of y from (i) and (ii) will clearly lead to a quadratic equation in x with rational coefficients.

Now, if the positive root of (ii) be substituted in (i) we

shall clearly obtain a positive value of x, and this will be the actual value of the given continued fraction.

Also, from the preceding article, the negative value of $\frac{1}{y}$ is $-\left\{\nu+\frac{1}{\mu+}\cdots\frac{1}{+\alpha}\right\}$; and, if this value be substituted in (i), we have

$$x=a+\frac{1}{b+}\cdots\frac{1}{+k-\nu}\;\frac{1}{-\mu+}\cdots:$$

we have to shew that this is positive. If $k>\nu$ the result is obvious; if $k<\nu$, $\frac{1}{k-\nu}\;\frac{1}{-\mu+}\cdots$ is negative but is less than 1, and therefore x is positive *provided one element at least precedes k*; also k cannot be equal to ν, for in that case the periodic part would really begin with k and not with a. Hence both values of x are positive in all cases.

Reduction of Quadratic Surds to Continued Fractions.

365. It is clear that a quadratic surd cannot be equal to a continued fraction with a finite number of elements; for every such continued fraction can be reduced to an ordinary fraction whose numerator and denominator are commensurable. It will be shewn that a quadratic surd can be reduced to a periodic continued fraction of the form $a+\frac{1}{b+}\frac{1}{c+}\cdots$, where $a, b, c, \ldots$ are positive integers. The process will be seen from the following example.

Ex. To reduce $\sqrt{8}$ to a continued fraction.

The integer next below $\sqrt{8}$ is 2; and we have

$$\sqrt{8}=2+\sqrt{8}-2=2+\frac{(\sqrt{8}-2)(\sqrt{8}+2)}{\sqrt{8}+2}=2+\frac{4}{\sqrt{8}+2}=2+\frac{1}{\frac{\sqrt{8}+2}{4}}$$

The integer next below $\frac{\sqrt{8}+2}{4}$ is 1; and we have

$$\frac{\sqrt{8}+2}{4}=1+\frac{\sqrt{8}-2}{4}=1+\frac{4}{4(\sqrt{8}+2)}=1+\frac{1}{\sqrt{8}+2}.$$

The integer next below $\sqrt{8}+2$ is 4; and we have

$$\sqrt{8}+2=4+\sqrt{8}-2=4+\frac{1}{\frac{\sqrt{8}+2}{4}}.$$

The steps now recur, so that

$$\sqrt{8}=2+\frac{1}{1+}\frac{1}{4+}\frac{1}{1+}\frac{1}{4+}\dots.$$

Thus $\sqrt{8}$ is equal to a periodic continued fraction with one non-periodic element, which is half the last quotient of the recurring portion; and it will be proved later on that this law holds good for every quadratic surd.

366. We now proceed to shew how to convert any quadratic surd into a continued fraction.

Let $\sqrt{N}$ be any quadratic surd, and let a be the integer next below $\sqrt{N}$; then

$$\sqrt{N}=a+\sqrt{N}-a=a+\frac{N-a^2}{\sqrt{N}+a}=a+\frac{1}{\frac{\sqrt{N}+a}{r_1}},$$

where $r_1=N-a^2$.

Since $\sqrt{N}-a$ is positive and less than 1, it follows that $\frac{\sqrt{N}+a}{r_1}$ is greater than 1. Let then b be the integer next below $\frac{\sqrt{N}+a}{r_1}$; then

$$\frac{\sqrt{N}+a}{r_1}=b+\frac{\sqrt{N}-(br_1-a)}{r_1}$$

$$=b+\frac{N-(br_1-a)^2}{r_1\{\sqrt{N}+(br_1-a)\}}=b+\frac{1}{\frac{\sqrt{N}+a_2}{r_2}},$$

where $a_2=br_1-a$ and $r_2=\frac{N-a_2^2}{r_1}$.

Then, as before, $\frac{\sqrt{N}+a_2}{r_2}$ is greater than unity; and if

c be the integer next *below* $\dfrac{\sqrt{N}+a_2}{r_2}$, we have

$$\frac{\sqrt{N}+a_2}{r_2}=c+\frac{\sqrt{N}-(cr_2-a_2)}{r_2}$$

$$=c+\frac{N-(cr_2-a_2)^2}{r_2\{\sqrt{N}+cr_2-a_2\}}=c+\frac{1}{\dfrac{\sqrt{N}+a_3}{r_3}},$$

where $a_3=cr_2-a_2$ and $r_3=\dfrac{N-a_3^2}{r_2}$.

The process can be continued in this way to any extent that may be desired. Thus $\sqrt{N}=a+\dfrac{1}{b+}\,\dfrac{1}{c+}\,\dfrac{1}{d+}\cdots\cdots$

367. *To shew that any quadratic surd is equal to a recurring continued fraction.*

It is first necessary to prove that the quantities which, in the preceding Article, are called $a, a_2, a_3, \ldots, r_1, r_2, r_3, \ldots$ are all positive integers.

It is known that N is a positive integer, and that $a, b, c, d, \ldots$ are all positive integers.

We have the following relations:

$$r_1=N-a^2, \qquad \qquad \ldots\ldots\ldots\ldots\text{(i)};$$

$$a_2=br_1-a, \quad r_1r_2=N-a_2^2 \quad \ldots\ldots\ldots\text{(ii)};$$

$$a_3=cr_2-a_2, \quad r_2r_3=N-a_3^2 \quad \ldots\ldots\ldots\text{(iii)};$$

$$a_4=dr_3-a_3, \quad r_3r_4=N-a_4^2 \quad \ldots\ldots\ldots\text{(iv)};$$

and so on.

Now it is obvious from (i) that r_1 is an integer.

From (ii) we have $r_2=\dfrac{N-(br_1-a)^2}{r_1}=1+2ab-b^2r_1$, since $N-a^2=r_1$.

Thus $a_2=br_1-a$, and $r_2=1+2ab-b^2r_1$; whence it follows that a_2 and r_2 are integers, since r_1 is an integer.

From (iii) we have similarly

$$a_3 = cr_2 - a_2 \text{ and } r_3 = r_1 + 2a_2c - c^2r_2;$$

whence it follows that a_3 and r_3 are integers, since a_2 and r_2 are integers.

Then again, from (iv) we have

$$a_4 = dr_3 - a_3 \text{ and } r_4 = r_2 + 2a_3d - d^2r_3;$$

whence it follows that a_4 and r_4 are integers, since a_3 and r_3 are integers.

And so on; so that a_n and r_n are *integral* for all values of n.

We have now to prove that a_n and r_n are *positive* for all values of n.

We know that a, b, c, &c. are the positive integers *next below* $\sqrt{N}$, $\frac{\sqrt{N}+a}{r_1}$, $\frac{\sqrt{N}+a_2}{r_2}$, &c. Hence $\sqrt{N}-a$, $\sqrt{N}-a_2$, $\sqrt{N}-a_3$, &c., and therefore also $N-a^2$, $N-a_2^2$, $N-a_3^2$, &c., are all positive. That is r_1, r_2, r_3, &c. are all *positive*.

Again, since b is the integer *next below* $\frac{\sqrt{N}+a}{r_1}$, it follows that $\sqrt{N}+a < br_1 + r_1$. Now, a cannot be equal or greater than br_1, for then $\sqrt{N} < r_1$, and therefore $a < r_1$; therefore $a < br_1$, since r_1 is positive and b a positive integer. Hence $a < br_1$, so that a_2 is positive.

Again, since c is the integer next below $\frac{\sqrt{N}+a_2}{r_2}$, it follows that $\sqrt{N}+a_2 < cr_2 + r_2$. And we cannot have $a_2 \overline{\gtrless} cr_2$, for then $\sqrt{N} < r_2$, and therefore $a_2 < r_2 < cr_2$, since r_2 is positive and c a positive integer. Thus $a_2 < cr_2$, so that a_3 is positive.

And so on; so that a_n is *positive* for all values of n.

Having shewn that the quantities r_1, r_2, r_3, &c. and a, a_2, a_3, &c., are all positive integers, it follows from the

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relation $r_n r_{n-1} = N - a_n^2$ that a_n is less than $\sqrt{N}$, so that $a_n \not> a$; hence the only possible values of a_n are 1, 2,..., a.

Then, from the relation $a_n + a_{n+1} = k \cdot r_n$, where k is a positive integer, it follows that r_n cannot be greater than $2a$.

Hence the expression $\dfrac{\sqrt{N}+a_n}{r_n}$ cannot have more than $2a \times a$ different values; and therefore after $2a^2$ quotients, at most, there must be a recurrence.

368. **Theorem.** *Any quadratic surd can be reduced to a periodic continued fraction with one non-recurring element, the last recurring quotient being twice the quotient which does not recur; also the quotients of the recurring period, exclusive of the last, are the same when read backwards or forwards.*

Let $\sqrt{N}$ be the quadratic surd.

Then, from the preceding Article, we know that $\sqrt{N}$ is equal to a *periodic* continued fraction.

We also know that any periodic continued fraction is equal to one of the roots of a quadratic equation with rational coefficients; and the only quadratic equation in x with rational coefficients of which $\sqrt{N}$ is one root is the equation $x^2 - N = 0$.

Now the roots of $x^2 - N = 0$ are *both greater* than unity in absolute magnitude, and the roots are of *different* signs: it therefore follows from Articles 363 and 364 that the continued fraction which is equal to $\sqrt{N}$ must be a *mixed recurring continued fraction with one non-recurring element.*

Hence we have

$$\sqrt{N} = a + \frac{1}{b+}\,\frac{1}{c+}\cdots\frac{1}{+h+}\,\frac{1}{k+}\,\frac{1}{l+}\,\frac{1}{b+}\cdots,$$

or $$\sqrt{N} - a = \frac{1}{b+}\,\frac{1}{c+}\cdots\frac{1}{+h+}\,\frac{1}{k+}\,\frac{1}{l+}\,\frac{1}{b+}\cdots.$$

Now $\frac{1}{b+}\frac{1}{c+}\cdots\frac{1}{+h+}\frac{1}{k+}\frac{1}{l+}\frac{1}{b+}\cdots$ is the positive root of a quadratic equation with rational coefficients; and as this positive root is $\sqrt{N}-a$, the negative root must be $-\sqrt{N}-a$. Hence, Art. 363, we have

$$\frac{1}{\sqrt{N}+a}=\frac{1}{l+}\frac{1}{k+}\frac{1}{h+}\cdots\frac{1}{+c}+\frac{1}{b}+\frac{1}{l+}\cdots,$$

$$\therefore \sqrt{N}+a=l+\frac{1}{k+}\frac{1}{h+}\cdots\frac{1}{+c+}\frac{1}{b+}\frac{1}{l+}\cdots.$$

Hence $$l-a+\frac{1}{k+}\frac{1}{h+}\cdots\frac{1}{+c+}\frac{1}{b+}\frac{1}{l+}\cdots$$

$$=a+\frac{1}{b+}\frac{1}{c+}\cdots\frac{1}{+h+}\frac{1}{k+}\frac{1}{l+}\cdots,$$

whence it is easy to see that $l-a=a$, $k=b$, $h=c$, ...

SERIES EXPRESSED AS CONTINUED FRACTIONS.

369. *To shew that any series can be expressed as a continued fraction.*

Let the series be

$$u_1+u_2+u_3+u_4+\ldots+u_n+\ldots\ldots\ldots\ldots\text{(i)}.$$

Then the sum of n terms of the series (i) is equal to the nth convergent of the continued fraction

$$\frac{u_1}{1}-\frac{u_2}{u_1+u_2-}\frac{u_1u_3}{u_2+u_3-}\frac{u_2u_4}{u_3+u_4-}\cdots\frac{u_{n-2}u_n}{-u_{n-1}+u_n-}\cdots\text{(ii)}.$$

This can be proved by induction, as follows.

Assume that the sum of the first n terms of (i) is equal to the nth convergent of (ii). Another term of the series is taken into account by changing u_n into u_n+u_{n+1}; and, by changing u_n into u_n+u_{n+1}, $\dfrac{u_{n-2}u_n}{u_{n-1}+u_n}$ will become

$\frac{u_{n-2}(u_n+u_{n+1})}{u_{n-1}+u_n+u_{n+1}}$, which is easily seen to be equal to $\frac{u_{n-2}u_n}{u_{n-1}+u_n-}\;\frac{u_{n-1}u_{n+1}}{u_n+u_{n+1}}$. Thus the sum of $n+1$ terms of (i) will be equal to the $(n+1)$th convergent of (ii) provided the sum of n terms of (i) is equal to the nth convergent of (ii). But it is easily seen that the theorem is true when n is 1 or 2 or 3: it is therefore true for all values of n.

Thus $u_1+u_2+u_3+u_4+\ldots$ to n terms

$$=\frac{u_1}{1-}\;\frac{u_2}{u_1+u_2-}\;\frac{u_1u_3}{u_2+u_3-}\;\frac{u_2u_4}{u_3+u_4-}\ldots \text{ to } n \text{ quotients}\ldots[\text{A}].$$

It can be proved in a precisely similar manner that

$$u_1-u_2+u_3-u_4+\ldots\ldots \text{ to } n \text{ terms}$$

$$=\frac{u_1}{1+}\;\frac{u_2}{u_1-u_2+}\;\frac{u_1u_3}{u_2-u_3+}\;\frac{u_2u_4}{u_3-u_4+}\ldots \text{ to } n \text{ quotients}\ldots[\text{B}].$$

The formula [B] can also be deduced from [A] by changing the signs of the alternate terms.

370. The following cases are of special interest:

$$\frac{a_1}{b_1}\pm\frac{a_1a_2}{b_1b_2}+\frac{a_1a_2a_3}{b_1b_2b_3}\pm\ldots \text{ to } n \text{ terms}$$

$$=\frac{a_1}{b_1\mp}\;\frac{b_1a_2}{b_2\pm a_2\mp}\;\frac{b_2a_3}{b_3\pm a_3\mp}\ldots \text{ to } n \text{ quotients}\ldots[\text{C}],$$

all the upper signs, or all the lower signs, being taken.

And $$\frac{1}{a_1}\pm\frac{1}{a_2}+\frac{1}{a_3}\pm\frac{1}{a_4}+\ldots \text{ to } n \text{ terms}$$

$$=\frac{1}{a_1\mp}\;\frac{a_1^2}{a_2\pm a_1\mp}\;\frac{a_2^2}{a_3\pm a_2\mp}\ldots \text{ to } n \text{ quotients}\ldots[\text{D}],$$

all the upper signs, or all the lower signs, being taken.

These can be proved by induction as in the preceding Article.

Thus to prove [C]. It is obvious by inspection that the theorem is true when $n=2$. Assume then that [C] is true for any particular value of n; then, to include another term of the series $\frac{a_n}{b_n}$ must be changed into $\frac{a_n}{b_n} \pm \frac{a_n a_{n+1}}{b_n b_{n+1}}$, and therefore $\frac{b_{n-1} a_n}{b_n \pm a_n}$ will become

$$\frac{b_{n-1}\left(\frac{a_n}{b_n} \pm \frac{a_n a_{n+1}}{b_n b_{n+1}}\right)}{1 \pm \left(\frac{a_n}{b_n} \pm \frac{a_n a_{n+1}}{b_n b_{n+1}}\right)},$$

which can easily be seen to be equal to $\frac{b_{n-1} a_n}{b_n \pm a_n \mp} \frac{b_n a_{n+1}}{b_{n+1} \pm a_{n+1}}$. Thus, if [C] be true for any value of n, it will be true for the next greater value; hence as [C] is true when $n=2$, it is true for all values of n.

The following are particular cases of [C].

$$a_1 \pm a_1 a_2 + a_1 a_2 a_3 \pm a_1 a_2 a_3 a_4 + \ldots\ldots$$

$$= \frac{a_1}{1 \mp} \frac{a_2}{1 \pm a_2 \mp} \frac{a_1 a_3}{1 \pm a_3 \mp} \frac{a_2 a_4}{1 \pm a_4 \mp} \ldots\ldots \text{[E]},$$

and

$$\frac{1}{a_1} \pm \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} \pm \frac{1}{a_1 a_2 a_3 a_4} + \ldots$$

$$= \frac{1}{a_1 \mp} \frac{a_1}{a_2 \pm 1 \mp} \frac{a_2}{a_3 \pm 1 \mp} \frac{a_3}{a_4 \pm 1 \mp} \ldots\ldots \text{[F]}.*$$

Ex. 1. Shew that

$$\frac{1}{1+} \frac{1^2}{2+} \frac{3^2}{2+} \frac{5^2}{2+} \ldots \text{ to infinity.}$$

$$= \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \text{ to infinity.} \quad \text{[Brouncker.]}$$

Put $a_1=1$, $a_2=3$, $a_3=5$, &c. in [D].

Ex. 2. Shew that

$$\frac{1}{1+} \frac{1^2}{1+} \frac{2^2}{1+} \frac{3^2}{1+} \ldots \text{ to infinity} = \log_e 2. \quad \text{[Euler.]}$$

Put $a_1=1$, $a_2=2$, $a_3=3$, &c. in [D].

* The formula [A] is due to Euler; [C] is given by Dr Glaisher in the *Proceedings of the London Mathematical Society*, Vol. v.

Ex. 3. Find the value of $\frac{1}{1+}\frac{1}{1+}\frac{2}{2+}\frac{3}{3+}\frac{4}{4+}\dots$ to infinity.

From [F] we see that

$$\frac{1}{1+}\frac{1}{1+}\frac{2}{2+}\frac{3}{3+}\text{ to infinity}$$

$$=\frac{1}{1}-\frac{1}{1\,.\,2}+\frac{1}{1\,.\,2\,.\,3}-\frac{1}{1\,.\,2\,.\,3\,.\,4}+\dots\text{ to infinity}=1-e^{-1}.$$

Ex. 4. Find the nth convergent of $\frac{1}{3+}\frac{3}{2+}\frac{3}{2+}\dots$.

From [F] we have

$$\frac{1}{3}-\frac{1}{3\,.\,3}+\frac{1}{3\,.\,3\,.\,3}-\dots=\frac{1}{3+}\frac{3}{2+}\frac{3}{2+}\dots.$$

Hence the nth convergent required $=\frac{1}{4}\left\{1-\left(-\frac{1}{3}\right)^n\right\}$.

Ex. 5. Shew that $\quad 1+\frac{r}{1-}\frac{r}{r+2-}\frac{2r}{r+3-}\frac{3r}{r+4-}\dots=e^r.$

$$e^r=1+\frac{r}{1}+\frac{r\,.\,r}{1\,.\,2}+\frac{r\,.\,r\,.\,r}{1\,.\,2\,.\,3}+\frac{r\,.\,r\,.\,r\,.\,r}{1\,.\,2\,.\,3\,.\,4}+\dots$$

$$=1+\frac{r}{1-}\frac{r}{r+2-}\frac{2r}{r+3-}\frac{3r}{r+4-}\dots,\text{ from [C].}$$

EXAMPLES XXXVII.

1. Find the continued fractions equivalent to the following quadratic surds:

(1) $\sqrt{17}$, (2) $\sqrt{140}$, (3) $\sqrt{33}$, (4) $\sqrt{43}$, (5) $\sqrt{(a^2+1)}$, (6) $\sqrt{(a^2+2a)}$.

2. Shew that $\sqrt{N}=a+\frac{b}{2a+}\frac{b}{2a+}\dots$, where a has any value whatever, and $b=N-a^2$.

3. Find the value of

(i) $1+\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\frac{1}{2+}\dots$ to infinity.

(ii) $1+\frac{1}{2+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\dots$ to infinity.

(iii) $\frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\frac{1}{5+}\frac{1}{4+}\frac{1}{5+}\dots$ to infinity.

4. Shew that

$$7+\frac{1}{14+}\frac{1}{14+}\dots \text{ to infinity}=5\left(1+\frac{1}{2+}\frac{1}{2+}\dots\text{to infinity}\right).$$

5. Shew that

$$\left(\frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\frac{1}{d+}\frac{1}{a+}\dots\right)\left(d+\frac{1}{c+}\frac{1}{b+}\frac{1}{a+}\frac{1}{d+}\dots\right)=\frac{b+d+bcd}{a+c+acb}.$$

6. Shew that, if $x=y+\frac{1}{2y+}\frac{1}{2y+}\dots$ to infinity, then $y=x-\frac{1}{2x-}\frac{1}{2x-}\dots$ to infinity.

7. Shew that, if

$$x=\frac{1}{a+}\frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\dots \text{ to infinity},$$

$$y=\frac{1}{2a+}\frac{1}{2b+}\frac{1}{2a+}\frac{1}{2b+}\dots \text{ to infinity},$$

and $$z=\frac{1}{3a+}\frac{1}{3b+}\frac{1}{3a+}\frac{1}{3b+}\dots \text{ to infinity};$$

then will $x(y^2-z^2)+2y(z^2-x^2)+3z(x^2-y^2)=0.$

8. Shew that, if n be any positive integer,

$$n=1+\frac{n^2-1^2}{3+}\frac{n^2-2^2}{5+}\frac{n^2-3^2}{7+}\dots$$

9. Shew that

$$\frac{1+a^2+a^4+\dots+a^{2n}}{a+a^3+a^5+\dots+a^{2n-1}}=a+\frac{1}{a}-\frac{1}{a+\frac{1}{a}-}\frac{1}{a+\frac{1}{a}-}\dots$$

to n quotients.

10. Shew that, if

$$x=\frac{a}{b+}\frac{c}{d+}\frac{a}{b+}\frac{c}{d+}\dots \text{ and } y=\frac{c}{d+}\frac{a}{b+}\frac{c}{d+}\frac{a}{b+}\dots,$$

then $bx-dy=a-c$.

11. Shew that the ratio of

$$a+\frac{1}{1+}\frac{1}{b+}\frac{1}{a+}\frac{1}{1+}\dots \text{ to } b+\frac{1}{1+}\frac{1}{a+}\frac{1}{b+}\frac{1}{1+}\dots$$

is $1+a:1+b$.

12. Shew that the nth convergent of

$$\frac{1}{3-}\frac{4}{3-}\frac{2}{3-}\frac{2}{3-}\dots \text{ is } \frac{2^n-1}{2^n+1}.$$

13. Shew that the nth convergent of

$$2+\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\dots \text{ is } \frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{(1+\sqrt{2})^n-(1-\sqrt{2})^n}.$$

14. Shew that the nth convergent of $\frac{1}{1-}\frac{2}{3-}\frac{2}{3-}\frac{2}{3-}\dots$ is 2^n-1.

15. Shew that the nth convergent of

$$\frac{1}{a+b-}\frac{ab}{a+b-}\frac{ab}{a+b-}\dots \text{ is } \frac{a^n-b^n}{a^{n+1}-b^{n+1}}.$$

16. Find the nth convergent of the continued fraction

$$\frac{2}{1-}\frac{3}{5-}\frac{8}{7-}\dots\frac{r^2-1}{-2r+1-}\dots.$$

17. In the series of fractions $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$, &c., where the law of formation is $p_r=q_{r-1}$, $q_r=(n^2-1)p_{r-1}+2q_{r-1}$; prove that the limit of $\frac{p_r}{q_r}$ when r is infinitely great is $\frac{1}{1+n}$.

18. Shew that in the continued fraction

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots\frac{a_n}{b_n+}\cdots,$$

the nth convergent is to the $(n-1)$th in the ratio

$$b_n+\frac{a_n}{b_{n-1}+}\cdots\frac{a_3}{+b_2} : b_n+\frac{a_n}{b_{n-1}+}\cdots\frac{a_2}{+b_1}.$$

19. Shew that, if y_1, y_2, &c., be the convergents of a simple periodic continued fraction found by taking 1, 2, &c., complete periods, and if $\frac{P}{Q}$, $\frac{P'}{Q'}$ be the two convergents immediately preceding y_1, then $y_n=\frac{P'y_{n-1}+P}{Q'y_{n-1}+Q}$.

20. If Z be any integer not a perfect square, and if $\sqrt{Z}$ be converted into a continued fraction

$$a+\frac{1}{b+}\frac{1}{c+}\cdots\frac{1}{+k+}\frac{1}{2a+}\frac{1}{b}+\ldots,$$

and if the convergents obtained by taking one, two, ..., i complete periods, each period terminating with k, be denoted by P_1, P_2, ..., P_i, prove that

$$\frac{P_i+\sqrt{Z}}{P_i-\sqrt{Z}}=\left(\frac{P_1+\sqrt{Z}}{P_1-\sqrt{Z}}\right)^i.$$

21. Find the nth convergent of the continued fraction

$$\frac{1}{a+a^{-1}-}\frac{1}{a+a^{-1}-}\frac{1}{a+a^{-1}-}\cdots;$$

and shew that the limit of the nth convergent when n is indefinitely increased is a or a^{-1} according as a is numerically less than or greater than unity.

22. Shew that the nth convergent of

$$\frac{1}{2+}\frac{2}{2+}\frac{3}{2+}\frac{3}{2+}\cdots \text{ is } \frac{3}{8}\left\{1-\left(-\frac{1}{3}\right)^n\right\}.$$

23. Shew that

$$e^{-x}=\frac{1}{1+}\,\frac{x}{1-x+}\,\frac{x}{2-x+}\,\frac{2x}{3-x+}\cdots\frac{(n-1)x}{n-x+}\cdots \text{ to infinity.}$$

24. Shew that

$$\frac{x}{a}-\frac{x^2}{ab}+\frac{x^3}{abc}-\ldots=\frac{x}{a+}\,\frac{ax}{b-x+}\,\frac{bx}{c-x+}\cdots.$$

25. Shew that

$$\frac{1}{1+}\,\frac{1}{1+}\,\frac{2}{3+}\,\frac{4}{5+}\,\frac{6}{7+}\cdots \text{ to infinity}=e^{-\frac{1}{2}}.$$

26. Find the value of

$$\frac{1}{1+}\,\frac{1}{2+}\,\frac{3}{5+}\,\frac{6}{8+}\cdots\frac{3(n-2)}{3n-4+}\cdots \text{ to infinity.}$$

27. Shew that

$$\frac{1}{3+}\,\frac{1^2.3^2}{5+}\,\frac{2^2.4^2}{7+}\,\frac{3^2.5^2}{9+}\cdots\frac{(n-1)^2(n+1)^2}{2n+1}$$

$$=\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-\ldots+(-1)^{n-1}\frac{1}{n(n+2)}.$$

28. Shew that the nth convergent of

$$\frac{2}{1-}\,\frac{2}{9-}\,\frac{2}{1-}\,\frac{2}{9-}\cdots \text{ is } 6\,\frac{2^n-1}{2^{n+1}-1}.$$

29. Shew that the nth convergent of

$$\frac{1}{3-}\,\frac{4}{3-}\,\frac{1}{3-}\,\frac{4}{3-}\cdots \text{ is } \frac{6n-1+(-1)^n}{6n+7+(-1)^n}.$$

30. Shew that $\dfrac{1^2}{3-}\,\dfrac{2^2}{5-}\,\dfrac{3^2}{7-}\cdots$ to infinity $=1$.

31. Shew that

$$\sqrt{2}=1+\frac{1}{2+}\,\frac{1.2}{3+}\,\frac{3.4}{3+}\,\frac{5.6}{3+}\cdots \text{ to infinity.}$$

32. Shew that

$$1-\frac{1}{\sqrt{2}}=\frac{1}{2+}\,\frac{2.3}{1+}\,\frac{4.5}{1+}\,\frac{6.7}{1+}\dots \text{ to infinity.}$$

33. Shew that

$$\frac{1}{4-}\,\frac{3.4}{9-}\,\frac{5.6}{13-}\,\frac{7.8}{17-}\,\frac{9.10}{21-}\dots \text{ to infinity}=1.$$

34. Shew that $(1+x)^n$

$$=1+\frac{nx}{1-}\,\frac{(n-1)x}{2+(n-1)x-}\,\frac{2(n-2)x}{3+(n-2)x-}\,\frac{3(n-3)x}{4+(n-3)x-}\dots$$

35. Shew that, if n be a positive integer,

$$2^n=1+\frac{n}{1-}\,\frac{n-1}{n+1-}\,\frac{2(n-2)}{n+1-}\,\frac{3(n-3)}{n+1-}\dots\frac{(n-1)1}{n+1}.$$

36. Shew that $\left(1+\frac{x}{a}\right)\left(1+\frac{x}{a'}\right)\left(1+\frac{x}{a''}\right)\dots$

$$=1+\frac{x}{a-}\,\frac{a(x+a)}{x+a+a'-}\,\frac{a'(x+a')}{x+a'+a''-}\dots$$

37. Shew that $\frac{1}{n-}\,\frac{n}{2n+1-}\,\frac{2n}{3n+1-}\dots$ to infinity

$$=\frac{1}{n-1+}\,\frac{n}{2n-1+}\,\frac{2n}{3n-1+}\dots \text{ to infinity.}$$

CHAPTER XXVIII.

THEORY OF NUMBERS.

371. Throughout the present chapter the word *number* will always denote a positive whole number; also the word *divide* will be used in its primitive meaning of division without remainder. The symbol $M(p)$ will often be used instead of '*a multiple of p.*'

Definitions. A number which can only be divided by itself and unity is called a *prime number*, or a *prime*.

A number which admits of other divisors besides itself and unity is called a *composite* number.

Two numbers which cannot both be divided by any number, except unity, are said to be *prime to one another*, and each is said to be prime to the other.

372. **The Sieve of Eratosthenes.** The different prime numbers can be found in order by the following method, called the *Sieve of Eratosthenes.*

Write down in order the natural numbers from 1 to any extent that may be required: thus

$1,\ 2,\ 3,\ \dot{4},\ 5,\ \ddot{6},\ 7,\ \dot{8},\ \dot{9},\ \ddot{10}$

$11,\ \ddot{12},\ 13,\ \ddot{14},\ \ddot{15},\ \dot{16},\ 17,\ \ddot{18},\ 19,\ \ddot{20}$

$\ddot{21},\ \ddot{22},\ 23,\ \ddot{24},\ \dot{25},\ \ddot{26},\ \dot{27},\ \ddot{28},\ 29,\ \dddot{30}$

$31,\ \dot{32},\ \ddot{33},\ \ddot{34},\ \ddot{35},\ \ddot{36},\ 37,\ \ddot{38},\ \ddot{39},\ \ddot{40}$ &c.

Now take the first prime number, 2, and over every second number from 2 place a dot; we thus mark all

multiples of 2. Then, leaving 3 unmarked, place a dot over every third number from 3: we thus mark all multiples of 3. The number next to 3 which is unmarked is 5; and leaving 5 unmarked, place a dot over every fifth number from 5: we thus mark all multiples of 5. And so for multiples of 7, &c.

Having done this, all the numbers which are left unmarked are primes, for no one of them is divisible by any number smaller than itself, except unity.

It should be here remarked that if a composite number be expressed as the product of two factors, one of these must be less and the other greater than the square root of the number, unless the number is a perfect square, in which case each of the factors may be equal to the square root. Hence every composite number is divisible by a prime not greater than its square root. On this account it is, for example, only necessary to reject as above multiples of the primes 2, 3, 5 and 7 in order to obtain the primes less than 121, for every composite number less than 121 is divisible by a prime less than 11.

373. Theorem. *The number of primes is infinite.*

For, if the number of primes be not infinite, there must be one particular prime which is greater than all others. Let then p be the greatest of all the prime numbers. Then $|\underline{p}$ will be divisible by p and by every prime less than p. Hence $|\underline{p}+1$ will not be divisible by p or by any smaller prime; therefore $|\underline{p}+1$ is either divisible by a prime greater than p, or it is itself a prime greater than p. Thus there cannot be a *greatest* prime number; and therefore the number of primes must be infinite.

Ex. Find n consecutive numbers none of which are primes.

The numbers are given by $|\underline{n+1}+r$, where r is any one of the numbers $2, 3, \ldots, (n+1)$.

374. Theorem. *No rational integral algebraical formula can represent prime numbers only.*

For, if possible, let the expression $a \pm bx \pm cx^2 \pm dx^3 \pm \ldots$ represent a prime number for *any* integral value of x, and for some particular constant integral values of $a, b, c, \ldots$. Give to x any value, m suppose, such that the whole expression is equal to p, where p is neither zero nor unity; then $p = a \pm bm \pm cm^2 \pm \ldots$. Now give to x any value $m + np$, where n is any positive integer; then the whole expression will be

$$a \pm b(m+np) \pm c(m+np)^2 \pm \ldots = a \pm bm \pm cm^2 \pm \ldots \\ + M(p) = p + M(p).$$

Thus an indefinite number of values can be given to x for each of which the expression $a \pm bx \pm cx^2 \pm \ldots$ is *not* a prime.

In connexion with the above theorem, the following formulae are noteworthy:—

(i) $x^2 + x + 41$, which is prime if $x < 40$. [Euler.]

(ii) $x^2 + x + 17$, which is prime if $x < 16$. [Barlow.]

(iii) $2x^2 + 29$, which is prime if $x < 29$. [Barlow.]

375. The student is already acquainted from Arithmetic with many properties of factors of numbers: these all depend upon the following fundamental

Theorem:—*If a number divide a product of two factors, and be prime to one of the factors, it will divide the other.*

For, let ab be divisible by x, and let a be prime to x. Reduce $\frac{a}{x}$ to a continued fraction, and let $\frac{p}{q}$ be the convergent which immediately precedes $\frac{a}{x}$; then [Art. 357, I.] $qa - px = \pm 1$; $\therefore qab - pxb = \pm b$. Now qab is, by supposition, divisible by p; and therefore $qab - pxb$ must be divisible by p, that is b must be divisible by p.

From the above theorem the following can easily be deduced:—

I. If a prime number divide the product of several factors it must divide one at least of the factors.

II. If a prime number divide a^n it will divide a.

III. If a be prime to each of α, β, γ, ... it will be prime to the product $\alpha\beta\gamma$....

IV. If a be prime to b, a^n will be prime to b^m.

V. If a number be divisible by several primes separately it will be divisible by the product of them all.

376. **Theorem.** *Every composite number can be resolved into prime factors; and this can be done in only one way.*

For, if N be not a prime number, it can be divided by some number, a suppose, which is neither N nor 1: thus $N = ab$. Again, if a and b be not primes, we have $a = c \times d$, $b = e \times f$, and therefore $N = cdef$. Proceeding in this way, since the factors diminish at every stage, we must at last come to numbers all of which are primes. Thus N can be expressed in the form $\alpha \times \beta \times \gamma \times \delta \times \ldots$, where α, β, γ, δ, ... are all primes but are not necessarily all different, so that N may be expressed in the form $\alpha^p\beta^q\gamma^r\ldots$, where α, β, γ,... are the different prime factors of N.

Next, to shew that there is only one way in which a number can be resolved into prime factors.

Suppose that $N = abcd\ldots$, where a, b, c, d,... are all primes but are not necessarily all different; suppose also that $N = \alpha\beta\gamma\delta\ldots$, where $\alpha, \beta, \gamma, \delta\ldots$ are also primes. Then we have $abcd\ldots = \alpha\beta\gamma\delta\ldots$. Hence a divides $\alpha\beta\gamma\delta\ldots$; and therefore, as all the letters represent prime numbers, a must be the same as one of the factors of $\alpha\beta\gamma\delta\ldots$. Let $a = \alpha$; then we have $bcd\ldots = \beta\gamma\delta\ldots$, from which it follows that b must be equal to one or other of β, γ, δ,...; and so on. Hence the prime factors a, b, c,... must be the same as the prime factors α, β, γ,....

Ex. Express 29645, 13689 and 90508 in terms of their prime factors.

Ans. $5 . 7^2 . 11^2$, $3^4 . 13^2$ and $2^2 . 11^3 . 17$.

377. *To find the highest power of a prime number contained in* $\underline{|n}$.

Let $I\left(\frac{x}{y}\right)$ denote the integral part of $\frac{x}{y}$; and let a be any prime number. Then the factors in $\underline{|n}$ which will be divisible by a are $a, 2a, 3a, \ldots, I\left(\frac{n}{a}\right).a$. Thus $I\left(\frac{n}{a}\right)$ factors of $\underline{|n}$ will be divisible by a. Similarly $I\left(\frac{n}{a^2}\right)$ factors will be divisible by a^2. And so on.

Hence the whole number of times the prime number a is contained in $\underline{|n}$ is $I\left(\frac{n}{a}\right)+I\left(\frac{n}{a^2}\right)+I\left(\frac{n}{a^3}\right)+\ldots.$

Ex. 1. Find the highest powers of 2 and 7 contained in $\underline{|50}$.

Here $I\left(\frac{50}{2}\right)=25$, $I\left(\frac{50}{2^2}\right)=12$, $I\left(\frac{50}{2^3}\right)=6$, $I\left(\frac{50}{2^4}\right)=3$, $I\left(\frac{50}{2^5}\right)=1$. Hence 2^{47} is the required highest power of 2.

Again, $I\left(\frac{50}{7}\right)=7$, $I\left(\frac{50}{7^2}\right)=1$. Hence 7^8 is the required highest power of 7.

Ex. 2. Find the highest powers of 3 and 5 which will divide $\underline{|80}$.

Ans. 3^{36}, 5^{19}.

Ex. 3. Find the highest power of 7 which will divide $\underline{|1000}$.

Ans. 7^{164}.

378. **Theorem.** *The product of any r consecutive numbers is divisible by* $\underline{|r}$.

Let n be the first of the r consecutive numbers; then we have to shew that $\dfrac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{\underline{|r}}$,

or $\dfrac{\underline{|n+r-1}}{\underline{|r}\,\underline{|n-1}}$, is an integer.

The theorem follows at once from the fact that $\dfrac{\underline{|n+r-1}}{\underline{|r}\,\underline{|n-1}}$ is ${}_{n+r-1}C_r$, and the number of combinations of $n+r-1$

things r together must be a whole number for all values of n and of r.

The theorem can also be proved at once from first principles by means of Art. 377.

For it is obvious that $I\left(\frac{n+r-1}{a}\right) \nless I\left(\frac{n-1}{a}\right) + I\left(\frac{r}{a}\right)$, $I\left(\frac{n+r-1}{a^2}\right) \nless I\left(\frac{n-1}{a^2}\right) + I\left(\frac{r}{a^2}\right)$, and so on. Hence from Art. 377 it follows that the number of times any prime number is contained in $\underline{|n+r-1}$ can never be *less*, although it may be greater, than the number of times the same prime number is contained in $\underline{|n-1} \times \underline{|r}$. Thus every prime number which occurs in $\underline{|n-1} \times \underline{|r}$, occurs to *at least as high* a power in $\underline{|n+r-1}$, which proves that $\underline{|n+r-1}$ is divisible by $\underline{|n-1} \times \underline{|r}$.

It can be proved in a similar manner that $\dfrac{\underline{|n}}{\underline{|\alpha}\,\underline{|\beta}\,\underline{|\gamma}\ldots}$ is an integer, where $\alpha + \beta + \gamma + \ldots = n$.

379. *If n be a prime number the coefficient of every term in the expansion of $(a + b)^n$ except the first and last terms is divisible by n.*

For, excluding the first and last terms, any coefficient is given by $\dfrac{n(n-1)\ldots(n-r+1)}{\underline{|r}}$, where r is any integer between 0 and n.

Now, by the preceding Article, $\dfrac{n(n-1)\ldots(n-r+1)}{\underline{|r}}$ is an integer; and, as n is a prime number greater than r, n must be prime to $\underline{|r}$; and therefore $\dfrac{(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}}$ must be an integer. Hence every coefficient, except the first and last, is divisible by n.

Similarly, if n be a prime number, the coefficient of

every term in the expansion of $(a+b+c+\ldots)^n$ which contains more than one of the letters, is divisible by n.

For the coefficient of any term which contains more than one of the letters is of the form $\dfrac{\underline{|n}}{\underline{|\alpha}\,\underline{|\beta}\,\underline{|\gamma}\ldots}$, where $\alpha+\beta+\gamma+\ldots=n$. Now $\dfrac{\underline{|n}}{\underline{|\alpha}\,\underline{|\beta}\,\underline{|\gamma}\ldots}$ is an integer; and, as n is a prime greater than any of the letters α, β, γ,..., n must be prime to $\underline{|\alpha}\,\underline{|\beta}\,\underline{|\gamma}\ldots$; and therefore the coefficient of every term which contains more than one letter is divisible by n.

Ex. 1. Shew that $n(n+1)(2n+1)$ is a multiple of 6.

Ex. 2. Shew that, if n be odd, $(n^2+3)(n^2+7)=M(32)$.

Ex. 3. Shew that, if n be odd, $n^4+4n^2+11=M(16)$.

Ex. 4. Shew that $1+7^{2n+1}=M(8)$.

Ex. 5. Shew that $19^{2n}-1=M(360)$.

Ex. 6. Shew that, if n be a prime number greater than 3,

$$n(n^2-1)(n^2-4)=M(360).$$

380. **Fermat's Theorem.** *If n be a prime number, and m any number prime to n; then $m^{n-1}-1$ will be divisible by n.*

We know that when n is a prime number, the coefficient of every term in the expansion of $(a_1+a_2+\ldots+a_m)^n$, which contains more than one of the letters, is divisible by n. Now there are m terms each of which contains only one letter and the coefficient of each of these terms is 1. Hence, putting $a_1=a_2=\ldots=1$, we have

$$m^n=m+M(n);\ \therefore m(m^{n-1}-1)=M(n).$$

Hence, if m be prime to n, $m^{n-1}-1$ will be a multiple of n.

Ex. 1. Shew that, if n be a prime number,

$$1^{n-1}+2^{n-1}+3^{n-1}+\ldots+(n-1)^{n-1}+1=M(n).$$

Ex. 2. Shew that, if a and b are both prime to the prime number n; then will $a^{n-1}-b^{n-1}$ be a multiple of n.

Ex. 3. Shew that $n^5-n=M(30)$.

Ex. 4. Shew that $n^7 - n = M(42)$.

Ex. 5. Shew that $x^{12} - y^{12} = M(1365)$, if x and y are prime to 1365.

Ex. 6. Shew that, if m and n are primes; then

$$m^{n-1} + n^{m-1} - 1 = M(mn).$$

Ex. 7. Shew that, if m, n and p are all primes; then

$$(np)^{m-1} + (pm)^{n-1} + (mn)^{p-1} - 1 = M(mnp).$$

Ex. 8. Shew that the 4th power of any number is of the form $5m$ or $5m+1$.

Ex. 9. Shew that the 12th power of any number is of the form $13m$ or $13m+1$.

Ex. 10. Shew that the 8th power of any number is of the form $17m$ or $17m \pm 1$.

381. *To find the number of divisors of a given number.*

Let the given number, N, expressed in prime factors, be $a^x b^y c^z$....... Then it is clear that N is divisible by every term of the continued product

$$(1 + a + a^2 + \ldots + a^x)(1 + b + b^2 + \ldots + b^y)(1 + c + c^2 + \ldots + c^z)\ldots$$

Hence the number of divisors of N, *including* N *and* 1, is

$$(x+1)(y+1)(z+1)\ldots\ldots$$

Ex. 1. The number of divisors of 600, that is of $2^3 \cdot 3 \cdot 5^2$, is

$$(3+1)(1+1)(2+1) = 24.$$

Ex. 2. Find the sum of the divisors of a given number.

The given number being $N = a^x b^y c^z \ldots$, the sum required is easily seen to be

$$\frac{(1-a^{x+1})(1-b^{y+1})(1-c^{z+1})\ldots}{(1-a)(1-b)(1-c)\ldots}.$$

Ex. 3. Find the number of divisors of 1000, 3600 and 14553.

Ans. 16, 45, 24.

Ex. 4. Shew that 6, 28 and 496 are *perfect* numbers. [A perfect number is one which is equal to the sum of all its divisors, not considering the number itself as a divisor.]

Ex. 5. Find the least number which has 6 divisors. *Ans.* 12.

Ex. 6. Find the least number which has 15 divisors. *Ans.* 144.

Ex. 7. Find the least number which has 20 divisors. *Ans.* 240.

Ex. 8. Find the least numbers by which 4725 must be multiplied in order that the product may be (i) a square, and (ii) a cube.

Ans. 21, 245.

382. *To find the number of pairs of factors, prime to each other, of a given number.*

Let the given number be $N = a^x b^y c^z \ldots$; then, if one of two factors prime to each other contains a, the other does not; and so for all the other different prime factors.

Hence the factors in question are the different terms in the product $(1 + a^x)(1 + b^y)(1 + c^z)\ldots$, the number of them being 2^n, where n is the number of different prime factors of N. The number of different *pairs* of factors prime to each other is therefore 2^{n-1}, in which result N and 1 are considered as one pair.

383. *To find the number of positive integers which are less than a given number and prime to it.*

Let the given number be $N = a^x b^y c^z \ldots$, where $a, b, c, \ldots$ are the different prime factors of N.

The terms of the series $1, 2, 3, \ldots, N$ which are divisible by a are $a, 2a, 3a, \ldots, \frac{N}{a}a$; and therefore there are $\frac{N}{a}$ numbers which are divisible by a. So also there are $\frac{N}{b}$ numbers divisible by b, $\frac{N}{bc}$ divisible by bc, $\frac{N}{abc}$ divisible by abc, and so on.

We will now shew that every integer which is less than N and *not* prime to N is counted once and once only in the series

$$\Sigma\frac{N}{a} - \Sigma\frac{N}{ab} + \Sigma\frac{N}{abc} - \Sigma\frac{N}{abcd} + \ldots\ldots\ldots(\alpha).$$

Suppose an integer is divisible by only one prime factor of N, a suppose; then that integer is counted *once* in (α), namely as one of the $\frac{N}{a}$ numbers which are divisible by a.

Next suppose an integer is divisible by r of the prime factors $a, b, c, \ldots$, then that integer will be counted r

times in $\Sigma \frac{N}{a}$, it will be counted $\frac{r(r-1)}{1.2}$ times in $\Sigma \frac{N}{ab}$, it will be counted $\frac{r(r-1)(r-2)}{1.2.3}$ times in $\Sigma \frac{N}{abc}$, and so on. Hence the whole number of times an integer divisible by r of the prime factors is counted, is

$$r - \frac{r(r-1)}{1.2} + \frac{r(r-1)(r-2)}{1.2.3} - \ldots + (-1)^{r-1} \frac{r(r-1)\ldots 1}{\lfloor r}$$
$$= 1 - (1-1)^r = 1.$$

Thus every number not prime to N is counted *once* in (α); and therefore the number of positive integers less than N and *not* prime to N is given by (α); *provided however that unity is considered to be prime to N.*

Hence the number of positive integers less than N and prime to N is

$$N - \Sigma \frac{N}{a} + \Sigma \frac{N}{ab} - \Sigma \frac{N}{abc} + \ldots$$

$$= N\left\{1 - \Sigma \frac{1}{a} + \Sigma \frac{1}{ab} - \Sigma \frac{1}{abc} + \ldots\right\}$$

$$= N\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\left(1 - \frac{1}{c}\right) \ldots\ldots \quad \text{[Art. 260].}$$

Ex. 1. Find the number of integers less than 100 and prime to it.

Since $100 = 2^2 . 5^2$, the number required is

$$100\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) - 1 = 39.$$

Ex. 2. Find the number of integers less than 1575 and prime to it.

Ans. 719.

Ex. 3. Shew that the number of integers, including unity, which are less than $N [N > 2]$ and prime to N is even, and that half these numbers are less than $\frac{N}{2}$.

[For if a be prime to N so also is $N - a$; and if $a > \frac{N}{2}$, then $N - a < \frac{N}{2}$.]

384. Forms of square numbers. Some of the different possible and impossible forms of square numbers will be seen from the following examples :—

Ex. 1. Shew that every square is of the form $3m$ or $3m+1$.

For every number is of the form $3m$ or $3m \pm 1$. Hence every square is of the form $9m$ or $3m+1$.

Ex. 2. Shew that every square is of the form $5m$ or $5m \pm 1$.

For every number is of the form $5m$, $5m \pm 1$ or $5m \pm 2$; and therefore every square is of the form $5m$, $5m+1$ or $5m+4$.

Ex. 3. Shew that, if $a^2+b^2=c^2$, where a, b, c are integers; then will abc be a multiple of 60.

First, every square is of the form $3m$ or $3m+1$; and therefore the sum of two squares neither of which is a multiple of 3 is of the form $3m+2$ which cannot be a square. Hence *either a or b must be a multiple of* 3.

Again, every square is of the form $5m$ or $5m \pm 1$. The sum of two squares neither of which is a multiple of 5 is therefore of one of the forms $5m$, or $5m \pm 2$. Now no square can be of the form $5m \pm 2$; and if a square be of the form $5m$, its root must be a multiple of 5. Hence, if ab is not a multiple of 5, c will be a multiple of 5. Thus, in any case, *abc is a multiple of* 5.

Lastly, since every number is of the form $4m$, $4m+1$, $4m+2$ or $4m+3$, every square is of the form $16m$, $8m+1$, $16m+4$. Now a and b cannot both be odd, for the sum of their squares would then be of the form $8m+2$ which cannot be a square. Also, if one is even and the other odd, the even number must be divisible by 4, for the sum of two squares of the forms $8m+1$ and $16m+4$ respectively is of the form $8m+5$ which cannot be a square. It therefore follows that *ab must be a multiple of* 4.

Thus abc is divisible by 3, by 5 and by 4; hence, as 3, 4 and 5 are prime to one another, $abc \doteq M(60)$.

Ex. 4. Shew that every cube is of the form $7m$ or $7m \pm 1$. Shew also that every cube is of the form $9m$ or $9m \pm 1$.

Ex. 5. Shew that every fourth power is of the form $5m$ or $5m+1$.

Ex. 6. Shew that no square number ends with 2, 3, 7 or 8.

Ex. 7. Shew that, if a square terminate with an odd digit, the last figure but one will be even.

Ex. 8. Shew that the last digit of any number is the same as the last digit of its $(4n+1)$th power.

Ex. 9. Shew that the product of four consecutive numbers cannot be a square.

EXAMPLES XXXVIII.

1. Shew that the difference of the squares of any two prime numbers greater than 3 is divisible by 24.

2. Shew that, if n be a prime greater than 3,

$$n(n^2-1)(n^2-4)(n^2-9)=M(2^7.3^2.5.7).$$

3. Shew that, if n be any odd number,

$$(n+2m)^n-(n+2m)=M(24).$$

4. Shew that $a^{4m+p}-a^{4n+p}=M(30)$.

5. Shew that, if $N-a^2=x$ and $(a+1)^2-N=y$, where x and y are positive; then $N-xy$ is a square.

6. How many numbers are there less than 1000 which are not divisible by 2, 3 or 5?

7. P, Q, R, p, q, r are integers, and p, q, r are prime to one another; prove that, if $\frac{P}{p}+\frac{Q}{q}+\frac{R}{r}$ be an integer, then $\frac{P}{p}$, $\frac{Q}{q}$ and $\frac{R}{r}$ will all be integers.

8. Shew that 284 and 220 are two 'amicable' numbers, that is two numbers such that each is equal to the sum of the divisors of the other.

9. Shew that, if 2^n-1 be a prime number, then $2^{n-1}(2^n-1)$ will be a 'perfect' number, that is a number which is equal to the sum of its divisors.

10. Find all the integral values of x less than 20 which make $x^{16}-1$ divisible by 680.

11. Shew that no number the sum of whose digits is 15 can be either a perfect square or a perfect cube.

12. Shew that every square can be expressed as the difference between two squares.

13. Find a general formula for all the numbers which when divided by 7, 8, 9 will leave remainders 1, 2, 3 respectively; and shew that 498 is the least of them.

14. If n be a prime number, and N prime to n, shew that

$$N^{n^2-n}-1=M(n^2), \text{ and that } N^{n^r-n^{r-1}}-1=M(n^r).$$

15. Shew that, if n be a prime number and N be prime to n, then will $N^{1+2+\ldots+(n-1)}\pm 1=M(n^2)$.

16. Shew that, if p be a prime number, and $(1+x)^{p-2}=1+a_1x+a_2x^2+a_3x^3+\ldots$; then a_1+2, a_2-3, a_3+4, &c. will be multiples of p.

17. Shew that if three prime numbers be in A. P. their common difference will be a multiple of 6, unless 3 be one of the primes.

18. Shew that $\dfrac{|\underline{2a}\;|\underline{2b}}{|\underline{a}\;|\underline{b}\;|\underline{a+b}}$ is an integer.

19. Shew that $\dfrac{|\underline{2n}}{|\underline{n+1}\;|\underline{n}}$ is an integer.

20. Shew that $\dfrac{|\underline{nr}}{|\underline{n}\,\{|\underline{r}\}^n}$ is an integer.

21. Each of two numbers is the sum of n squares; shew that the product of the two numbers can be expressed as the sum of $\frac{1}{2}n(n-1)+1$ squares.

22. Shew that a^2+b^2 cannot be divisible by 3, unless both a and b are divisible by 3; shew also that the same result holds good for the numbers 7 and 11.

23. Shew that, if $a^2+b^2=c^2$, then $ab(a^2-b^2)$ will be a multiple of 84.

24. Shew that no rational values of a, b, c, d can be found which will satisfy either of the relations $a^2+b^2=3(c^2+d^2)$, $a^2+b^2=7(c^2+d^2)$ or $a^2+b^2=11(c^2+d^2)$.

25. Shew that, if $a^2+c^2=2b^2$, then $a^2-b^2=M(24)$.

CONGRUENCES.

385. **Definition.** If two numbers a and b leave the same remainder when divided by a third number c, they are said to be *congruent* with respect to the *modulus* c; and this is expressed by the notation $a \equiv b$ (mod. c), which is called a *congruence*.

For example, $21 \equiv 1$ (mod. 10), and $(a+1)^2 \equiv 1$ (mod. a).

The congruence $a \equiv b$ (mod. c) shews that $a-b$ is a multiple of c, which can be expressed by

$$a-b \equiv 0 \text{ (mod. } c).$$

386. **Theorem.** *If* $a_1 \equiv b_1$ *(mod.* x*), and* $a_2 \equiv b_2$*(mod.* x*); then will* $a_1 + a_2 \equiv b_1 + b_2$ *(mod.* x*), and* $a_1a_2 \equiv b_1b_2$ *(mod.* x*).*

For let $a_1 = m_1x + r_1$, and $a_2 = m_2x + r_2$; then, by supposition, $b_1 = n_1x + r_1$ and $b_2 = n_2x + r_2$.

Hence $a_1 + a_2 - (b_1 + b_2) = (m_1 + m_2 - n_1 - n_2)\, x$;

$$\therefore (a_1 + a_2) - (b_1 + b_2) \equiv 0 \text{ (mod. } x),$$

or $$a_1 + a_2 \equiv b_1 + b_2 \text{ (mod. } x).$$

Again, it is easily seen that $a_1a_2 - b_1b_2 =$ a multiple of x, and therefore $a_1a_2 \equiv b_1b_2$ (mod. x).

The proposition will clearly hold good for any number of congruences to the same modulus.

387. Congruences have many properties analogous to equations. For example, if the congruence

$$Ax^2 + Bx + C \equiv 0 \text{ (mod. } p),$$

wherein A, B, C have constant integral values, be satisfied by the *three* values a, b, c of x, which are such that $a-b$ is unity or prime to p, and so for every other pair, then the congruence will hold good for all integral values of x, and A, B, C will all be multiples of p.

For we have

$$Aa^2 + Ba + C \equiv 0 \text{ (mod. } p),$$

and $$Ab^2 + Bb + C \equiv 0 \text{ (mod. } p);$$

$\therefore$ by subtraction

$$(a-b)\{A(a+b)+B\} \equiv 0 \pmod{p}.$$

Hence, as $a-b$ is unity or prime to p, we have

$$A(a+b)+B \equiv 0 \pmod{p}.$$

Similarly, $\quad A(a+c)+B \equiv 0 \pmod{p}.$

Hence, by subtraction, $A(b-c) \equiv 0 \pmod{p}$.

Therefore $A \equiv 0 \pmod{p}$, since $b-c$ is unity or prime to p.

Then, since $A \equiv 0 \pmod{p}$, it follows that $B \equiv 0 \pmod{p}$, and then that $C \equiv 0 \pmod{p}$.

Then, since A, B, C are all multiples of p, it follows that $Ax^2 + Bx + C$ is also a multiple of p for all integral values of x.

We can prove in a similar manner the general theorem, namely:—

If a congruence of the nth degree in x be satisfied by more than n values of x, which are such that the difference between any two is unity or is prime to the modulus, then the congruence will be satisfied for all integral values of x, and the coefficients of all the different powers of x will be multiples of the modulus.

388. **Theorem.** *If a and b are prime to one another, the numbers a, $2a$, $3a$,..., $(b-1)a$ will all leave different remainders when divided by b.*

For suppose that ra and sa leave the same remainder when divided by b.

Then $ra - sa = M(b)$; but if b divide $(r-s)a$, and be prime to a, it must divide $r-s$, which is impossible if r and s are both less than b.

Hence the remainders obtained by dividing a, $2a$, ..., $(b-1)a$ by b are all different; and since there are $b-1$ of these remainders, they must be the numbers 1, 2, ..., $(b-1)$ in some order or other.

If a be *not* prime to b the remainders obtained by dividing a, $2a$, $3a$, ..., $(b-1)\,a$ by b will *not* be all different. For let k be a common factor of a and b, and let $a = k\alpha$ and $b = k\beta$. Then it is easily seen that $(r+\beta)\,a$ and ra will leave the same remainder when divided by b, and $(r+\beta)\,a$ and ra are both included in the sreies a, $2a$, ..., $(b-1)\,a$ provided $r+\beta < b-1$.

COR. If a be prime to b, and n be any integer whatever, the remainders obtained by dividing n, $n+a$, $n+2a, \ldots, n+(b-1)\,a$ by b will all be different, and will therefore be the numbers $0, 1, 2, \ldots, (b-1)$.

389. **Fermat's Theorem.** From the result of the preceding article, Fermat's theorem can be easily deduced. For, if a and b are prime to each other, the numbers a, $2a, \ldots, (b-1)\,a$ will leave, in some order or other, the remainders $1, 2, \ldots, (b-1)$, when divided by b. Hence we have

$$a\,.\,2a\,.\,3a\ldots(\overline{b-1}\,.\,a) \equiv 1\,.\,2\,.\,3\ldots(b-1)\ (\text{mod. } b).$$

that is $\underline{|b-1}\,(a^{b-1}-1) \equiv 0$ (mod. b).

Now, if b be a prime number, $\underline{|b-1}$ will be prime to b; and we have $a^{b-1}-1 \equiv 0$ (mod. b), which is Fermat's theorem.

390. **Wilson's Theorem.** *If n be a prime number, $1+\underline{|n-1}$ will be divisible by n.*

If a be any number less than the prime number n, a will be prime to n, and hence, from Art. 388, the remainders obtained by dividing $a, 2a, \ldots, (n-1)\,a$ by n will be the numbers $1, 2, \ldots, (n-1)$; hence one and only one of the remainders will be unity. Let then ab be the multiple of a which gives rise to the remainder 1; then, if b were equal to a, we should have $a^2 = 1 + M(n)$, or $(a+1)(a-1) = M(n)$, and this can only be the case, since n is a prime, if $a=1$ or $a=n-1$. Hence the numbers $2, 3, \ldots (n-3), (n-2)$ can be taken in pairs in such a way that the product of each pair, and therefore the product of all the pairs, is of the form $M(n)+1$.

Thus $2\,.\,3\,.\,4\ldots(n-2) = M(n)+1$;

$$\therefore\ \underline{|n-1} = M(n)\times(n-1)+n-1.$$

Hence $\qquad \underline{|n-1} + 1 = M(n).$

Wilson's theorem may also be proved as follows:—

From Art. 305, we have

$$(n-1)^{n-1} - (n-1)(n-2)^{n-1} + \frac{(n-1)(n-2)}{1\,.\,2}(n-3)^{n-1} - \ldots$$
$$+ (-1)^{n-2}\frac{(n-1)(n-2)\ldots 2}{\underline{|n-2}}1^{n-1} = \underline{|n-1}.$$

Now by Fermat's theorem $(n-1)^{n-1} = 1 + M(n)$, $(n-2)^{n-1} = 1 + M(n)$, &c.

Hence we have

$$1-(n-1)+\frac{(n-1)(n-2)}{1\,.\,2} - \ldots + (-1)^{n-2}(n-1) + M(n)$$
$$= \underline{|n-1},$$

that is $(1-1)^{n-1} - (-1)^{n-1} + M(n) = \underline{|n-1}$; hence, as $n-1$ is even, $\underline{|n-1} + 1 = M(n)$.

Wilson's theorem is important on account of its expressing a distinctive property of prime numbers; for $1 + \underline{|n-1}$ is not divisible by n unless n is a prime. For if any number less than n divide n it will divide $\underline{|n-1}$ and therefore cannot divide $\underline{|n-1} + 1$.

391. **Theorem.** *If the number of integers less than any number n and prime to n be denoted by $\phi(n)$; then, if $a, b, c, \ldots$ are prime to each other,*

$$\phi(abc\ldots) = \phi(a) \times \phi(b) \times \phi(c) \ldots\ldots,$$

provided that unity is considered to be prime to any greater number.

First take the case of two numbers a, b and their product ab.

Arrange the ab numbers as under:

1,	2,	3,	......,	k,	......	a
$a+1$,	$a+2$,	$a+3$,	......,	$a+k$,	......	$2a$
$2a+1$,	$2a+2$,	$2a+3$,	......,	$2a+k$,	......	$3a$
......	......	......	......	......	......	......
$(b-1)a+1$,	$(b-1)a+2$,	$(b-1)a+3$,	...,	$(b-1)a+k$,	...	ba.

Then it is clear that all the integers in the kth vertical column will or will not be prime to a according as k is or is not prime to a. Hence there are $\phi(a)$ columns of integers, including the first, all of which are prime to a. Then again, we know from Art. 388 that since a is prime to b, the remainders obtained by dividing the numbers k, $a+k, \ldots, (b-1)a+k$ by b are the numbers $0, 1, 2, \ldots (b-1)$; and it is clear that a number is or is not prime to b according as the remainder obtained by dividing the number by b is or is not prime to b. Hence there are as many integers prime to b in any one column as there are in the series $0, 1, 2, \ldots (b-1)$, that is to say, there are in each column $\phi(b)$ integers prime to b. Thus there are $\phi(a)$ columns of integers prime to a and each column contains $\phi(b)$ integers which are also prime to b. But all integers which are prime to a and also to b are prime to $a \times b$. Hence the number of integers less than ab and prime to ab is $\phi(a) \times \phi(b)$, so that $\phi(ab) = \phi(a) \times \phi(b)$.

The proposition can at once be extended, for we have

$$\begin{aligned}\phi(abc\ldots) = \phi(a \times bc\ldots) &= \phi(a) \times \phi(bc\ldots) \\ &= \phi(a)\,\phi(b)\,\phi(c\ldots) \\ &= \phi(a)\,.\,\phi(b)\,.\,\phi(c)\ldots\end{aligned}$$

392. The number of integers less than a given number and prime to it can be found by means of the theorem in the preceding article.

For let the number be $N = a^\alpha b^\beta c^\gamma \ldots$, where $a, b, c, \ldots$ are the different prime factors of N.

To find the number of integers less than a^α and prime to it, (unity being considered as one of these numbers) we must subtract $a^{\alpha-1}$ from a^α; for the numbers $a, 2a, 3a, \ldots, a^{\alpha-1}\,.\,a$ are *not* prime to a, and these are the only numbers which are not prime to a; thus

$$\phi(a^\alpha) = a^\alpha - a^{\alpha-1} = a^\alpha\left(1 - \frac{1}{a}\right).$$

Similarly $\phi(b^\beta) = b^\beta\left(1 - \frac{1}{b}\right)$, $\phi(c^\gamma) = c^\gamma\left(1 - \frac{1}{c}\right)$, &c.

But, by the preceding article,

$$\phi(a^\alpha . b^\beta . c^\gamma \ldots) = \phi(a^\alpha) . \phi(b^\beta) . \phi(c^\gamma) \ldots;$$

$$= a^\alpha\left(1 - \frac{1}{a}\right) . b^\beta\left(1 - \frac{1}{b}\right) . c^\gamma\left(1 - \frac{1}{c}\right) \ldots$$

Hence $\quad \phi(N) = N\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\left(1 - \frac{1}{c}\right) \ldots,$

where $a, b, c, \ldots$ are the different prime factors of N, and unity is considered to be prime to a, b, c, &c.

393. The following is an extension of Fermat's Theorem :—

If a and m are two numbers prime to one another, and $\phi(m)$ the number of integers, including unity, which are less than m and prime to m; then $a^{\phi(m)} - 1 \equiv 0$ (mod. m).

Let the $\phi(m)$ integers less than m and prime to m be denoted by $1, \alpha, \beta, \gamma, \ldots, (m-1)$. Then the products $a . 1$, $a\alpha, a\beta, a\gamma, \ldots, a(m-1)$ must all leave different remainders when divided by m, for if any two, ra and sa suppose, left the same remainder, $(r-s)a$ would be a multiple of m, which is impossible since a is prime to m and $r-s$ is less than m. Moreover the remainders must all be prime to m, since the two factors of any one of the products are both prime to m; and therefore as the $\phi(m)$ remainders are all different, and are all prime to m, they must be, in some order or other, the $\phi(m)$ numbers $1, \alpha, \beta, \gamma \ldots$

Hence

$$a . a\alpha . a\beta \ldots\ldots a(m-1) \equiv 1 . \alpha . \beta . \gamma \ldots (m-1) \text{ (mod. } m);$$

$$\therefore \{a^{\phi(m)} - 1\} 1 . \alpha . \beta \ldots (m-1) \equiv 0 \text{ (mod. } m).$$

Hence as $1 . \alpha . \beta \ldots (m-1)$ is prime to m, we have $a^{\phi(m)} - 1 \equiv 0$ (mod. m).

If m be a prime number, $\phi(m) = m - 1$, and we have Fermat's Theorem.

394. **Lagrange's Theorem.** *If p be a prime number, the sum of all the products r together of the numbers* 1, 2, 3, ..., $p-1$, *is divisible by p, r being any integer not greater than $p-2$.*

Consider the identity

$$(x-1)(x-2)\dots(x-p+1) = x^{p-1} - S_1 x^{p-2} + S_2 x^{p-3} + \dots + (-1)^{p-1} S_{p-1}.$$

Change x into $x-1$; then

$$(x-2)(x-3)\dots(x-p) = (x-1)^{p-1} - S_1(x-1)^{p-2} + \dots + (-1)^{p-1} S_{p-1}.$$

Hence

$$(x-p)\{x^{p-1} - S_1 x^{p-2} + S_2 x^{p-3} + \dots + (-1)^{p-1} S_{p-1}\}$$
$$= (x-1)\{(x-1)^{p-1} - S_1(x-1)^{p-2} + \dots + (-1)^{p-1} S_{p-1}\}.$$

Equate the coefficients of the different powers of x in the above identity; and we have

$$1 \,.\, S_1 = \frac{p(p-1)}{1 \,.\, 2},$$

$$2 \,.\, S_2 = \frac{p(p-1)(p-2)}{1 \,.\, 2 \,.\, 3} + S_1 \,.\, \frac{(p-1)(p-2)}{1 \,.\, 2},$$

$$3 \,.\, S_3 = \frac{p(p-1)(p-2)(p-3)}{1 \,.\, 2 \,.\, 3 \,.\, 4} + S_1 \,.\, \frac{(p-1)(p-2)(p-3)}{1 \,.\, 2 \,.\, 3} + S_2 \,.\, \frac{(p-2)(p-3)}{1 \,.\, 2},$$

$$\dots\dots = \dots\dots\dots\dots,$$

$$(p-2) \,.\, S_{p-2} = \frac{p(p-1)\dots 2}{1 \,.\, 2 \dots (p-1)} + S_1 \,.\, \frac{(p-1)(p-2)\dots 2}{1 \,.\, 2 \dots (p-2)} + S_2 \,.\, \frac{(p-2)\dots 2}{1 \,.\, 2 \dots (p-3)} + \dots + S_{p-3} \,.\, \frac{3 \,.\, 2}{1 \,.\, 2}.$$

Since p is a prime the first term in each right-hand member is divisible by p; whence it follows from the first equation that S_1 is a multiple of p, and then that S_2 is a multiple of p, and so on to S_{p-2}.

Lagrange's Theorem may also be deduced from the Theorem of Art. 387, assuming that Fermat's Theorem is known.

For the congruence

$$(x-1)(x-2)\dots(x-p+1)-x^{p-1}+1\equiv 0 \pmod{p},$$

is of the $(n-2)$th degree in x, and by Fermat's Theorem it is satisfied by the $p-1$ values $1, 2, \dots, p-1$, which are such that the difference between any pair is either unity or is prime to p. Hence, by Art. 387 it is true for all integral values of x, and the coefficients of all the different powers of x are multiples of p.

It should be remarked that Wilson's Theorem follows at once by putting $x=0$.

395. **Reduction of fractions to circulating decimals.**

It is obvious that a fraction whose denominator contains only the factors 2 and 5 can be reduced to a terminating decimal, for

$$\frac{a}{2^p 5^q}=\frac{a \cdot 5^p \cdot 2^q}{10^{p+q}}.$$

If, however, the denominator contains any factor which is prime to 10, the fraction can only be reduced to a circulating decimal.

Let the fraction in its lowest terms be $\dfrac{a}{2^p 5^q \cdot b}$, where b is prime to 10. Let this fraction be equivalent to a circulating decimal with α recurring and β non-recurring figures.

Then

$$\frac{a}{2^p \cdot 5^q \cdot b}=\frac{a \cdot 5^p \cdot 2^q}{10^{p+q} \cdot b}=\frac{N}{10^\beta(10^\alpha-1)};$$

$$\therefore\ 10^{p+q} \cdot b \cdot N=a \cdot 5^p \cdot 2^q \cdot 10^\beta(10^\alpha-1).$$

Hence, as b is prime to a and to 10, $10^\alpha-1=M(b)$, *and α is the lowest power of* 10 *for which this is true,* for

otherwise the fraction could be expressed as a circulating decimal with fewer than a recurring figures.

It should be noticed that the number of recurring figures in the circulating decimal depends only on b and is not affected by the presence of 2^p5^q in the denominator, for the number is a, where a is the lowest power of 10 which is equal to $M(b)+1$.

We will now prove that a is either equal to $\phi(b)$ or to one of its sub-multiples.

By the extension of Fermat's Theorem [Art. 393] we have

$$10^{\phi(b)}-1=M(b).$$

We have also $$10^a-1=M(b).$$

Hence, if a be not $\phi(b)$ or one of its sub-multiples, let

$$\phi(b)=ka+r, \text{ where } r<a.$$

Then $$10^{\phi(b)}-1=10^{ak}.10^r-1$$

$$=\{M(b)+1\}^k.10^r-1=M(b)+10^r-1;$$

$$\therefore \quad 10^r-1=M(b),$$

which is impossible since $r<a$, and a is the lowest power of 10 which is equal to $M(b)+1$.

Hence, if b be the factor of the denominator of a fraction which is prime to 10, *the number of recurring figures in the equivalent decimal is either $\phi(b)$ or one of its sub-multiples.*

396. We shall conclude this chapter by considering the following examples:—

Ex. 1. Shew that $3^{2n+2}-8n-9$ is a multiple of 64.

We have

$$3^{2n+2}-8n-9=(1+8)^{n+1}-8n-9=1+(n+1)8+M(8^2)-8n-9=M(8^2).$$

Ex. 2. Shew that $3^{2n}-32n^2+24n-1\equiv 0 \pmod{512}$.

Let $$u_n=3^{2n}-32n^2+24n-1;$$

then $$u_{n+1}=3^{2n+2}-32(n+1)^2+24(n+1)-1.$$

Hence $u_{n+1}-9u_n=256n^2-256n=256n(n-1)=M(512)$,

since $n(n-1)$ is divisible by 2.

And since $u_{n+1} - 9u_n \equiv 0$ (mod. 512), it follows that $u_{n+1} \equiv 0$ (mod. 512) provided $u_n \equiv 0$ (mod. 512). The theorem is therefore true for all values of n provided it is true for $n=1$, which is the case since $u_1=0$.

Ex. 3. Shew that no prime factor of n^2+1 can be of the form $4m-1$.

Every prime number, except 2, is of the form $2k+1$. Let then $2k+1$ be a prime factor of n^2+1. Then n is prime to $2k+1$, and therefore by Fermat's theorem $n^{2k} = M(2k+1)+1$.

But, by supposition, $n^2+1 = M(2k+1)$;

$$\therefore \qquad n^{2k} = \{M(2k+1)-1\}^k = M(2k+1)+(-1)^k.$$

Since $n^{2k} = M(2k+1)+1$ and $n^{2k} = M(2k+1)+(-1)^k$ it follows that k must be *even*, and therefore every prime factor of n^2+1 is of the form $4m+1$, and therefore no prime factor can be of the form $4m-1$.

Since the product of any number of factors of the form $4m+1$ is of the same form, it follows that every odd divisor of n^2+1 is of the form $4m+1$.

Ex. 4. Shew that every whole number is a divisor of a series of nines followed by zeros.

Divide the successive powers of 10 by the number, n suppose, then there can only be n different remainders including zero, and hence any particular remainder must recur. Let then 10^x and 10^y leave the same remainder when divided by n: then $10^x - 10^y$ is divisible by n and is of the required form.

EXAMPLES XXXIX.

1. Prove the following:—

(i) $2^{3n+1} - 9n^2 + 3n - 2 = M(54)$.

(ii) $5^{2n+1} + n^5 - 5n^3 + 4n - 5 = M(120)$.

(iii) $4^{2n+1} + 3^{n+2} \equiv 0$ (mod. 13).

(iv) $3^{4n+2} + 2 \,.\, 4^{3n+1} \equiv 0$ (mod. 17).

2. Shew that, if a be a prime number, and b be prime to a; then 1^2b^2, 2^2b^2, ..., $\left(\frac{a-1}{2}\right)^2 b^2$ will give different remainders when divided by a.

3. Shew that, if $4n+1$ be a prime number, it will be a factor of $\{\underline{|2n}\}^2+1$; and that, if $4n-1$ be a prime, it will be a factor of $\{\underline{|2n-1}\}^2-1$.

4. Shew that, if n be a prime number, and r be less than n; then will $\underline{|r-1}\,\underline{|n-r} + (-1)^{r-1} = M(n)$.

5. Shew that, if m and n are prime to one another, every odd divisor of m^2+n^2 is of the form $4k+1$.

6. Shew that $\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\dots$ to infinity

$$=\left(1-\frac{1}{2^n}\right)^{-1}\left(1-\frac{1}{3^n}\right)^{-1}\left(1-\frac{1}{5^n}\right)^{-1}\dots\dots,$$

where 2, 3, 5,... are the prime numbers in order.

7. Shew that the arithmetic mean of all numbers less than n and prime to it (including unity) is $\frac{1}{2}n$.

8. Shew that, if N be any number, and $a, b, c, \dots$ be its different prime factors; then the sum of all the numbers less than N and prime to N is $\frac{N^2}{2}\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\dots$, and the sum of the squares of all such numbers is

$$\frac{N^3}{3}\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\dots+\frac{N}{6}(1-a)(1-b)\dots$$

9. If $\phi(m)$ denote the number of integers less than m and prime to it; and if $d_1, d_2, d_3, \dots$ be the different divisors of n; then will $\Sigma\phi(d)=n$.

10. Shew that, if a fraction $\frac{a}{b}$, where b is prime and prime to 10, be reduced to a decimal, and if the number of figures in the recurring period is even; then the sum of the first half of the figures added to the last half will consist wholly of nines.

11. If $\frac{1}{p}$ be converted to a circulating decimal with $p-1$ figures in its recurring period, shew that p must be prime and that the recurring period being multiplied by 2, 3,......$(p-1)$ will reproduce its own digits in the same order.

12. Shew that, if $\frac{1}{P}$ has a circulating period of p figures, $\frac{1}{Q}$ of q figures, and $\frac{1}{R}$ of r figures,..., and if $P, Q, R, \dots$ are prime, then $\frac{1}{PQR\dots}$ will have a circulating period of n figures, where n is the L.C.M. of $p, q, r\dots.$

CHAPTER XXIX.

INDETERMINATE EQUATIONS.

397. WE have already seen that a single equation with more than one unknown quantity, or n equations with more than n unknown quantities, can be satisfied in an indefinite number of ways, provided there is no restriction on the values which the unknown quantities may have. If, however, the values of the unknown quantities are subject to any restriction, n equations may suffice to determine the values of more than n unknown quantities.

We shall in the present chapter consider some cases of equations in which the unknown quantities are restricted to integral values.

398. It is clear that every equation of the first degree with two unknown quantities x and y can be reduced to one or other of the forms $ax+by=\pm c$, $ax-by=\pm c$, where a, b, c are positive integers.

By changing x into $-x$ and y into $-y$, $ax+by=c$ will become $ax+by=-c$, and $ax-by=c$ will become $-ax+by=c$; hence in order to shew how to find integral solutions of any equation of the first degree in x and y, it is only necessary to consider the two types

$$ax+by=c \text{ and } ax-by=c.$$

Now, it is evident that the equation $ax \pm by=c$ cannot be satisfied by integral values of x and y, if a and b have any common factor which is not also a factor of c; and, if a, b and c have any common factor, the equation can be

divided throughout by that factor. In what follows it will therefore be supposed that a and b are prime to one another.

399. *To shew that integral values can always be found which will satisfy the equation $ax \pm by = c$, provided a and b are prime to one another.*

Let $\frac{a}{b}$ be reduced to a continued fraction, and let $\frac{p}{q}$ be the convergent immediately preceding $\frac{a}{b}$. Then, from Art. 357,

$$aq - pb = \pm 1;$$

$$\therefore\ a(\pm cq) - b(\pm cp) = c \text{ ....(i)},$$

and $$a(\pm cq) + b(\mp cp) = c \text{..............(ii)}.$$

Hence it follows from (i) that either $x = cq$, $y = cp$ or $x = -cq$, $y = -cp$ is a solution of the equation $ax - by = c$; and from (ii) that either $x = cq$, $y = -cp$ or $x = -cq$, $y = cp$ is a solution of the equation $ax + by = c$.

Hence at least one set of integral values of x and y can always be found which will satisfy the equation $ax \pm by = c$.

The above investigation fails when a or b is unity. But the equation $ax \pm y = c$ is obviously satisfied by the values $x = \alpha$, $\pm y = c - a\alpha$, where α is any integer. So also $x \pm by = c$ is satisfied by the values $x = c \mp b\beta$, $y = \beta$, where β is any integer.

Hence the equation $ax \pm by = c$ always admits of at least *one* set of integral values.

400. *Having given one set of integral values which satisfy the equation $ax - by = c$, to find all other possible integral solutions.*

Let $x = \alpha$, $y = \beta$ be *one* solution of the equation $ax - by = c$; then $a\alpha - b\beta = c$. Hence, by subtraction,

$$a(x - \alpha) - b(y - \beta) = 0.$$

Now since a divides $a(x-\alpha)$, it must also divide $b(y-\beta)$; a must therefore be a factor of $y-\beta$, since it is prime to b.

Let then $y-\beta=ma$, where m is any integer; then $a(x-\alpha)=mba$, and therefore $x=\alpha+mb$.

Hence, if $x=\alpha$, $y=\beta$ be one solution in integers of the equation $ax-by=c$, all other solutions are given by

$$x=\alpha+mb, \quad y=\beta+ma,$$

where m is any integer.

It is clear from the above that there are an indefinite number of sets of integral values which satisfy the equation $ax-by=c$, provided there is one such set; and, from the preceding article, we know that there is one set of integral values.

It is also clear that, whether α and β are positive or not, an indefinite number of values can be given to m which will make $\alpha+mb$ and $\beta+ma$ both *positive*.

Hence there are an infinite number of *positive integral* solutions of the equation $ax-by=c$.

401. *Having given one set of integral values which satisfy the equation $ax+by=c$, to find all other possible integral solutions.*

Let $x=\alpha$, $y=\beta$ be *one* integral solution of the equation $ax+by=c$; then $a\alpha+b\beta=c$. Hence, by subtraction, $a(x-\alpha)+b(y-\beta)=0$.

Now, since a divides $a(x-\alpha)$, it must also divide $b(y-\beta)$; a must therefore be a factor of $y-\beta$, since it is prime to b.

Let then $y-\beta=ma$, where m is any integer; then $a(x-\alpha)=-b(y-\beta)=-mab$; and therefore $x=\alpha-mb$.

Hence, if $x=\alpha$, $y=\beta$ be one solution in integers of the equation $ax-by=c$, all other integral solutions are given by

$$x=\alpha-mb, \quad y=\beta+ma,$$

where m is any integer.

From the above, together with Art. 399, it follows that there are an indefinite number of sets of *integral* values which satisfy the equation $ax+by=c$. The number of *positive integral* solutions of the equation is, however, limited in number.

402. *To find the number of positive integral solutions of the equation* $ax+by=c$.

We have proved in Art. 399, that the equation $ax+by=c$ is satisfied by the values $x=cq$, $y=-cp$, or by the values $x=-cq$, $y=cp$, where p/q is the penultimate convergent to a/b.

First suppose that $x=cq$, $y=-cp$ satisfy the equation; then all other integral values which satisfy the equation are given by

$$x=cq-mb, \quad y=-cp+ma \;\ldots\ldots\ldots\ldots\text{(i)},$$

where m is any integer.

From (i) it is clear that in order that x and y may both be *positive*, and not *zero*, m must be a positive integer, and that the greatest permissible value of m is $I\left(\frac{cq}{b}\right)$ and its least value $I\left(\frac{cp}{a}\right)+1$, so that the number of different values of m is $I\left(\frac{cq}{b}\right)-I\left(\frac{cp}{a}\right)$. Hence, as one set of values of x and y corresponds to each value of m, the number of solutions is $I\left(\frac{cq}{b}\right)-I\left(\frac{cp}{a}\right)$.

Let $\frac{cq}{b}=I_1+f_1$ and $\frac{cp}{a}=I_2+f_2$; then $\frac{c}{ab}=\frac{a(cq)-b(cp)}{ab}$ $=\frac{cq}{b}-\frac{cp}{a}=I_1-I_2+f_1-f_2$. Hence $I\left(\frac{c}{ab}\right)$ is I_1-I_2 or I_1-I_2-1 according as f_1 is not or is less than f_2.

Thus the number of solutions is $I\left(\frac{c}{ab}\right)+1$ or $I\left(\frac{c}{ab}\right)$

according as the fractional part of $\frac{cq}{b}$ is or is not less than the fractional part of $\frac{cp}{a}$.

It can be shewn in a similar manner that if $x = -cq$, $y = cp$ satisfy the equation, the number of solutions in positive integers is $I\left(\frac{c}{ab}\right) + 1$ or $I\left(\frac{c}{ab}\right)$ according as the fractional part of $\frac{cp}{a}$ is or is not less than the fractional part of $\frac{cq}{b}$.

Ex. 1. Find the positive integral values of x and y which satisfy the equation $7x - 13y = 26$.

We have $\frac{7}{13} = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6}$, the penultimate convergent is therefore $\frac{1}{2}$. Then $7 \cdot 2 - 13 \cdot 1 = 1$; $\therefore\ 7\,(2 \times 26) - 13\,(26) = 26$.

Hence one solution is $x = 52$, $y = 26$; the general solution is therefore $x = 52 + 13m$, $y = 26 + 7m$.

[In this case the solution $x = 0$, $y = -2$ can be seen by inspection; and hence the general solution is $x = 13m$, $y = -2 + 7m$, which is easily seen to agree with the previous result.]

Ex. 2. Find the positive integral values of x and y which satisfy the equation $7x + 10y = 280$.

Here $\frac{7}{10} = \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3}$, the penultimate convergent being $\frac{2}{3}$. Then $7 \cdot 3 - 10 \cdot 2 = 1$; $\therefore\ 7\,(3 \cdot 280) + 10\,(-2 \cdot 280) = 280$.

Hence $x = 840$, $y = -560$ is one solution in integers. The general solution in integers is therefore $x = 840 - 10m$, $y = -560 + 7m$; and, in order that x and y may be *positive* $m \ngtr 84$ and $m \nless 80$. Thus the only values are $x = 40$, $y = 0$; $x = 30$, $y = 7$; $x = 20$, $y = 14$; $x = 10$, $y = 21$; $x = 0$, $y = 28$.

Ex. 3. Find the number of solutions in positive integers of the equation $3x + 5y = 1306$.

Here $\frac{3}{5} = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2}$, whence $3 \cdot 2 - 5 \cdot 1 = 1$;

$$\therefore\ 3 \cdot (2 \times 1306) + 5\,(-1306) = 1306.$$

Hence the general solution is $x = 2612 - 5m$, $y = 3m - 1306$.

For positive values of x and y we must have $m > 435$ and $m \ngtr 522$.

Hence the number of solutions is $522 - 435 = 87$.

403. Integral solutions of the two equations

$$ax + by + cz = d, \; a'x + b'y + c'z = d'$$

can be obtained as follows.

Eliminate one of the variables, z suppose; we then have the equation

$$(ac' - a'c)\, x + (bc' - b'c)\, y = dc' - d'c \ldots\ldots\ldots \text{(i)},$$

and this equation has integral solutions provided $ac' - a'c$ and $bc' - b'c$ are prime to one another, or will become prime to one another after division by any common factor which is also a factor of $dc' - d'c$.

Hence from (i) we obtain, as in the preceding articles, the general solution

$$x = \alpha + (bc' - b'c)\, n, \; y = \beta - (ac' - a'c)\, n,$$

where $x = \alpha$, $y = \beta$ is any integral solution, and n is any integer.

Now substitute these values of x and y in either of the original equations: we then obtain an equation of the form $Az + Bn = C$, from which we can obtain integral solutions of the form $z = \gamma + Bm$, $n = \delta - Am$, provided A and B are prime to one another, or will become so after division by any common factor which is also a factor of C.

Ex. Find integral solutions of the simultaneous equations

$$5x + 7y + 2z = 24, \quad 3x - y - 4z = 4.$$

Eliminating z, we have $13x + 13y = 52$, or $x + y = 4$. Whence $x = 2 + n$, $y = 2 - n$. Then $5(2+n) + 7(2-n) + 2z = 24$, that is $z - n = 0$.

Hence the general solution is $x = 2 + n$, $y = 2 - n$, $z = n$.

If x, y and z are to be *positive*, the only solutions are $x = 4$, $y = 0$, $z = 2$; $x = 3$, $y = 1$, $z = 1$; and $x = 2$, $y = 2$, $z = 0$; and, if zero values are excluded, there is only one solution, namely $x = 3$, $y = 1$, $z = 1$.

404. The following are examples of some other forms of indeterminate equations. Other cases will be found in Barlow's Theory of Numbers.

Ex. 1. Find the positive integral solutions (excluding zero values) of the equation $3x + 2y + 8z = 40$.

It is clear that z cannot be greater than 4, if zero and negative values of x and y are inadmissible.

Hence we have the following equations:

$$z=4, \quad 3x+2y=8;$$
$$z=3, \quad 3x+2y=16;$$
$$z=2, \quad 3x+2y=24;$$
$$z=1, \quad 3x+2y=32.$$

And it will be found that all the solutions required are 2, 1, 4; 4, 2, 3; 2, 5, 3; 6, 3, 2; 4, 6, 2; 2, 9, 2; 10, 1, 1; 8, 4, 1; 6, 7, 1; 4, 10, 1; and 2, 13, 1.

Ex. 2. Find the positive integral solutions of the equation

$$6x^2-13xy+6y^2=16.$$

We have $(3x-2y)(2x-3y)=16$; hence, as x and y are integers, $3x-2y$ must be an integer, and must therefore be a factor of 16. Thus one or other of the following simultaneous equations must hold good

$$3x-2y=\pm 16, \quad 2x-3y=\pm 1 \ldots\ldots \text{(i)};$$
$$3x-2y=\pm 8, \quad 2x-3y=\pm 2 \ldots\ldots \text{(ii)};$$
$$3x-2y=\pm 4, \quad 2x-3y=\pm 4 \ldots\ldots \text{(iii)};$$
$$3x-2y=\pm 2, \quad 2x-3y=\pm 8 \ldots\ldots \text{(iv)};$$
$$3x-2y=\pm 1, \quad 2x-3y=\pm 16 \ldots\ldots \text{(v)}.$$

Whence we find that $5x$ must be $\pm(48-2)$, $\pm(24-4)$, $\pm(12-8)$, $\pm(6-16)$ or $\pm(3-32)$.

Hence the only integral values of x are 4 and 2, the corresponding values of y being 2 and 4.

Ex. 3. Solve in positive integers the equation

$$3x^2+7xy-2x-5y-35=0.$$

We have $y(7x-5)+3x^2-2x-35=0$;

$$\therefore\ y+\frac{3x^2-2x-35}{7x-5}=0;$$

$$\therefore\ 7y+3x+\frac{x-245}{7x-5}=0;$$

$$\therefore\ 49y+21x+1-\frac{1710}{7x-5}=0.$$

Hence $\frac{1710}{7x-5}$ must be an integer, and therefore $7x-5$ must be a factor of 1710. Whence it will be found that the only positive integral solutions are $x=2$, $y=3$ and $x=1$, $y=17$.

EXAMPLES XL.

1. Find all the positive integral solutions of the equations:

(1) $7x + 15y = 59$. (2) $8x + 13y = 138$.
(3) $7x + 9y = 100$. (4) $15x + 71y = 10653$.

2. Find the number of positive integral solutions of

$$2x + 3y = 133 \text{ and of } 7x + 11y = 2312.$$

3. Find the general integral solutions of the equations

(1) $7x - 13y = 15$. (2) $9x - 11y = 4$.
(3) $119x - 105y = 217$. (4) $49x - 69y = 100$.

4. Find the positive integral solutions (excluding zero) of the equations

(1) $2x + 3y + 7z = 23$. (2) $7x + 4y + 18z = 109$.
(3) $5x + y + 7z = 39$, $2x + 4y + 9z = 63$. (4) $3x + 2y + 3z = 250$, $9x - 4y + 5z = 170$.

5. Solve in positive integers (excluding zero) the equations:

(i) $2xy - 3x + 2y = 1329$.
(ii) $x^2 - xy + 2x - 3y = 11$.
(iii) $2x^2 + 5xy - 12y^2 = 28$.
(iv) $2x^2 - xy - y^2 + 2x + 7y = 84$.

6. Shew that integral values of x, y and z which satisfy the equation $ax + by + cz = d$, form three arithmetical progressions.

7. Divide 316 into two parts so that one part may be divisible by 13 and the other by 11.

8. In how many ways can £1. 6*s.* 6*d.* be paid with half-crowns and florins?

9. In how many ways can £100 be made up of guineas and crowns?

10. In how many ways can a man who has only 8 crown pieces pay 11 shillings to another who has only florins?

11. Find the greatest and least sums of money which can be paid in eight ways and no more with half-crowns and florins, both sorts of coins being used.

12. Find all the different sums of money which can be paid in three ways and no more with four-penny pieces and three-penny pieces, both sorts of coins being used.

13. Find all the numbers of two digits which are multiples of the product of their digits.

14. Two numbers each of two digits, and which end with the same digit, are such that when divided by 9 the quotient of each is the remainder of the other. Find all the sets of numbers which satisfy the conditions.

15. A man's age in 1887 was equal to the sum of the digits in the year of his birth: how old was he?

16. Shew that, if

$$\frac{1}{(1-x^{a_1})(1-x^{a_2})\dots(1-x^{a_n})} = 1 + A_1 x + \dots + A_n x^n + \dots,$$

then the number of solutions in positive integers (including zero) of the equation $a_1x_1 + a_2x_2 + \dots + a_nx_n = m$, is A_m, $a_1, a_2, \dots, a_n$ being all integers.

The number of solutions of the equations $x + 2y = n$ is $\frac{1}{4}\{2n + 3 + (-1)^n\}$.

At an entertainment the prices of admission were 1*s.*, 2*s.* and £5, and the total receipts £1000; shew that there are 1005201 ways in which the audience might have been made up.

17. The money paid for admission to a concert was £300, the prices of admission being 5*s.*, 3*s.* and 1*s.*; shew that the number of ways in which the audience may have been made up is 1201801.

CHAPTER XXX.

PROBABILITY.

405. THE following is generally given as the definitions of *probability* or *chance*:—

Definition. If an event can happen in a ways and fail in b ways, and all these ways are *equally likely* to occur, then the probability of its happening is $\frac{a}{a+b}$ and the probability of its failing is $\frac{b}{a+b}$.

To make the above definition complete it is necessary to explain what is meant by 'equally likely.' Events are said to be *equally likely* when we have no reason to expect any one rather than any other. For example, if we have to draw a ball from a bag which we know contains unknown numbers of black and white balls, and none of any other colour, we have just as much reason to expect a black ball as a white; the drawing of a black ball and of a white one are thus equally likely. Hence, as either a black ball or a white ball must be chosen, the probability of drawing either is $\frac{1}{2}$, for there are two equally likely cases, in one of which the event happens and in the other it fails. Again, if we have to draw a ball from a bag which we know contains only black, white and red balls, but in unknown proportions, we have just as much reason to expect one colour as to expect either of the others, so that the drawing of a black, of a white and of a red ball

are all equally likely; and hence the probability of drawing any particular colour is $\frac{1}{3}$, for there are three equally likely cases, and any particular colour is drawn in one case and is not drawn in the other two cases.

Another meaning may however be given to 'equally likely;' for events may be said to be equally likely when they occur equally often, in the long run. For example, if a coin be tossed up, we may know that in a very great number of trials, although the number of 'heads' is by no means necessarily the same as the number of 'tails,' yet the ratio of these numbers becomes more and more nearly equal to unity as the number of trials is increased, and that the ratio of the number of heads to the number of tails will differ from unity by a very small fraction when the number of trials is very great; and this is what is meant by saying that heads and tails occur equally often *in the long run.*

Now, if each of the a ways in which an event can happen and each of the b ways in which it can fail occur equally often, in the long run, it follows that the event happens, in the long run, a times and fails b times out of every $a+b$ cases. We may therefore say, consistently with the former definition, that *the probability of an event is the ratio of the number of times in which the event occurs, in the long run, to the sum of the number of times in which events of that description occur and in which they fail to occur.*

Thus, if it be known that, in the long run, out of every 41 children born, there are 21 boys and 20 girls, the probability of any particular birth being that of a boy is $\frac{21}{41}$.

Again, if one of two players at any game win, in the long run, 5 games out of every 8, the probability of his winning any particular game is $\frac{5}{8}$.

We may remark that, in the great majority of cases, including all the cases of practical utility, such as the data used by Assurance Companies, the only way in which probability can be estimated is by the last method, namely, by finding the ratio of the actual number of times the event

occurs, in a large number of cases, to the whole number of times in which it occurs and in which it fails.

406. If an event is *certain* it will occur without fail in every case: its probability is therefore unity.

It follows at once from the definition of probability that if p be the probability that any event should occur, $1-p$ will be the probability of its failing to occur.

When the probability of the happening of an event is to the probability of its failure as a is to b, the *odds* are said to be a to b *for* the event, or b to a *against* it, according as a is greater or less than b.

407. **Exclusive events.** Events are said to be *mutually exclusive* when the supposition that any one takes place is incompatible with the supposition that any other takes place.

When different events are mutually exclusive the chance that one or other of the different events occurs is the sum of the chances of the separate events.

It will be sufficient to consider three events.

Let the respective probabilities of the three events, expressed as fractions with the same denominator, be

$$\frac{a_1}{d},\ \frac{a_2}{d} \text{ and } \frac{a_3}{d}.$$

Then, out of d equally likely ways, the three events can happen in a_1, a_2 and a_3 ways respectively.

Hence, as the events never concur, one or other of them will happen in $a_1+a_2+a_3$ out of d equally likely ways. Hence the probability of one or other of the three events happening is

$$\frac{a_1+a_2+a_3}{d}, \text{ that is } \frac{a_1}{d}+\frac{a_2}{d}+\frac{a_3}{d}.$$

This proves the proposition for three mutually exclusive events; and any other case can be proved in a similar manner.

Ex. 1. Find the chance of throwing 3 with an ordinary six-faced die.

Since any one face is as likely to be exposed as any other face, there is one favourable and five unfavourable cases which are all equally likely; hence the required probability is $\frac{1}{6}$.

Ex. 2. Find the chance of throwing an odd number with an ordinary die. *Ans.* $\frac{1}{2}$.

Ex. 3. Find the chance of drawing a red ball from a bag which contains 5 white and 7 red balls.

Here any one ball is as likely to be drawn as any other; thus there are 7 favourable and 5 unfavourable cases which are all equally likely; the required probability is therefore $\frac{7}{12}$.

Ex. 4. Two balls are to be drawn from a bag containing 5 red and 7 white balls; find the chance that they will both be white.

Here any one pair of balls is as likely to be drawn as any other pair. The total number of pairs is ${}_{12}C_2$, and the number of pairs which are both white is ${}_7C_2$: the required chance is therefore

$$\frac{7.6}{1.2} \Big/ \frac{12.11}{1.2} = \frac{7}{22}.$$

Ex. 5. Shew that the odds are 7 to 3 against drawing 2 red balls from a bag containing 3 red and 2 white balls.

Ex. 6. Three balls are to be drawn from a bag containing 2 black, 2 white and 2 red balls; shew that the odds are 3 to 2 against drawing a ball of each colour, and 4 to 1 against drawing 2 white balls.

Ex. 7. A party of n persons take their seats at random at a round table: shew that it is $n-3$ to 2 against two specified persons sitting together.

408. **Independent Events.** *The probability that two independent events should both happen is the product of the separate probabilities of their happening.*

Suppose that the first event can happen in a_1 and fail in b_1 equally likely ways; and suppose that the second event can happen in a_2 and fail in b_2 equally likely ways. Then each of the a_1+b_1 cases may be associated with each of the a_2+b_2 cases to make $(a_1+b_1)(a_2+b_2)$ compound cases which are all equally likely; and in a_1a_2 of these compound cases both events happen. Hence the proba-

bility that both events happen is $\dfrac{a_1a_2}{(a_1+b_1)(a_2+b_2)}$, that is $\dfrac{a_1}{a_1+b_1} \times \dfrac{a_2}{a_2+b_2}$, which proves the proposition.

Thus the probability of the concurrence of two independent events whose respective probabilities are p_1 and p_2 is $p_1 \times p_2$.

Cor. If p_1 and p_2 be the probabilities of two independent events, the chance that they will both fail is $(1-p_1)(1-p_2)$, the chance that the first happens and the second fails is $p_1(1-p_2)$, and the chance that the second happens and the first fails is $(1-p_1)p_2$.

It can be shewn in a similar manner that, if $p_1, p_2, p_3, \ldots$ be the probabilities of any number of independent events, then the probability that they all happen will be $p_1 . p_2 . p_3 \ldots$, and that they all fail $(1-p_1)(1-p_2)(1-p_3)\ldots$, &c.

409. **Dependent Events.** If two events are not independent, but the probability of the second is different when the first happens from what it is when the first fails, the reasoning of the previous article will still hold good provided that p_2 is the probability that the second event happens when the first is known to have happened. Thus if p_1 be the probability of any event, and p_2 the probability of any other event on the supposition that the first has happened; then the probability that both events will happen in the order specified will be $p_1 \times p_2$. And similarly for any number of dependent events.

Ex. 1. Find the probability of throwing two heads with two throws of a coin.

The probability of throwing heads is $\frac{1}{2}$ for each throw; hence the required probability is, by Art. 408, $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Ex. 2. Find the probability of throwing one 6 at least in six throws with a die.

The probability of not throwing 6 is $\frac{5}{6}$ in each throw. Hence the probability of not throwing a 6 in six throws is, by Art. 408, $\left(\frac{5}{6}\right)^6$, and therefore the probability of throwing one six at least is $1-\left(\frac{5}{6}\right)^6$.

Ex. 3. Find the chance of drawing 2 white balls in succession from a bag containing 5 red and 7 white balls, the balls drawn not being replaced.

The chance of drawing a white ball the first time is $\frac{7}{12}$; and, having drawn a white ball the first time, there will be 5 red and 6 white balls left, and therefore the chance of drawing a white ball the second time will be $\frac{6}{11}$. Hence, from Art. 409, the chance of drawing two white balls in succession will be $\frac{7}{12}\times\frac{6}{11}=\frac{7}{22}$.

[Compare Ex. 4, Art. 407.]

Ex. 4. There are two bags, one of which contains 5 red and 7 white balls and the other 3 red and 12 white balls, and a ball is to be drawn from one or other of the two bags; find the chance of drawing a red ball.

The chance of choosing the first bag is $\frac{1}{2}$, and if the first bag be chosen the chance of drawing a red ball from it is $\frac{5}{12}$; hence the chance of drawing a red ball from the first bag is $\frac{1}{2}\times\frac{5}{12}=\frac{5}{24}$. Similarly the chance of drawing a red ball from the second bag is $\frac{1}{2}\times\frac{3}{15}=\frac{1}{10}$. Hence, as these events are mutually exclusive, the chance required is $\frac{5}{24}+\frac{1}{10}=\frac{37}{120}$.

Ex. 5. In two bags there are to be put altogether 2 red and 10 white balls, neither bag being empty. How must the balls be divided so as to give to a person who draws one ball from either bag, (1) the least chance and (2) the greatest chance of drawing a red ball.

[The least chance is when one bag contains only one white ball, and the greatest chance is when one bag contains only one red ball, the chances being $\frac{1}{11}$ and $\frac{6}{11}$ respectively.]

410. When the probability of the happening of an event in one trial is known, the probability of its happening exactly once, twice, three times, &c. in n trials can be at once written down.

For, if p be the probability of the happening of the event, the probability of its failing is $1-p=q$. Hence, from Art. 408, the probability of its happening r times and failing $n-r$ times *in any specified order is* $p^r q^{n-r}$. But the whole number of ways in which the event could happen r times exactly in n trials is ${}_nC_r$, and these ways are all equally probable and are mutually exclusive. Hence the probability of the event happening r times exactly in n trials is ${}_nC_r p^r q^{n-r}$.

Thus, if $(p+q)^n$ be expanded by the binomial theorem, the successive terms will be the probability of the happening of the event exactly n times, $n-1$ times, $n-2$ times, &c. in n trials.

Cor. I. To find the most probable number of successes and failures in n trials it is only necessary to find the greatest term in the expansion of $(p+q)^n$.

Cor. II. The probability of the event happening *at least* r times in n trials is

$$p^n + n.p^{n-1}q + \frac{n(n-1)}{1.2}p^{n-2}q^2 + \ldots + \frac{\lfloor n}{\lfloor r \lfloor n-r}p^r q^{n-r}.$$

Ex. 1. Find the chance of throwing 10 with 4 dice.

The whole number of different throws is 6^4, for any one of six numbers can be exposed on each die; also the number of ways of throwing 10 is the coefficient of x^{10} in $(x+x^2+\ldots+x^6)^4$, for this coefficient gives the number of ways in which 10 can be made up by the addition of four of the numbers 1, 2, ..., 6, repetitions being allowed. Now the coefficient of x^{10} in $(x+x^2+\ldots+x^6)^4$, that is in $x^4\left(\frac{1-x^6}{1-x}\right)^4$, is easily found to be 80. Hence the required chance is

$$\frac{80}{6.6.6.6}=\frac{5}{81}.$$

Ex. 2. Find the chance of throwing 8 with two dice. *Ans.* $\frac{5}{36}$.

Ex. 3. Find the chance of throwing 10 with two dice. *Ans.* $\frac{1}{12}$.

Ex. 4. Find the chance of throwing 15 with three dice. *Ans.* $\frac{5}{108}$.

Ex. 5. A and B each throws a die; shew that it is 7 : 5 that A's throw is not greater than B's.

Ex. 6. A and B each throw with two dice: find the chance that their throws are equal. *Ans.* $\frac{73}{648}$.

Ex. 7. A and B have equal chances of winning a single game at tennis: find the chance of A winning the 'set' (1) when A has won 5 games and B has won 4, (2) when A has won 5 games and B has won 3, and (3) when A has won 4 games and B has won 2.

Ans. (1) $\frac{3}{4}$, (2) $\frac{7}{8}$, (3) $\frac{13}{16}$.

Ex. 8. A and B have equal chances of winning a single game; and A wants 2 games and B wants 3 games to win a match: shew that it is 11 to 5 that A wins the match.

Ex. 9. A and B have equal chances of winning a single game; and A wants n games and B wants $n+1$ games to win a match: shew that the odds on A are $1+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}$ to $1-\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}$.

Ex. 10. A's chance of winning a single game against B is $\frac{3}{5}$: find the chance of his winning at least 2 games out of 3. *Ans.* $\frac{81}{125}$.

Ex. 11. A's chance of winning a single game against B is $\frac{2}{3}$: find the chance of his winning at least 3 games out of 5. *Ans.* $\frac{192}{243}$.

Ex. 12. What is the chance of throwing at least 2 sixes in 6 throws with a die? *Ans.* $\frac{12281}{46656}$.

v Ex. 13. A coin is tossed five times in succession: shew that it is an even chance that three consecutive throws will be the same.

Ex. 14. Three men toss in succession for a prize which is to be given to the first who gets 'heads'. Find their respective chances.

Ans. $\frac{4}{7}$, $\frac{2}{7}$, $\frac{1}{7}$.

411. The value of a given chance of obtaining a given sum of money is called the *expectation*.

If $\dfrac{a}{a+b}$ is the chance of obtaining a sum of money M, then the expectation is $M \times \dfrac{a}{a+b}$.

For if E be the expectation in one trial, $E(a+b)$ will be the expectation in $a+b$ trials. But the chance being $\dfrac{a}{a+b}$, the sum M will, on the average, be won a times in every $a+b$ trials; and hence the expectation in $a+b$ trials is Ma. Hence $E(a+b) = Ma$; therefore

$$E = M \times \frac{a}{a+b}.$$

Thus the expectation is the sum which may be won multiplied by the chance of winning it.

Ex. 1. A bag contains 5 white balls and 7 black ones. Find the expectation of a man who is allowed to draw a ball from the bag and who is to receive one shilling if he draws a black ball, and a crown if he draws a white one.

The chance of drawing a black ball is $\frac{7}{12}$; and therefore the expectation from drawing a black ball is 7*d*. The chance of drawing a white ball is $\frac{5}{12}$; and therefore the expectation from drawing a white ball is 2*s*. 1*d*. Hence, as these events are exclusive, the whole expectation is 2*s*. 8*d*.

Ex. 2. A purse contains 2 sovereigns, 3 half-crowns and 7 shillings. What should be paid for permission to draw (1) one coin and (2) two coins? *Ans.* (1) 4*s*. 6½*d*. (2) 9*s*. 1*d*.

Ex. 3. Two persons toss a shilling alternately on condition that the first who gets 'heads' wins the shilling: find their expectations.

Ans. 8*d*., 4*d*.

Ex. 4. Two persons throw a die alternately, and the first who throws 6 is to receive 11 shillings: find their expectations.

Ans. 6*s*., 5*s*.

412. Inverse Probability. When it is known that an event has happened and that it must have followed from some one of a certain number of causes, the determination of the probabilities of the different possible causes is said to be a problem of *inverse* probability.

For example, it may be known that a black ball was drawn from one or other of two bags, one of which was known to contain 2 black and 7 white balls and the other 5 black and 4 white balls; and it may be required to determine the probability that the ball was drawn from the first bag.

Now, if we suppose a great number, $2N$, of drawings to be made, there will in the long run be N from each bag. But in N drawings from the first bag there are, on the average, $\frac{2}{9}N$ which give a black ball; and in N drawings from the second bag there are $\frac{5}{9}N$ which give a black ball. Hence, in the long run, $\frac{2}{9}N$ out of a total of $\frac{2}{9}N+\frac{5}{9}N$ black balls are due to drawings from the first bag; thus the probability that the ball was drawn from the first bag is $\frac{2}{9}N\div\left(\frac{2}{9}N+\frac{5}{9}N\right)$, that is $\frac{2}{7}$.

We now proceed to the general proposition:—

Let $P_1, P_2, \ldots, P_n$ be the probabilities of the existence of n causes, which are mutually exclusive and are such that a certain event must have followed from one of them; and let $p_1, p_2, \ldots, p_n$ be the respective probabilities that when one of the causes $P_1, P_2, \ldots, P_n$ exists it will be followed by the event in question; then on any occasion when the event is known to have occurred the probability of the rth cause is

$$P_r p_r \div (P_1 p_1 + P_2 p_2 + \ldots + P_n p_n).$$

Let a great number N of trials be made; then the first cause will exist in $N.P_1$ cases, and the event will follow in $N.P_1.p_1$ cases. So also the second cause exists and the event follows in $N.P_2.p_2$ cases; and so on.

Hence the event is due to the rth cause in $N.P_r.p_r$

cases out of a total of $N(P_1p_1+P_2p_2+\ldots+P_np_n)$; the probability of the rth cause is therefore $\dfrac{P_rp_r}{\Sigma Pp}$.

Having found the probability of the existence of each of the different causes, the probability that the event would occur on a second trial can be at once found.

For let P_r' be the probability of the existence of the rth cause; then p_r is the probability that the event will happen when the rth cause exists; and therefore $P_r'.p_r$ is the probability that the event will happen from the rth cause.

Hence, as the causes are mutually exclusive, the probability that the event would happen on a second trial is

$$P_1'.p_1+P_2'.p_2+\ldots+P_n'.p_n.$$

Ex. 1. There are 3 bags which are known to contain 2 white and 3 black, 4 white and 1 black, and 3 white and 7 black balls respectively. A ball was drawn at random from one of the bags and found to be a black ball. Find the chance that it was drawn from the bag containing the most black balls.

Here $P_1=P_2=P_3=\frac{1}{3}$. Also $p_1=\frac{3}{5}$, $p_2=\frac{1}{5}$ and $p_3=\frac{7}{10}$.

Hence the required probability is $\dfrac{\frac{1}{3}.\frac{7}{10}}{\frac{1}{3}.\frac{3}{5}+\frac{1}{3}.\frac{1}{5}+\frac{1}{3}.\frac{7}{10}}=\dfrac{7}{15}$.

Ex. 2. From a bag which is known to contain 4 balls each of which is just as likely to be black as white, a ball is drawn at random and found to be white. Find the chance that the bag contained 3 white and 1 black balls.

The bag may have contained (1) 4 white, (2) 3 white and 1 black, (3) 2 white and 2 black, (4) 1 white and 3 black, and (5) 4 black; and the chances of these are respectively $\frac{1}{16}$, $\frac{4}{16}$, $\frac{6}{16}$, $\frac{4}{16}$ and $\frac{1}{16}$. Art. 410. Also the chances of drawing a white ball in these different cases will be 1, $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{4}$ and 0 respectively.

Hence the required probability $=\dfrac{\frac{4}{16}.\frac{3}{4}}{\frac{1}{16}+\frac{3}{4}.\frac{4}{16}+\frac{1}{2}.\frac{6}{16}+\frac{1}{4}.\frac{4}{16}}=\dfrac{3}{8}$.

413. **Probability of testimony.** The method of dealing with questions relating to the credibility of witnesses will be seen from the following examples:

Ex. 1. A ball has been drawn at random from a bag containing 99 black balls and 1 white ball; and a man whose statements are accurate 9 times out of 10 asserts that the white ball was drawn. Find the chance that the white ball was really drawn.

The probability that the white ball will really be drawn in any case is $\frac{1}{100}$, and therefore the probability that the man will *truly* assert that the white ball is drawn is $\frac{1}{100} \times \frac{9}{10}$.

The probability that the white ball will not be drawn is $\frac{99}{100}$, and therefore the probability that the man will *falsely* assert that the white ball is drawn is $\frac{99}{100} \times \frac{1}{10}$.

Hence as in Art. 412 the required probability is

$$\frac{\frac{1}{100} \times \frac{9}{10}}{\frac{1}{100} \times \frac{9}{10} + \frac{99}{100} \times \frac{1}{10}} = \frac{1}{12}.$$

Ex. 2. From a bag containing 100 tickets numbered 1, 2, ..., 100 respectively, a ticket has been drawn at random; and a witness, whose statements are accurate 9 times out of 10, asserts that a particular ticket has been drawn. Find the chance that this ticket was really drawn.

In $1000N$ trials the ticket in question will be drawn $10N$ times; and the witness will *correctly* assert that it has been drawn $9N$ times. The ticket will not be drawn in $990N$ cases, and the witness will make a wrong assertion in $99N$ of these cases; but there are 99 ways of making a wrong assertion and these may all be supposed to be equally likely; hence the witness will *wrongly* assert that the particular ticket has been drawn in N cases. Hence the required probability is $\frac{9}{10}$, so that the probability is in this case equal to the probability of the witness speaking the truth.

Ex. 3. A speaks the truth three times out of four, and B five times out of six; and they agree in stating that a white ball has been drawn from a bag which was known to contain 1 white and 9 black balls. Find the chance that the white ball was really drawn.

The probability that the white ball will be drawn in any case is

$\frac{1}{10}$, and therefore the probability that A and B will agree in truly asserting that a white ball is drawn is $\frac{1}{10} \times \frac{3}{4} \times \frac{5}{6}$.

The probability that a black ball will really be drawn in any case is $\frac{9}{10}$; and therefore the probability that A and B will agree in falsely asserting that a white ball is drawn is $\frac{9}{10} \times \frac{1}{4} \times \frac{1}{6}$.

Hence, as in Art. 412, the required probability is

$$\frac{\frac{1}{10} \times \frac{3}{4} \times \frac{5}{6}}{\frac{1}{10} \times \frac{3}{4} \times \frac{5}{6} + \frac{9}{10} \times \frac{1}{4} \times \frac{1}{6}} = \frac{5}{8}.$$

Ex. 4. A speaks truth three times out of four, and B five times out of six; and they agree in stating that a white ball has been drawn from a bag which was known to contain 10 balls all of different colours, white being one. What is the chance that a white ball was really drawn?

The probability that the white ball will really be drawn in any case is $\frac{1}{10}$, and therefore the probability that A and B will agree in truly asserting that the white ball is drawn is $\frac{1}{10} \times \frac{3}{4} \times \frac{5}{6} = \frac{1}{16}$.

The probability that the white ball will not be drawn in any case is $\frac{9}{10}$. The probability that A will make a wrong statement is $\frac{1}{4}$; hence, as there are nine ways of making a wrong statement which may all be supposed to be equally likely, the chance that A will wrongly assert that a white ball is drawn is $\frac{1}{4} \times \frac{1}{9}$. Therefore the chance that A and B will agree in falsely asserting that a white ball is drawn is

$$\frac{9}{10} \times \frac{1}{4 \times 9} \times \frac{1}{6 \times 9} = \frac{1}{2160}.$$

Hence the required probability is $$\frac{\frac{1}{16}}{\frac{1}{16} + \frac{1}{2160}} = \frac{135}{136}.$$

Ex. 5. It is 3 to 1 that A speaks truth, 4 to 1 that B does and 6 to 1 that C does: find the probability that an event really took place which A and B assert to have happened and which C denies; the event being, independently of this evidence, as likely to have happened as not. *Ans.* $\frac{2}{3}$.

414. We shall conclude this chapter by considering the following examples, referring the reader who wishes for fuller information on the subject of Probabilities to the article in the Encyclopaedia Britannica, and to Todhunter's History of the Mathematical Theory of Probability.

Ex. 1. A bag contains n balls, and all numbers of white balls from 0 to n are equally likely; find the chance that r white balls in succession will be drawn, the balls not being replaced.

The chance that the bag contains s white balls is $\frac{1}{n+1}$; and the chance that r balls in succession will be drawn from a bag containing n balls of which s are white is $\frac{s(s-1)\dots(s-r+1)}{n(n-1)\dots(n-r+1)}$.

Hence the chance required is

$$\frac{1}{n+1}\left\{\frac{n(n-1)\dots(n-r+1)}{n(n-1)\dots(n-r+1)}+\frac{(n-1)(n-2)\dots(n-r)}{n(n-1)\dots(n-r+1)}+\dots\right.$$
$$\left.\dots+\frac{r(r-1)\dots1}{n(n-1)\dots(n-r+1)}\right\}.$$

Now $\{1.2\dots r\}+\{2.3\dots(r+1)\}+\dots+\{(n-r+1)\dots(n-1)n\}$

$$=\frac{(n-r+1)(n-r+2)\dots n(n+1)}{r+1}, \text{ by Art. 318.}$$

Hence the required chance is $\frac{1}{r+1}$, which is independent of the whole number of balls in the bag.

If it be known that r white balls in succession have been drawn, the probability of the next drawing giving a white ball can be at once found from the preceding result.

For in a great number N, of cases, there will be r white balls in succession in $\frac{N}{r+1}$ cases, and $r+1$ white balls in succession in $\frac{N}{r+2}$ cases. Hence the required chance is $\frac{N}{r+2}\div\frac{N}{r+1}=\frac{r+1}{r+2}$.

Ex. 2. Two men A and B, who have a and b counters respectively to begin with, play a match consisting of separate games, none of which can be drawn, and the winner of a game receives a counter from the loser. Find their respective chances of winning the match, which is supposed to be continued until one of the players has no more counters, the odds being $p:q$ that A wins any particular game.

Let A's chance of ultimate success when he has n counters be u_n. Then A's chance of winning the next game is $\frac{p}{p+q}$, and his chance of ultimate success will then be u_{n+1}; also A's chance of losing the next game is $\frac{q}{p+q}$, and his chance of ultimate success will then be u_{n-1}.

Hence $u_n = \frac{p}{p+q} u_{n+1} + \frac{q}{p+q} u_{n-1}$;

$\therefore pu_{n+1} - (p+q) u_n + qu_{n-1} = 0$, from which it follows that u_n will be the coefficient of x^n in the expansion of $\frac{A+Bx}{p-(p+q)x+qx^2}$, provided A and B be properly chosen.

Now $\frac{A+Bx}{p-(p+q)x+qx^2}$ can be expressed in the form $\frac{C}{p-qx} + \frac{D}{1-x}$; and hence the coefficient of x^n is $D + \frac{C}{p}\left(\frac{q}{p}\right)^n$.

Thus $u_n = D + \frac{C}{p}\left(\frac{q}{p}\right)^n$, where C and D have to be determined. But it is obvious that A's chance of winning is zero if he has no counters and unity if he has $a+b$, so that $u_0 = 0$ and $u_{a+b} = 1$; hence $0 = D + \frac{C}{p}$, and $1 = D + \frac{C}{p}\left(\frac{q}{p}\right)^{a+b}$, whence the values of C and D are found, and we have

$$u_n = \left\{1 - \left(\frac{q}{p}\right)^n\right\} \bigg/ \left\{1 - \left(\frac{q}{p}\right)^{a+b}\right\}.$$

Hence A's chance of winning the game is

$$\left\{1 - \left(\frac{q}{p}\right)^a\right\} \bigg/ \left\{1 - \left(\frac{q}{p}\right)^{a+b}\right\}.$$

Similarly B's chance of winning the game is

$$\left\{1 - \left(\frac{p}{q}\right)^b\right\} \bigg/ \left\{1 - \left(\frac{p}{q}\right)^{a+b}\right\}.$$

EXAMPLES XLI.

1. A and B throw alternately with two dice, and a prize is to be won by the one who first throws 8. Find their respective chances of winning if A throws first.

2. A, B and C throw alternately with three dice, and a prize is to be won by the one who first throws 6. Find their respective chances of winning if they throw in the order A, B, C.

3. Three white balls and five black are placed in a bag, and three men draw a ball in succession (the balls drawn not being replaced) until a white ball is drawn: shew that their respective chances are as 27 : 18 : 11.

4. What is the most likely number of sixes in 50 throws of a die?

5. Shew that with two dice the chance of throwing more than 7 is equal to the chance of throwing less than 7.

6. In a bag there are three tickets numbered 1, 2, 3. A ticket is drawn at random and put back; and this is done four times: shew that it is 41 to 40 that the sum of the numbers drawn is even.

7. From a bag containing 100 tickets numbered 1, 2, 3,... 100, two tickets are drawn at random; shew that it is 50 to 49 that the sum of the numbers on the tickets will be odd.

8. There are n tickets in a bag numbered 1, 2, ..., n. A man draws two tickets together at random, and is to receive a number of shillings equal to the product of the numbers he draws: find the value of his expectation.

9. An event is known to have happened n times in n years: shew that the chance that it did not happen in a particular year is $\left(1-\frac{1}{n}\right)^n$.

10. If p things be distributed at random among p persons; shew that the chance that one at least of the persons will be void is $\frac{p^p - \lfloor p}{p^p}$.

11. A writes a letter to B and does not get an answer; assuming that one letter in m is lost in passing through the post, shew that the chance that B received the letter is $\frac{m-1}{2m-1}$, it being considered certain that B would have answered the letter if he had received it.

12. From a bag containing 3 sovereigns and 3 shillings, four coins are drawn at random and placed in a purse; two coins are then drawn out of the purse and found to be both sovereigns. Shew that the value of the expectation of the remaining coins in the purse is 11*s.* 6*d.*

13. From a bag containing 4 sovereigns and 4 shillings, four coins are drawn at random and placed in a purse; two coins are then drawn out of the purse and found to be both sovereigns. Shew that the probable value of the coins left in the bag is $29\frac{1}{3}$ shillings.

14. If three points are taken at random on a circle the chance of their lying on the same semi-circle is $\frac{3}{4}$.

15. A rod is broken at random into three pieces: find the chance that no one of the pieces is greater than the sum of the other two.

16. A rod is broken at random into four pieces: find the chance that no one of the pieces is greater than the sum of the other three.

17. Three of the sides of a regular polygon of $4n$ sides are chosen at random; prove that the chance that they being produced will form an acute-angled triangle which will contain the polygon is $\dfrac{(n-1)(n-2)}{(4n-1)(4n-2)}$.

18. Out of m persons who are sitting in a circle three are selected at random; prove that the chance that no two of those selected are sitting next one another is $\dfrac{(m-4)(m-5)}{(m-1)(m-2)}$.

19. If m odd integers and n even integers be written down at random, shew that the chance that no two odd numbers are adjacent to one another is $\dfrac{n!\,(n+1)!}{(m+n)!\,(n-m+1)!}$, m being $\not> n+1$.

20. If m things are distributed amongst a men and b women, shew that the chance that the number of things received by the group of men is odd, is $\dfrac{1}{2}\,\dfrac{(b+a)^m-(b-a)^m}{(b+a)^m}$.

21. The sum of two whole numbers is 100; find the chance that their product is greater than 1000.

22. The sum of two positive quantities is given; prove that it is an even chance that their product will not be less than three-fourths of their greatest product; prove also that the chance of their product being less than one-half their greatest product is $1 - \frac{1}{\sqrt{2}}$.

23. Two men A and B have a and b counters respectively, and they play a match consisting of separate games, none of which can be drawn, and the winner of a game receives a counter from the loser. The two players have an equal chance of winning any single game, and the match is continued until one of the players has no more counters. Shew that A's chance of winning the match is $\frac{a}{a+b}$.

24. An urn contains a number of balls which are known to be either white or black, and all numbers are equally likely. If the result of $p+q$ drawings (the balls not being replaced) is to give p white and q black balls, shew that the chance that the next drawing will give a black ball is $\frac{q+1}{p+q+2}$.

25. Two sides play at a game in which the total number of points that can be scored is $2m+1$; and the chances of any point being scored by one side or the other are as $2m+1-x$ to $2m+1-y$, where x and y are the points already scored by the respective sides. Shew that the chance that the side which scores the first point will just win the game is

$$\frac{(2m!\ 2m+1!)^2}{(m!)^3\, m+1!\ 4m+1!}.$$

CHAPTER XXXI.

DETERMINANTS.

415. IF there are nine quantities arranged in a square as under:

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}$$

then all the possible products of the quantities three together, subject to the condition that of the three quantities in each product one and only one is taken from each of the rows and one and only one from each of the columns, will be

$$a_1b_2c_3,\ a_1b_3c_2,\ a_2b_3c_1,\ a_2b_1c_3,\ a_3b_1c_2, \text{ and } a_3b_2c_1.$$

Let now these products be considered to be positive or negative according as there is an even or an odd number of *inversions* of the natural order in the suffixes; then the algebraic sum of all the products will be

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \ldots\ldots (A);$$

for there are no inversions in $a_1b_2c_3$, there is one inversion in $a_1b_3c_2$ since 3 precedes 2, there are two inversions in $a_2b_3c_1$ since 2 and 3 both precede 1, there is one inversion in $a_2b_1c_3$ since 2 precedes 1, there are two inversions in $a_3b_1c_2$ since 3 precedes both 1 and 2, and there are three inversions in $a_3b_2c_1$ since 3 precedes both 1 and 2 and 2 precedes 1.

The expression (A) is called the *determinant* of the nine quantities a_1, a_2, &c., which are called its *elements;* and the products $a_1b_2c_3$, $a_1b_3c_2$, &c. are called the *terms* of the *determinant.*

416. **Definition.** If there are n^2 quantities arranged in a square as under:

$$\begin{vmatrix} a_1 & a_2 & a_3 & \cdots\cdots & a_n \\ b_1 & b_2 & b_3 & \cdots\cdots & b_n \\ c_1 & c_2 & c_3 & \cdots\cdots & c_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_1 & m_2 & m_3 & \cdots\cdots & m_n \end{vmatrix},$$

the members of the same row being distinguished by the same letter, and the members of the same column by the same suffix; and if all the possible products of the quantities n at a time are taken subject to the condition that of the n quantities in each product one and only one is taken from every row and one and only one from every column, and if the sign of each product is considered to be positive or negative according as there is an even or an odd number of *inversions* of the natural order in the suffixes; then the algebraic sum of all the products so formed is called the *determinant* of the n^2 quantities or *elements.*

To denote that the n^2 quantities are to be operated upon in the manner above described, they are enclosed by two lines, as in the above scheme.

The diagonal through the left-hand top corner is called the *principal diagonal;* and the product of the n elements a_1, b_2, c_3,......, m_n which lie along it, is called the *principal term* of the determinant.

All the other terms can be formed in order from the principal term by taking the letters in their alphabetical order and permuting the suffixes in every possible way: on this account a determinant is sometimes represented by enclosing its principal term in brackets; thus the above determinant would be written $[a_1b_2c_3...m_n]$, the

determinant is also often represented by the notation $\Sigma(\pm a_1 b_2 c_3 \ldots m_n)$.

When only one determinant is considered it is generally denoted by the symbol Δ.

A determinant is said to be of the nth *order* when there are n elements in each of its rows or columns, and therefore also n elements in each of its terms.

417. Since there are as many terms in a determinant of the nth order as there are permutations of the n suffixes, it follows that there are $\underline{|n}$ terms in a determinant of the nth order. There are, for example, six terms in a determinant of the third order.

418. The law by which the sign of any term of a determinant is found is equivalent to the following:

Take the elements in order from the successive rows beginning at the first; then the sign of any term is positive or negative according as there is an even or an odd number of inversions in the order of the columns from which the elements are taken.

We will now shew that the words *row* and *column* may be interchanged in the above law. To prove this, consider any product, for example, $a_5 b_3 c_1 d_4 e_6 f_2$ and its equivalent $c_1 f_2 b_3 d_4 a_5 e_6$, where in the first form the letters follow the alphabetical order and in the second form the numbers follow the natural order.

We have to shew that the number of inversions in the suffixes in the first form is the same as the number of inversions of the alphabetical order in the second form. This follows immediately from the fact that if, in the first form, any suffix follow r suffixes greater than itself; then, in the second form, the letter corresponding to that suffix must precede r letters earlier than itself in alphabetical order. Thus, in the example, 2 follows four suffixes greater than itself in $a_5 b_3 c_1 d_4 e_6 f_2$, and f precedes four letters earlier than itself in $c_1 f_2 b_3 d_4 a_5 e_6$.

Since the words rows and columns are interchangeable in the law which determines the sign of any term, we have the following

Theorem. *A determinant is unaltered by changing its rows into columns and its columns into rows.*

For example
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Ex. 1. Count the number of inversions in 2314, 3142 and 4231.

Ans. 2, 3, 5.

Ex. 2. Count the number of inversions in 4132, 35142 and 531264.

Ans. 4, 6, 7.

Ex. 3. What are the signs of the terms bfg, cdh and ceg in the determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$?

[The order of the columns is 231, 312 and 321.]

Ans. +, +, −.

Ex. 4. What are the signs of the terms $bgiq$, $celn$ and $dfkm$ in the determinant $\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{vmatrix}$?

[The order of the columns is 2314, 3142 and 4231.]

Ans. +, −, −.

419. **Theorem I.** *If in any term of a determinant any two suffixes be interchanged, another term of the determinant will be obtained whose sign is opposite to that of the original term.*

Let $P.h_\alpha.k_\beta$ be any term of a determinant, P being the product of all the elements except h_α and k_β; then, by interchanging α and β we have $P.h_\beta.k_\alpha$. Now since $P.h_\alpha.k_\beta$ is a term of the determinant, P can contain no element from the rows of h's and k's and no element from

the α or β columns; and this is a sufficient condition that $Ph_\beta k_\alpha$ should also be a term of the determinant.

We have now to shew that the two terms have different signs.

First suppose that two consecutive suffixes are interchanged.

Consider the term $Ah_\alpha k_\beta B$ where A denotes the product of all the elements which precede h_α and B the product of all the elements which follow k_β. By interchanging α and β we have $Ah_\beta k_\alpha B$, which we have already found is a term of the determinant.

Now the number of inversions in the two terms must be the same so far as the suffixes contained in A, or in B, are concerned, whether compared with one another or with α and β; but there must be an inversion in one or other of $\alpha\beta$ and $\beta\alpha$ but not in both. Hence the numbers of the inversions in the two terms differ by unity, and therefore the signs of the terms must be different.

Now suppose that two non-consecutive suffixes are interchanged; and let there be r elements between the two whose suffixes, α and β suppose, are to be interchanged.

Then α will be brought into the place of β by $r+1$ interchanges of consecutive suffixes, and β can then be brought into the original place occupied by α by r interchanges of consecutive suffixes; and therefore the interchange of α and β can be made by means of $2r+1$, that is by an *odd* number, of interchanges of successive suffixes. But, by the first case, each such interchange gives rise to a loss or gain of one inversion; and hence there must on the whole be a loss or gain of an *odd* number of inversions: the sign of the new term will therefore be different from the sign of the original term.

420. **Theorem II.** *A determinant is unaltered in absolute value, but is changed in sign, by the interchange of any two columns or any two rows.*

Suppose that in any determinant the rows in which the letters h and k occur are interchanged. Then, if

$A . h_\alpha . B . k_\beta . C$ be any term of the original determinant, the term of the new determinant formed by the elements which occur in the same places as before will be $Ak_\alpha Bh_\beta C$; and these two terms must have the same sign in the two determinants. Now by Art. 419 we know that $A . k_\alpha . B . h_\beta . C$ is a term of the original determinant and that its sign is *different* from that of $A . h_\alpha . B . k_\beta . C$. Hence any term of the new determinant is also a term of the original determinant but the sign of the term is different: the two determinants must therefore be equal in absolute magnitude but different in sign.

The proposition being true for *rows* is, from Art. 418, true also for *columns*.

For example

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & a_3 & a_2 \\ b_1 & b_3 & b_2 \\ c_1 & c_3 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_3 & a_2 \\ c_1 & c_3 & c_2 \\ b_1 & b_3 & b_2 \end{vmatrix}.$$

421. **Theorem III.** *A determinant, in which two rows or two columns are identical, is equal to zero.*

When two rows (or two columns) are identical, the determinant is unaltered either in sign or magnitude by the interchange of these two rows (or columns). But, by Theorem II, the interchange of any two rows (or columns) of a determinant changes its sign. Thus the determinant is not altered in value by changing its sign: its value must therefore be zero.

Ex. 1. Find the value of $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

It is obvious that two rows would become identical, and therefore the determinant would vanish, if $a=b$. Hence Δ must be equal to an expression which has $a-b$ as a factor. Similarly $b-c$ and $c-a$ must be factors of Δ. But Δ is by inspection seen to be of the third degree in a, b, c; hence $\Delta = L(b-c)(c-a)(a-b)$, where L is numerical. The principal term of Δ is bc^2 and this is the only term which gives bc^2, and the coefficient of bc^2 in $L(b-c)(c-a)(a-b)$ is L; therefore $L=1$. Thus $\Delta=(b-c)(c-a)(a-b)$.

Ex. 2. Find the value of $\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$.

Ans. $-(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)$.

Ex. 3. Find the value of $\begin{vmatrix} 1 & a & a^2 & a^4 \\ 1 & b & b^2 & b^4 \\ 1 & c & c^2 & c^4 \\ 1 & d & d^2 & d^4 \end{vmatrix}$.

Ans. $-(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)(a+b+c+d)$.

422. **Theorem IV.** *If all the elements of one row or of one column of a determinant be multiplied by the same quantity, the whole determinant will be multiplied by that quantity.*

For every term of the determinant contains one element and only one from each column and from each row; and it therefore follows that if all the terms of one row or of one column be multiplied by the same quantity, every term of the determinant, and therefore the sum of all the terms, will be multiplied by that quantity.

Cor. From the above, together with Theorem III, it follows that if two rows or two columns of a determinant only differ by a constant factor, the determinant must vanish.

For example

$$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ na_2 & nb_2 & nc_2 \\ pa_3 & pb_3 & pc_3 \end{vmatrix} = mnp \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} ma_1 & nb_1 & pc_1 \\ ma_2 & nb_2 & pc_2 \\ ma_3 & nb_3 & pc_3 \end{vmatrix}.$$

Also
$$\begin{vmatrix} ma & na & 1 \\ mb & nb & 1 \\ mc & nc & 1 \end{vmatrix} = mn \begin{vmatrix} a & a & 1 \\ b & b & 1 \\ c & c & 1 \end{vmatrix} = 0.$$

423. **Minor determinants.** When any number of columns and the same number of rows of a determinant are suppressed, the determinant formed by the remaining elements is called a *minor determinant.*

A minor determinant is said to be of the *first order*, or to be a *first minor*, when one column and one row are suppressed; it is said to be of the *second order*, or to be a *second minor*, when two columns and two rows are suppressed; and so on.

The determinant obtained by suppressing the line and the column through any particular element is called the *minor of that element*, and will be denoted by Δ_x where x is the element in question.

Thus $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$ and $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ are first minors of $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, and are Δ_{c_3}, Δ_{b_2} and Δ_{a_1} respectively.

424. Development of determinants. Consider the determinant of the fourth order

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

A certain number of the terms of Δ will contain a_1; let the sum of all these terms be $a_1 . A_1$. Similarly let the sum of all the terms which contain a_2, a_3 and a_4, be respectively $a_2 . A_2$, $a_3 . A_3$ and $a_4 . A_4$. Then, since no term can contain more than one of the letters a_1, a_2, a_3, a_4 we have

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 \text{............(i).}$$

Now, since no term of Δ which contains a_1 can contain any element from the column or the row through a_1, it follows that every term of Δ which contains a_1 is the product of a_1 and some term of Δ_{a_1}; conversely the product of a_1 and any term, T, of Δ_{a_1} will be a term of Δ, and the sign of the term $a_1 . T$ of Δ will be the same as the sign of the term T of Δ_{a_1}, for there is no change in the number of

inversions. Hence the sum of all the terms of Δ which contain a_1 is $a_1 . \Delta_{a_1}$.

So also, every term of Δ which contains a_2 is the product of a_2 and some term of Δ_{a_2}, and the product of a_2 and any term, T, of Δ_{a_2} will be a term of Δ, but there is one more inversion in the term $a_2 . T$ of Δ than there is in the term T of Δ_{a_1}, since 2 precedes 1. Hence the sum of all the terms in Δ which contain a_2 is $-a_2 . \Delta_{a_2}$.

Similarly the sum of all the terms of Δ which contain a_3 are $a_3 . \Delta_{a_3}$; and the sum of all the terms which contain a_4 are $-a_4 . \Delta_{a_4}$.

Hence

$$\Delta = a_1 . \Delta_{a_1} - a_2 . \Delta_{a_2} + a_3 . \Delta_{a_3} - a_4 . \Delta_{a_4} \ldots\ldots\ldots \text{(ii)}.$$

By means of Articles 419 and 420, we can shew in a similar manner that

$$\begin{aligned} \Delta &= -b_1 \Delta_{b_1} + b_2 \Delta_{b_2} - b_3 \Delta_{b_3} + b_4 \Delta_{b_4} \\ &= a_1 \Delta_{a_1} - b_1 \Delta_{b_1} + c_1 \Delta_{c_1} - d_1 \Delta_{d_1} = \&\text{c.} \end{aligned}$$

Cor. By comparing (i) and (ii) we see that the *cofactors of the elements a_1, a_2, &c., are equal in absolute magnitude to the minors of the same elements.*

425. We have in the previous article considered the case of a determinant of the fourth order; the reasoning is however perfectly general, so that if Δ be a determinant of the nth order having $a_1, a_2, \ldots, a_n$ for the elements of its first row or column; then will

$$\Delta = a_1 . \Delta_{a_1} - a_2 \Delta_{a_2} + \ldots + (-1)^{n-1} a_n \Delta_{a_n}.$$

So also

$$\Delta = (-1)^{r-1} \{k_1 . \Delta_{k_1} - k_2 . \Delta_{k_2} + \ldots + (-1)^{n-1} k_n \Delta_{k_n}\}.$$

Where $k_1, k_2, \ldots, k_n$ are the elements of the rth row.

For example

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1).$$

Prove the following:

1. $$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 4.$$

2. $$\begin{vmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{vmatrix} = 27.$$

3. $$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 18.$$

4. $$\begin{vmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{vmatrix} = 16.$$

5. $$\begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix} = 2abc.$$

6. $$\begin{vmatrix} a & a & a \\ a & b & b \\ a & b & c \end{vmatrix} = a(b-c)(a-b).$$

7. $$\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} = 4abc.$$

8. $$\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = 4abc.$$

9. $$\begin{vmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{vmatrix} = 9.$$

add 2nd, 3d, + 4th to first

10. $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} = 1.$$

11. Write down the co-factors of a, f and c in the expansion of the determinant $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

Shew that, if A, B, &c. are the co-factors of a, b, &c. in the above determinant, and A', B', &c., the co-factors of A, B, &c. in the determinant

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}; \text{ then } \frac{A'}{a} = \frac{B'}{b} = \ldots\ldots = \Delta.$$

426. **Theorem V.** *If the elements of one column of a determinant be multiplied in order by the co-factors of the corresponding elements of any other column; then the sum of the products will be zero.*

Let the elements of the rth column be multiplied by the co-factors of the corresponding elements in the sth column; then the sum of the products will be

$$a_r . A_s + b_r . B_s + \ldots$$

Now consider the determinant which differs from the original determinant only in having the sth column identical with the rth; then A_s, B_s, &c. will be the same in the new determinant as in the original one.

The value of the new determinant will therefore, by Art. 425, be equal to

$$\pm \{a_s . A_s + b_s . B_s + \ldots\}$$
$$= \pm \{a_r . A_s + b_r . B_s + \ldots\},$$

since $a_r = a_s$, $b_r = b_s$, &c.

But, from Art. 421, we know that the new determinant is zero.

Hence $a_r . A_s + b_r . B_s + \ldots = 0$.

Thus in the determinant $\Delta = [a_1 b_2 c_3 d_4]$, we have

$$\Delta = a_1 A_1 + a_2 A_2 + a_3 A_3 + a_4 A_4;$$

also

$$0 = b_1 A_1 + b_2 A_2 + b_3 A_3 + b_4 A_4,$$
$$0 = a_1 A_3 + b_1 B_3 + c_1 C_3 + d_1 D_3, \text{ \&c.}$$

427. **Theorem VI.** *If each element of any row (or column) of a determinant be the sum of two quantities, the determinant can be expressed as the sum of two determinants of the same order.*

It will be sufficient to take as an example the determinant

$$\begin{vmatrix} a_1 + \alpha_1 & b_1 & c_1 \\ a_2 + \alpha_2 & b_2 & c_2 \\ a_3 + \alpha_3 & b_3 & c_3 \end{vmatrix}.$$

By Art. 424, we have, if A_1, A_2, A_3, be the co-factors of the elements of the first column,

$$\Delta = (a_1 + \alpha_1) A_1 + (a_2 + \alpha_2) A_2 + (a_3 + \alpha_3) A_3$$
$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}.$$

Similarly it can be proved that

$$\begin{vmatrix} a_1+\alpha_1 & b_1-\beta_1 & c_1 \\ a_2+\alpha_2 & b_2-\beta_2 & c_2 \\ a_3+\alpha_3 & b_3-\beta_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$
$$- \begin{vmatrix} a_1 & \beta_1 & c_1 \\ a_2 & \beta_2 & c_2 \\ a_3 & \beta_3 & c_3 \end{vmatrix} - \begin{vmatrix} \alpha_1 & \beta_1 & c_1 \\ \alpha_2 & \beta_2 & c_2 \\ \alpha_3 & \beta_3 & c_3 \end{vmatrix}.$$

428. **Theorem VII.** *A determinant is not altered in value by adding to all the elements of any column (or row) the same multiples of the corresponding elements of any number of other columns (or rows).*

Take as an example a determinant of the third order: we have to shew that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+mb_1+nc_1 & b_1 & c_1 \\ a_2+mb_2+nc_2 & b_2 & c_2 \\ a_3+mb_3+nc_3 & b_3 & c_3 \end{vmatrix}.$$

By Theorem VI, the last determinant is equal to

$$\begin{vmatrix} a_1 & b_1 & c \\ a_2 & b_2 & c \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mb_1 & b_1 & c_1 \\ mb_2 & b_2 & c_2 \\ mb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} nc_1 & b_1 & c_1 \\ nc_2 & b_2 & c_2 \\ nc_3 & b_3 & c_3 \end{vmatrix}.$$

But each of the last two determinants is zero [Art. 422, Cor.]; this proves the theorem.

Ex. 1. Shew that $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0.$

Add the second column to the third; then

$$\Delta = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = 0,$$

since two columns are now identical.

Ex. 2. Shew that $\begin{vmatrix} a & b & c & d \\ -a & b & c & d \\ -a & -b & c & d \\ -a & -b & -c & d \end{vmatrix} = 8abcd.$

Add the first row to each of the others; then

$$=\Delta\begin{vmatrix} a & b & c & d \\ 0 & 2b & 2c & 2d \\ 0 & 0 & 2c & 2d \\ 0 & 0 & 0 & 2d \end{vmatrix} = a\begin{vmatrix} 2b & 2c & 2d \\ 0 & 2c & 2d \\ 0 & 0 & 2d \end{vmatrix} = 2ab\begin{vmatrix} 2c & 2d \\ 0 & 2d \end{vmatrix} = 8abcd.$$

Ex. 3. Shew that $\begin{vmatrix} a+2b & a+4b & a+6b \\ a+3b & a+5b & a+7b \\ a+4b & a+6b & a+8b \end{vmatrix} = 0.$

Take the second row from the third, and then the first from the second.

Ex. 4. Shew that $\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8abc.$

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} = 2abc$$

Ex. 5. Find the value of $\begin{vmatrix} 1 & 15 & 14 & 4 \\ 12 & 6 & 7 & 9 \\ 8 & 10 & 11 & 5 \\ 13 & 3 & 2 & 16 \end{vmatrix}$.

$$\Delta = \begin{vmatrix} 1 & 15 & 14 & 4 \\ 12 & 6 & 7 & 9 \\ -4 & 4 & 4 & -4 \\ 12 & -12 & -12 & 12 \end{vmatrix} = 48\begin{vmatrix} 1 & 15 & 14 & 4 \\ 12 & 6 & 7 & 9 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix}$$

$$= 48\begin{vmatrix} 1 & 15 & 14 & 4 \\ 12 & 6 & 7 & 9 \\ -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0.$$

Ex. 6. Find the values of

$$\begin{vmatrix} 0 & -1 & -1 & 1 \\ 4 & 5 & 1 & 1 \\ 3 & 9 & 4 & 1 \\ -4 & 4 & 4 & 1 \end{vmatrix} \text{ and } \begin{vmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{vmatrix}.$$

Ans. 0, 9.

429. The following is an important example.

To shew that

$$\begin{vmatrix} a_1 & b_1 & c_1 & l & m & n \\ a_2 & b_2 & c_2 & p & q & r \\ a_3 & b_3 & c_3 & s & t & u \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & \gamma_1 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 & \gamma_2 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

It is in the first place clear that a term of the determinant of the sixth order will be obtained by taking *any* term of $[a_1b_2c_3]$ with *any* term of $[\alpha_1\beta_2\gamma_3]$. Thus $\Delta=[a_1b_2c_3]\cdot[\alpha_1\beta_2\gamma_3]$ together with terms involving l, m, n, &c.; and we have to shew that all terms involving any of the letters l, m, n, &c. will vanish.

Now, in every term of the minor of l, three elements must be chosen from the last three rows, and two only of these can be chosen from the last two columns; hence one of the three elements must be zero, and therefore every term of Δ_l is zero. Hence the minor of l, and so also the minor of each of the elements m, n, &c. is zero; this proves that there are no terms involving any of the letters l, m, n, &c.

It can be proved in a similar manner that any determinant of the $2n$th order is the product of two determinants of the nth order, provided every element of one of the nth minors of the original determinant is zero.

430. **Multiplication of determinants.** We shall consider the case of two determinants of the third order: the method is however perfectly general.

To express as a determinant of the third order, the product of the two determinants.

$$\Delta_1=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \Delta_2=\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

We know from Art. 429 that

$$\Delta_1\Delta_2=\begin{vmatrix} a_1 & b_1 & c_1 & -1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & -1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & \gamma_1 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 & \gamma_2 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \text{............(A).}$$

Multiply the first three rows by α_1, β_1, γ_1 and add the products to the fourth row; then multiply the first three rows by α_2, β_2, γ_2 respectively and add the products to the fifth row; and then multiply the first three rows by α_3, β_3, γ_3 respectively and add the products to the sixth row. We shall then have the equivalent determinant

$$\begin{vmatrix} a_1, & b_1, & c_1, & -1, & 0, & 0 \\ a_2, & b_2, & c_2, & 0, & -1, & 0 \\ a_3, & b_3, & c_3; & 0, & 0, & -1 \\ a_1\alpha_1+a_2\beta_1+a_3\gamma_1, & b_1\alpha_1+b_2\beta_1+b_3\gamma_1, & c_1\alpha_1+c_2\beta_1+c_3\gamma_1, & 0, & 0, & 0 \\ a_1\alpha_2+a_2\beta_2+a_3\gamma_2, & b_1\alpha_2+b_2\beta_2+b_3\gamma_2, & c_1\alpha_2+c_2\beta_2+c_3\gamma_2, & 0, & 0, & 0 \\ a_1\alpha_3+a_2\beta_3+a_3\gamma_3, & b_1\alpha_3+b_2\beta_3+b_3\gamma_3, & c_1\alpha_3+c_2\beta_3+c_3\gamma_3, & 0, & 0, & 0 \end{vmatrix}$$

which is by Art. 429 equivalent to the product of

0

$$-\begin{vmatrix} -1 & 0 & 0 \\ \cancel{1} & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \text{ that is 1, and}$$

$$\begin{vmatrix} a_1\alpha_1+a_2\beta_1+a_3\gamma_1, & b_1\alpha_1+b_2\beta_1+b_3\gamma_1, & c_1\alpha_1+c_2\beta_1+c_3\gamma_1 \\ a_1\alpha_2+a_2\beta_2+a_3\gamma_2, & b_1\alpha_2+b_2\beta_2+b_3\gamma_2, & c_1\alpha_2+c_2\beta_2+c_3\gamma_2 \\ a_1\alpha_3+a_2\beta_3+a_3\gamma_3, & b_1\alpha_3+b_2\beta_3+b_3\gamma_3, & c_1\alpha_3+c_2\beta_3+c_3\gamma_3 \end{vmatrix}.$$

Hence the required product is the determinant last written.

Ex. 1. Multiply $\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$ by $\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}$.

The required product is $\begin{vmatrix} X & Y & Z \\ Z & X & Y \\ Y & Z & X \end{vmatrix}$, where $X=ax+by+cz$, $Y=ay+bz+cx$, and $Z=az+bx+cy$.

Since $\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}=x^3+y^3+z^3-3xyz$, and the other determinants are of the same form, we see that the product of any two expressions of the form $x^3+y^3+z^3-3xyz$ can be expressed in the same form. [See Art. 156, Ex. 4.]

Ex. 2. Shew that $\begin{vmatrix} 2bc-a^2, & c^2, & b^2 \\ c^2, & 2ac-b^2, & a^2 \\ b^2, & a^2, & 2ab-c^2 \end{vmatrix}=(a^3+b^3+c^3-3abc)^2.$

Form the product of $\begin{vmatrix} a, & -b, & c \\ c, & -a, & b \\ b, & -c, & a \end{vmatrix}$ and $\begin{vmatrix} -a, & b & c \\ -c, & a & b \\ -b, & c & a \end{vmatrix}$.

Ex. 3. Shew that $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$, where A_1, B_1, &c. are the co-factors of a_1, b_1, &c. in the expansion of the determinant $[a_1b_2c_3]$.

For $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} [a_1b_2c_3] & 0 & 0 \\ 0 & [a_1b_2c_3] & 0 \\ 0 & 0 & [a_1b_2c_3] \end{vmatrix}$.

since $\quad A_1a_1 + A_2a_2 + A_3a_3 = B_1b_1 + B_2b_2 + B_3b_3$
$= C_1c_1 + C_2c_2 + C_3c_3 = [a_1b_2c_3]$,

and $\quad A_1b_1 + A_2b_2 + A_3b_3 = \&c. = 0$ [Art. 426].

Hence $\quad [A_1B_2C_3] \cdot [a_1b_2c_3] = [a_1b_2c_3]^3$.

431. The notation

$$\begin{Vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{Vmatrix}$$

is employed to denote the system of four determinants obtained by omitting any one of the columns.

432. We conclude with the following important applications of determinants.

Simultaneous Equations of the First degree. The solution of any number of simultaneous equations of the first degree can be at once obtained by means of the foregoing properties of determinants.

First take the case of the three equations

$$a_1x + b_1y + c_1z = k_1,$$
$$a_2x + b_2y + c_2z = k_2,$$
$$a_3x + b_3y + c_3z = k_3.$$

Multiply the equations in order by A_1, A_2, A_3, where A_1, A_2, A_3 are the co-factors of a_1, a_2, a_3 respectively in the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Then we have by addition

$$(a_1A_1 + a_2A_2 + a_3A_3)\,x + (b_1A_1 + b_2A_2 + b_3A_3)\,y$$
$$+ (c_1A_1 + c_2A_2 + c_3A_3)\,z = k_1A_1 + k_2A_2 + k_3A_3;$$

that is $$[a_1\, b_2\, c_3]\, x = [k_1\, b_2\, c_3],$$
for from Art. 426 the coefficients of y and z are zero.

Similarly we obtain
$$[a_1\, b_2\, c_3]\, y = [a_1\, k_2\, c_3],$$
and $$[a_1\, b_2\, c_3]\, z = [a_1\, b_2\, k_3].$$

Now consider n equations of the form
$$a_1x_1 + b_1x_2 + c_1x_3 + d_1x_4 + \ldots\ldots = k_1.$$

As before, multiply the equations in order by A_1, A_2, A_3, &c. the co-factors respectively of a_1, a_2, a_3, &c. in the determinant $[a_1\, b_2\, c_3 \ldots]$; then we have by addition
$$(a_1A_1 + a_2A_2 + a_3A_3 + \ldots)\, x = k_1A_1 + k_2A_2 + k_3A_3 + \ldots,$$
the coefficients of y, z, &c. being all zero by Art. 426.

Hence $$x = \frac{[k_1\, b_2\, c_3 \ldots]}{[a_1\, b_2\, c_3 \ldots]}.$$

So also $$y = \frac{[a_1\, k_2\, c_3 \ldots]}{[a_1\, b_2\, c_3 \ldots]}, \text{ \&c.}$$

Ex. 1. Solve the equations
$$x + 2y + 3z = 6,$$
$$2x + 4y + z = 7,$$
$$3x + 2y + 9z = 14.$$

The values of x, y, z are respectively
$$\frac{\begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}}, \quad \frac{\begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}} \text{ and } \frac{\begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}}.$$

and it will be found that each determinant is -20, so that $x = y = z = 1$.

Ex. 2. Solve the equations
$$x + y + z + w + k = 0,$$
$$ax + by + cz + dw + k^2 = 0,$$
$$a^2x + b^2y + c^2z + d^2w + k^3 = 0,$$
$$a^3x + b^3y + c^3z + d^3w + k^4 = 0.$$

We have

$$x = \begin{vmatrix} 1 & 1 & 1 & k \\ b & c & d & k^2 \\ b^2 & c^2 & d^2 & k^3 \\ b^3 & c^3 & d^3 & k^4 \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$$

$$= \frac{k(c-d)(d-b)(b-c)(k-b)(k-c)(k-d)}{(c-d)(d-b)(b-c)(a-b)(a-c)(a-d)};$$

$$\therefore \quad x = \frac{k(k-b)(k-c)(k-d)}{(a-b)(a-c)(a-d)},$$

and the values of y, z and w can be written down from that of x.

433. **Elimination.** *To find the condition that the three equations*

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0,$$
$$a_3x + b_3y + c_3 = 0,$$

may be simultaneously true.

Multiply the equations in order by C_1, C_2, C_3, the co-factors of c_1, c_2, c_3 respectively in the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then by addition we have

$$(a_1C_1 + a_2C_2 + a_3C_3)x + (b_1C_1 + b_2C_2 + b_3C_3)y + c_1C_1 + c_2C_2 + c_3C_3 = 0,$$

that is, from Art. 426,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0,$$

which is the required condition.

The three homogeneous equations $a_1x + b_1y + c_1z = a_2x + b_2y + c_2z = a_3x + b_3y + c_3z = 0$ are obviously satisfied by the values $x = y = z = 0$. If however x, y, z are not all zero, it follows from the above that the condition $[a_1b_2c_3] = 0$ must hold good.

It can be shewn in a similar manner that the condition that n equations of the form $a_1x + b_1y + \ldots + k_1 = 0$, with $(n-1)$ unknown quantities, may be simultaneously true is $[a_1 b_2 c_3 \ldots k_n] = 0$.

434. Sylvester's method of Elimination. This is a method by which x can be eliminated from any two rational and integral equations in x. The method will be understood from the following examples.

Ex. 1. Eliminate x from the equations

$$ax^2+bx+c=0 \text{ and } px^2+qx+r=0.$$

From the given equations we have

$$\begin{aligned} ax^3+bx^2+cx \quad &=0, \\ ax^2+bx+c&=0, \\ px^3+qx^2+rx \quad &=0, \\ \text{and} \qquad px^2+qx+r&=0. \end{aligned}$$

Now we may consider the different powers of x as so many different unknown quantities; and the result of eliminating x^3, x^2 and x from the four last equations is by Art. 433

$$\begin{vmatrix} a & b & c & 0 \\ 0 & a & b & c \\ p & q & r & 0 \\ 0 & p & q & r \end{vmatrix} = 0.$$

[This result is equivalent to that obtained in Art. 153, Ex. 3.]

Ex. 2. Eliminate x from the equations $ax^3+bx^2+cx+d=0$ and $px^2+qx+r=0$.

From the given equations we have

$$\begin{aligned} ax^4+bx^3+cx^2+dx \quad &=0, \\ ax^3+bx^2+cx+d&=0, \\ px^4+qx^3+rx^2 \qquad &=0, \\ px^3+qx^2+rx \quad &=0, \\ px^2+qx+r&=0. \end{aligned}$$

Eliminating x^4, x^3, x^2, x from the five last equations as if the different powers of x were so many different unknown quantities, we have the condition

$$\begin{vmatrix} a & b & c & d & 0 \\ 0 & a & b & c & d \\ p & q & r & 0 & 0 \\ 0 & p & q & r & 0 \\ 0 & 0 & p & q & r \end{vmatrix} = 0.$$

EXAMPLES XLII.

1. Shew that $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2.$

$4 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = (2abc)$

2. Shew that $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0.$

3. Shew that $\begin{vmatrix} b+c & c+a & a+b \\ b'+c' & c'+a' & a'+b' \\ b''+c'' & c''+a'' & a''+b'' \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}.$

4. Shew that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3.$

5. Shew that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ba & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$

6. Shew that $\begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = 2(bc+ca+ab)^3.$

7. Shew that $\begin{vmatrix} -bc & bc+b^2 & bc+c^2 \\ ca+a^2 & -ca & ca+c^2 \\ ab+a^2 & ab+b^2 & -ab \end{vmatrix} = (bc+ca+ab)^3.$

8. Shew that $\begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3.$

9. Shew that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$

10. Shew that $\begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}$

$$= -(a+b+c)(-a+b+c)(-b+c+a)(-c+a+b).$$

11. Prove that $\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & \gamma^2 & \beta^2 \\ b^2 & \gamma^2 & 0 & \alpha^2 \\ c^2 & \beta^2 & \alpha^2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & a\alpha & b\beta & c\gamma \\ a\alpha & 0 & c\gamma & b\beta \\ b\beta & c\gamma & 0 & a\alpha \\ c\gamma & b\beta & a\alpha & 0 \end{vmatrix}.$

12. Shew that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix} = abc.$

13. Shew that

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

14. Shew that $\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$

$$=(a+b+c+d)\,(a+b-c-d)\,(a+c-b-d)\,(a+d-b-c).$$

15. Shew that $\begin{vmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix} = x^3\,(x+10).$

16. Shew that $\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2.$

17. Shew that $\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = 0.$

18. Shew that $\begin{vmatrix} a & a & a & a \\ a & b & a & a \\ a & a & b & a \\ a & a & a & b \end{vmatrix} = a\,(b-a)^3.$

19. Shew that $\begin{vmatrix} a & b & b & b \\ a & b & a & a \\ b & b & a & b \\ a & a & a & b \end{vmatrix} = -(a-b)^4.$

20. Shew that $\begin{vmatrix} ax-by-cz & ay+bx & cx+az \\ ay+bx & by-cz-ax & bz+cy \\ cx+az & bz+cy & cz-ax-by \end{vmatrix}$

$$=(a^2+b^2+c^2)\,(x^2+y^2+z^2)\,(ax+by+cz).$$

21. Shew that

$$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)(a^2+b^2+c^2).$$

22. Shew that

$$\begin{vmatrix} (b-c)^2 & (a-b)^2 & (a-c)^2 \\ (b-a)^2 & (c-a)^2 & (b-c)^2 \\ (c-a)^2 & (c-b)^2 & (a-b)^2 \end{vmatrix} = -2(a^2+b^2+c^2-bc-ca-ab)^3.$$

23. Shew that, if any determinant vanishes, the minors of any one row will be proportional to the minors of any other row.

24. Shew that $\begin{vmatrix} a^2+1 & ab & ac & ad \\ ba & b^2+1 & bc & bd \\ ca & cb & c^2+1 & cd \\ da & db & dc & d^2+1 \end{vmatrix} = a^2+b^2+c^2+d^2+1.$

25. Shew that

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & a^2+\alpha^2 & ab+\alpha\beta & ac+\alpha\gamma \\ 1 & ab+\alpha\beta & b^2+\beta^2 & bc+\beta\gamma \\ 1 & ac+\alpha\gamma & bc+\beta\gamma & c^2+\gamma^2 \end{vmatrix} = (b\gamma-c\beta+c\alpha-a\gamma+a\beta-b\alpha)^2.$$

26. Shew that the determinants

$$\left\|\begin{matrix} 0 & 0 & 0 & a & b & c \\ 0 & 0 & z & a & b & 0 \\ 0 & y & 0 & a & 0 & c \\ x & 0 & 0 & 0 & b & c \\ x & y & z & 0 & 0 & 0 \end{matrix}\right\|$$

are all zero.

27. Shew that $\begin{vmatrix} x^2-yz & y^2-zx & z^2-xy \\ z^2-xy & x^2-yz & y^2-zx \\ y^2-zx & z^2-xy & x^2-yz \end{vmatrix} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2.$

28. Shew that

$$\begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix} \begin{vmatrix} a^2+\lambda^2 & ab+\lambda c & ac-\lambda b \\ ab-\lambda c & b^2+\lambda^2 & bc+\lambda a \\ ac+\lambda b & bc-\lambda a & c^2+\lambda^2 \end{vmatrix} = \lambda^3(\lambda^2+a^2+b^2+c^2)^3.$$

29. Shew that

$$\begin{vmatrix} x & y & z & w \\ a & b & c & d \\ d & c & b & a \\ w & z & y & x \end{vmatrix} = \begin{vmatrix} x+w & y+z \\ a+d & b+c \end{vmatrix} \cdot \begin{vmatrix} x-w & y-z \\ a-d & b-c \end{vmatrix}.$$

CHAPTER XXXII.

THEORY OF EQUATIONS.

435. ANY algebraical expression which contains x is called a *function* of x, and is denoted for brevity by $f(x)$, $F(x)$, $\phi(x)$, or some similar symbol.

The most general rational and integral expression [Art. 75] of the nth degree in x may be written

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n,$$

where $a_0, a_1, a_2, \dots$ do not contain x.

Since all the terms of any equation can be transposed to one side, every equation of the nth degree in x can be written in the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0,$$

where n is any integer, and the coefficients $a_0, a_1, a_2 \dots$ do not contain x.

Now any equation in x is equivalent to that obtained by dividing every one of its terms by any quantity which does not contain x; and, if we divide the left side of the above equation by a_0, the coefficient of x^n, we shall obtain the equation of the nth degree in its simplest form, namely

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0,$$

where $p_1, p_2, p_3, \dots$ do not contain x, but are otherwise unrestricted.

436. If we assume the fundamental theorem* that *every equation has a root* real or imaginary, it is easy to prove that an equation of the nth degree has n roots.

For suppose the equation to be $f(x)=0$, where

$$f(x) \equiv x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n.$$

Since $f(x)=0$ has a root, a_1 suppose, we have $f(a_1)=0$, and therefore [Art. 88] $f(x)$ must be divisible by $x-a_1$, so that $f(x)=(x-a_1)\phi(x)$, where $\phi(x)$ is an integral function of x and of the $(n-1)$th degree. Similarly, since the equation $\phi(x)=0$ has a root, a_2 suppose, we have $\phi(x)=(x-a_2)\psi(x)$, where $\psi(x)$ is an integral function of x of the $(n-2)$th degree. Hence

$$f(x)=(x-a_1)(x-a_2)\psi(x).$$

Proceeding in this way we shall find n factors of $f(x)$ of the form $x-a_1$, and we have finally

$$f(x)=(x-a_1)(x-a_2)\dots(x-a_n).$$

It is now clear that $a_1, a_2, \dots, a_n$ are roots of the equation $f(x)=0$; also no other value of x will make $f(x)$ vanish, so that the equation can only have these n roots.

In the above the quantities $a_1, a_2, a_3, \dots$ need not be all different from one another; but if the factors $x-a_1$, $x-a_2$, $x-a_3$, &c. be repeated p, q, r, &c. times respectively in $f(x)$, we must have

$$f(x)=(x-a_1)^p(x-a_2)^q(x-a_3)^r\dots,$$

where $$p+q+r+\dots=n.$$

The equation $f(x)=0$ has in this case p roots each a_1, q roots each a_2, &c., the whole number of roots being

$$p+q+r+\dots=n.$$

* Proofs of this fundamental proposition have been given by Cauchy, Clifford and others the proofs are however, long and difficult.

437. Relations between the roots and the coefficients of an equation.

We have seen that if a_1, a_2, a_3,... be the roots of the equation $f(x)=0$; then

$$f(x) \equiv (x-a_1)(x-a_2)\dots(x-a_n).$$

Hence [Art. 260]

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n$$
$$\equiv x^n - S_1 . x^{n-1} + S_2 . x^{n-2} - \dots + (-1)^n S_n,$$

where S_r is the sum of all the products of a_1, a_2, a_3,... taken r together.

Equating the coefficients of the different powers of x on the two sides of the above identity, we have

$$S_1 = -p_1,\ S_2 = p_2, \dots S_r = (-1)^r p_r, \dots S_n = (-1)^n p_n.$$

438. By means of the relations obtained in Art. 437, which give the values of certain symmetrical functions of the roots of an equation in terms of its coefficients, the values of many other symmetrical functions of the roots can be easily obtained without knowing the roots themselves.

The following are simple examples:

Ex. 1. If a, b, c be the roots of the equation

$$x^3+px^2+qx+r=0,$$

find the value of (i) Σa^2 and (ii) Σa^2b^2.

We have $\quad a+b+c=-p,$

$$bc+ca+ab=q$$

and $\quad abc=-r.$

Hence

$$a^2+b^2+c^2=(a+b+c)^2-2(bc+ca+ab)=p^2-2q.$$

Also,

$$\Sigma b^2c^2=(bc+ca+ab)^2-2abc(a+b+c)=q^2-2pr.$$

Ex. 2. If a, b, c,... be the roots of $x^n+p_1x^{n-1}+p_2x^{n-2}+\dots+p_n=0$, find the values of Σa^2 and Σa^3.

We know that $\Sigma a=-p_1$, $\Sigma ab=p_2$ and $\Sigma abc=-p_3$.

Now $(\Sigma a)^2=(a+b+c+\dots)^2=\Sigma a^2+2\Sigma ab$ [Art. 65];

$\therefore \quad \Sigma a^2=(\Sigma a)^2-2\Sigma ab=p_1^2-2p_2.$

Again $\quad \Sigma a^2 . \Sigma a=\Sigma a^3+\Sigma a^2b,$

and $\quad \Sigma a^2b=\Sigma ab . \Sigma a-3\Sigma abc.$

[For in $\Sigma ab \,.\, \Sigma a$ there can only be terms of the types a^2b and abc; of these the term a^2b will occur once, but the term abc will occur *three* times, for we can take either a or b or c from Σa and multiply by bc, ca or ab respectively from Σab. Thus $\Sigma ab \,.\, \Sigma a = \Sigma a^2 b + 3\Sigma abc$.]

Hence

$$\Sigma a^3 = \Sigma a^2 \,.\, \Sigma a - \Sigma ab \,.\, \Sigma a + 3\Sigma abc = (p_1^2 - 2p_2)(-p_1) - p_2(-p_1) - 3p_3.$$

439. **Theorem.** If there are any n quantities a_1, a_2, a_3, &c., and m be any positive integer not greater than n; then will

$$\Sigma a_1^m = \Sigma a_1^{m-1} \,.\, \Sigma a_1 - \Sigma a_1^{m-2} \,.\, \Sigma a_1 a_2 + \Sigma a_1^{m-3} \,.\, \Sigma a_1 a_2 a_3 - \dots \mp \Sigma a_1 \,.\, \Sigma a_1 a_2 \dots a_{m-1} \pm m \,.\, \Sigma a_1 a_2 \dots a_m.$$

The following relations hold good:

$$\left.\begin{aligned} \Sigma a_1^m &= \Sigma a_1 \Sigma a_1^{m-1} - \Sigma a_1^{m-1} a_2, \\ \Sigma a_1^{m-1} a_2 &= \Sigma a_1 a_2 \,.\, \Sigma a_1^{m-2} - \Sigma a_1^{m-2} a_2 a_3, \\ \Sigma a_1^{m-2} a_2 a_3 &= \Sigma a_1 a_2 a_3 \,.\, \Sigma a_1^{m-3} - \Sigma a_1^{m-3} a_2 a_3 a_4, \\ \dots\dots\dots &= \dots\dots\dots\dots\dots\dots\dots\dots \\ \Sigma a_1^2 a_2 a_3 \dots a_{m-1} &= \Sigma a_1 a_2 \dots a_{m-1} \,.\, \Sigma a_1 - m \Sigma a_1 a_2 \dots a_m \end{aligned}\right\} \dots (A).$$

To prove the first relation it is only necessary to notice that the product $\Sigma a_1 \,.\, \Sigma a_1^{m-1}$ can only give rise to terms of the types a_1^m and $a_1^{m-1} a_2$; also every term of either type will occur, and no term can occur more than once.

Thus $$\Sigma a_1 \,.\, \Sigma a_1^{m-1} = \Sigma a_1^m + \Sigma a_1^{m-1} a_2.$$

The other relations, except the last, will be seen to be true in a similar manner.

Also, the product $\Sigma a_1 a_2 \dots a_{m-1} \,.\, \Sigma a_1$ can only give rise to terms of the types $a_1^2 a_2 a_3 \dots a_{m-1}$ and $a_1 a_2 \dots a_m$; the first of these terms can only occur once, namely as $a_1 a_2 a_3 \dots a_{m-1} \times a_1$; the second term will, however, occur m times, for we get the term by taking any one of the m factors it contains from Σa_1 and multiplying this by the proper term of $\Sigma a_1 a_2 \dots a_{m-1}$.

Hence

$$\Sigma a_1 a_2 \dots a_{m-1} \,.\, \Sigma a_1 = \Sigma a_1^2 a_2 a_3 \dots a_{m-1} + m \,.\, \Sigma a_1 a_2 \dots a_m.$$

From the relations [A], we have at once

$$\Sigma a_1^m = \Sigma a_1^{m-1} \, . \, \Sigma a_1 - \Sigma a_1^{m-2} \, . \, \Sigma a_1 a_2 + \Sigma a_1^{m-3} \Sigma a_1 a_2 a_3 - \dots$$
$$\pm\, m \, . \, \Sigma a_1 a_2 \dots a_m \dots\dots\dots\dots\dots\dots [\text{B}].$$

If now a_1, a_2, a_3, &c. be the n roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = 0,$$

we know that

$$\Sigma a_1 = -p_1, \quad \Sigma a_1 a_2 = p_2, \quad \Sigma a_1 a_2 a_3 = -p_3, \quad \text{\&c.}$$

Hence, by substituting in [B] and transposing we have

$$\Sigma a_1^m + p_1 \, . \, \Sigma a_1^{m-1} + p_2 \Sigma a_1^{m-2} + \dots + p_{m-1} \Sigma a_1$$
$$+ p_m \, . \, m = 0 \dots\dots\dots\dots [\text{C}].$$

The formula [C] gives the sum of the mth powers of the roots of an equation of the nth degree [$m \not> n$] in terms of the coefficients and the sums of lower powers of the roots.

The sum of the *m*th powers of the roots of an equation can therefore be obtained from the formulae

$$\Sigma a_1 + p_1 = 0,$$
$$\Sigma a_1^2 + p_1 \Sigma a_1 + 2p_2 = 0,$$
$$\Sigma a_1^3 + p_1 \Sigma a_1^2 + p_2 \Sigma a_1 + 3p_3 = 0,$$
$$\Sigma a_2^4 + p_1 \Sigma a_1^3 + p_2 \Sigma a_1^2 + p_3 \, . \, \Sigma a_1 + 4p_4 = 0,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

If we eliminate Σa_1^2 and Σa_1 from the first three equations we have

$$\begin{vmatrix} p_1 & p_2 & 3p_3 + \Sigma a_1^3 \\ 1 & p_1 & 2p_2 \\ 0 & 1 & p_1 \end{vmatrix} = 0; \qquad \therefore \; \Sigma a_1^3 + \begin{vmatrix} p_1 & p_2 & 3p_3 \\ 1 & p_1 & 2p_2 \\ 0 & 1 & p_1 \end{vmatrix} = 0.$$

To find Σa_1^m we must eliminate Σa_1^{m-1}, Σa_1^{m-2},..., Σa_1 from the first m equations, and we have

$$\begin{vmatrix} p_1 & p_2 & p_3 \dots p_{m-1} & m \,.\, p_m + \Sigma a_1^m \\ 1 & p_1 & p_2 \dots p_{m-2} & (m-1) p_{m-1} \\ 0 & 1 & p_1 \dots p_{m-3} & (m-2) p_{m-2} \\ 0 & 0 & 1 \dots p_{m-4} & (m-3) p_{m-3} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \dots \;\; 1 & p_1 \end{vmatrix} = 0.$$

The coefficient of Σa_1^m is a determinant of which all the elements on one side of its principal diagonal are zeros, the elements along the principal diagonal being all equal to 1; the determinant is therefore equal to 1. Hence Σa_1^m is equal to an *integral* function of p_1, p_2, &c.

If m be greater than n the relation corresponding to [C] can be very easily obtained. For, since a_1, a_2,... are roots of $f(x) = 0$, we have n equations of the type

$$a_1^n + p_1 a_1^{n-1} + p_2 a_1^{n-2} + \dots + p_n = 0.$$

Multiply by a_1^{m-n}, a_2^{m-n},... respectively; then we shall have n equations of the type

$$a_1^m + p_1 a_1^{m-1} + p_2 a_1^{m-2} + \dots + p_n a_1^{m-n} = 0.$$

Hence, by addition, we have

$$\Sigma a_1^m + p_1 \Sigma a_1^{m-1} + p_2 \Sigma a_1^{m-2} + \dots + p_n \Sigma a_1^{m-n} = 0 \dots [D].$$

By means of the relations [C] and [D], which were first given by Newton, it is easily seen that the *sum of the* m*th powers of the roots of any equation can be expressed as a rational and integral function of the coefficients*, m being any integer.

440. Any rational and integral symmetrical function of the roots of an equation can be expressed in terms of the coefficients by means of the relations

$$\Sigma a_1 = -p_1, \quad \Sigma a_1 a_2 = p_2, \quad \Sigma a_1 a_2 a_3 = -p_3, \ \&c.$$

Consider the symmetric functions of the third degree.

It is easily seen that

$$-p_1^3 = \Sigma a_1^3 + 3\Sigma a_1^2 a_2 + 6\Sigma a_1 a_2 a_3,$$
$$-p_1 p_2 = \Sigma a_1^2 a_2 + 3\Sigma a_1 a_2 a_3,$$
$$-p_3 = \Sigma a_1 a_2 a_3.$$

Thus we have three equations to determine Σa_1^3, $\Sigma a_1^2 a_2$ and $\Sigma a_1 a_2 a_3$, and these are the only symmetrical functions of the third degree.

Similarly each of the products p_1^4, $p_1^2 p_2$, $p_1 p_3$, p_2^2 and p_4 can be expressed in terms of symmetric functions of the fourth degree, and there will be *as many such equations as there are symmetric functions* of the fourth degree.

The same will hold good with respect to symmetric functions of any other degree.

Transformation of Equations.

441. We now consider some cases in which an equation is to be found such that its roots are connected with the roots of a given equation in some specified manner.

I. *To find an equation whose roots are those of a given equation with contrary signs.*

If the given equation be $f(x) = 0$, the required equation will be $f(-y) = 0$. For, if a be any root of the given equation so that $f(a) = 0$, then $-a$ will be a root of $f(-y) = 0$.

Thus if the given equation be

$$p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_n = 0,$$

the required equation will be

$$p_0(-y)^n + p_1(-y)^{n-1} + p_2(-y)^{n-2} + \ldots\ldots + p_n = 0,$$

or $$p_0 y^n - p_1 y^{n-1} + p_2 y^{n-2} - \ldots\ldots + (-1)^n p_n = 0.$$

II. *To find an equation whose roots are those of a given equation each multiplied by a given quantity.*

Let $f(x) = 0$ be the given equation, and let c be the quantity by which each of its roots is to be multiplied.

Let $y = cx$, or $\frac{y}{c} = x$; then $f\left(\frac{y}{c}\right) = 0$ is the equation required. For, if a be any root of $f(x) = 0$, so that $f(a) = 0$, ac will be a root of $f\left(\frac{y}{c}\right) = 0$.

Thus, if the given equation be

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_n = 0,$$

the required equation will be

$$p_0\left(\frac{y}{c}\right)^n + p_1\left(\frac{y}{c}\right)^{n-1} + p_2\left(\frac{y}{c}\right)^{n-2} + \ldots\ldots + p_n = 0,$$

or $$p_0y^n + p_1cy^{n-1} + p_2c^2y^{n-2} + \ldots\ldots + p_nc^n = 0.$$

The above transformation is useful for getting rid of fractional coefficients.

Ex. Find the equation whose roots are the roots of

$$x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{12}x + \tfrac{1}{18} = 0$$

each multiplied by c.

The required equation is

$$y^3 - \tfrac{1}{2}cy^2 + \tfrac{1}{12}c^2y + \tfrac{1}{18}c^3 = 0.$$

We can now choose c so that all the coefficients may be integers; the smallest possible value of c is easily seen to be 6.

III. *To find an equation whose roots are those of a given equation each diminished by the same given quantity.*

Let $f(x) = 0$ be the given equation, and let c be the quantity by which each of its roots is to be diminished.

Let $y = x - c$, or $x = y + c$; then $f(y + c) = 0$ will be the equation required. For, if a be any root of $f(x) = 0$, so that $f(a) = 0$, $a - c$ will be a root of $f(y + c) = 0$.

An expeditious method of finding $f(y + c)$ will be given later on. [Art. 471.]

The chief use of above transformation is in finding approximate solutions of numerical equations; it can also be used to obtain from any given equation another equation in which a particular term is absent.

Ex. Find the equation whose roots are those of $x^3 - 3x^2 - 9x + 5 = 0$ each diminished by c, and find what c must be in order that in the transformed equation (i) the sum of the roots, and (ii) the sum of the products two together of the roots, may be zero.

The equation required is $f(y+c)=0$, that is

$$(y+c)^3 - 3(y+c)^2 - 9(y+c) + 5 = 0,$$

or $$y^3 + (3c-3)y^2 + (3c^2 - 6c - 9)y + c^3 - 3c^2 - 9c + 5 = 0.$$

The sum of the roots will be zero if the coefficient of y^2 be zero; that is, if $c=1$.

The sum of the products two together of the roots will be zero if the coefficient of y be zero; that is, if $c^2 - 2c - 3 = 0$, or $(c-3)(c+1)=0$.

IV. *To find an equation whose roots are the reciprocals of the roots of a given equation.*

Let $f(x)=0$ be the given equation. Then the equation $f\left(\frac{1}{x}\right)=0$ is satisfied by the reciprocal of any value of x which satisfies the original equation.

This transformation enables us to find the sum of any negative power of the roots of the equation $f(x)=0$, for we have only to find the sum of the corresponding positive power of the roots of the equation $f\left(\frac{1}{x}\right)=0$.

442. A **reciprocal equation** is one in which the reciprocal of any root is also a root.

To find the conditions that an equation may be a reciprocal equation.

Let the equation be

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots\dots + p_n = 0.$$

Then the equation whose roots are the reciprocals of the roots of the given equation is

$$p_0\left(\frac{1}{x}\right)^n + p_1\left(\frac{1}{x}\right)^{n-1} + p_2\left(\frac{1}{x}\right)^{n-2} + \dots\dots + p_n = 0,$$

or, multiplying by x^n,

$$p_0 + p_1x + p_2x^2 + \dots\dots + p_nx^n = 0.$$

The equation last written must be the same as the original equation, the ratio of corresponding coefficients must therefore be the same throughout. Thus

$$\frac{p_0}{p_n} = \frac{p_1}{p_{n-1}} = \frac{p_2}{p_{n-2}} = \ldots\ldots = \frac{p_n}{p_0}.$$

From the first and last we have $p_n^2 = p_0^2$, so that $p_n = \pm p_0$, whence it follows that the coefficients are the same when read backwards as forwards, or else that all the coefficients read in order backwards differ in sign only from the coefficients read in order forwards. These two forms of reciprocal equations are often said to be of the first and of the second *class* respectively.

443. The following important properties of reciprocal equations can easily be proved.

I. A reciprocal equation of the first class and of odd degree has one root equal to -1.

II. A reciprocal equation of the second class and of odd degree has one root equal to $+1$.

III. A reciprocal equation of the second class and of even degree has the two roots ± 1.

[These follow at once from Art. 87.]

IV. After rejecting the factor corresponding to the roots given in I, II, III, we are in all cases left with a reciprocal equation of the *first* class and of *even* degree.

V. The problem of solving a reciprocal equation of the first class and of even degree can, by means of the substitution $x + x^{-1} = y$, be reduced to that of solving an equation of half the dimensions. For the equation may be written

$$a_0(x^{2n} + 1) + a_1(x^{2n-1} + x) + \ldots = 0.$$

Divide by x^n; then

$$a_0(x^n + x^{-n}) + a_1(x^{n-1} + x^{-n+1}) + \ldots = 0.$$

Now, if $\quad x + x^{-1} = y, \quad x^2 + x^{-2} = y^2 - 2$;

and, from the general relation

$$x^n + x^{-n} = (x^{n-1} + x^{-n+1})(x + x^{-1}) - (x^{n-2} + x^{-n+2}),$$

it follows by induction that $x^n + x^{-n}$ can be expressed as a rational and integral expression of the nth degree in y.

Ex. Solve the equation $6x^6-25x^5+31x^4-31x^2+25x-6=0$.

As in III, the expression on the left has the factor x^2-1 corresponding to the roots ± 1. Thus we have

$$6(x^6-1)-25x(x^4-1)+31x^2(x^2-1)=0.$$

Hence the required roots are ± 1 and the roots of

$$6x^4-25x^3+37x^2-25x+6=0.$$

Divide by x^2; then

$$6\left(x^2+\frac{1}{x^2}\right)-25\left(x+\frac{1}{x}\right)+37=0.$$

Put $\qquad x+\dfrac{1}{x}=y\,; \qquad \therefore\ x^2+\dfrac{1}{x^2}=y^2-2.$

Hence $\qquad 6y^2-25y+25=0;$

$\therefore \qquad y=\dfrac{5}{2}$ or $y=\dfrac{5}{3}$.

From $x+\dfrac{1}{x}=\dfrac{5}{2}$, we have $x=2$ or $\dfrac{1}{2}$.

From $x+\dfrac{1}{x}=\dfrac{5}{3}$, we have $x=\dfrac{1}{6}(5\pm\sqrt{-11})$.

Thus the required roots are $\pm 1,\ 2,\ \dfrac{1}{2},\ \dfrac{1}{6}(5\pm\sqrt{-11})$.

444. The method of dealing with other cases of transformation will be seen from the following examples.

Ex. 1. If a, b, c be the roots of the equation $x^3+px^2+qx+r=0$, find the equation whose roots are bc, ca, ab.

Since $bc=\dfrac{abc}{a}=\dfrac{r}{a}$, if we put $y=\dfrac{r}{x}$, the three values of y corresponding to the values a, b, c of x will be bc, ca, ab. Hence the equation required will be obtained by substituting $\dfrac{r}{y}$ for x in the given equation, so that the required equation is

$$\left(\frac{r}{y}\right)^3+p\left(\frac{r}{y}\right)^2+q\left(\frac{r}{y}\right)+r=0,$$

or $$r^2+pry+qy^2+y^3=0.$$

Ex. 2. Find the equation whose roots are the squares of the roots of the equation $x^3+px^2+qx+r=0$.

We have $\qquad x(x^2+q)=-p(x^2+r);$

$\therefore \qquad x^2(x^2+q)^2=p^2(x^2+r)^2.$

Now put $y=x^2$, and we have the required equation, namely

$$y(y+q)^2=p^2(y+r)^2.$$

Ex. 3. If a, b, c be the roots of $x^3+px^2+qx+r=0$, find the equation whose roots are $a(b+c)$, $b(c+a)$, $c(a+b)$.

$$a(b+c)=a(-p-a)\text{ ; \&c.}$$

Hence, if we put $y=x(-p-x)$, y will have the values required provided x is restricted to the three values a, b, c; that is provided x satisfies the equation

$$x^3+px^2+qx+r=0.$$

Thus if we eliminate x between the given equation and the equation

$$x^2+px+y=0,$$

we shall get the required equation in y.

Multiply the second equation by y and subtract; then $(y-q)x=r$. Now substitute for x in the second equation, and we obtain the equation required, namely

$$r^2+pr(y-q)+y(y-q)^2=0.$$

EXAMPLES XLIII.

1. If a_1, a_2, a_3 be the roots of the equation $x^3+px+q=0$, find the values of

(i) $(a_2+a_3)(a_3+a_1)(a_1+a_2)$. (ii) $(a_2+a_3-2a_1)(a_3+a_1-2a_2)(a_1+a_2-2a_3)$.

(iii) Σa_1^2. (iv) Σa_1^3. (v) Σa_1^4. (vi) $\Sigma a_1^2a_2$. (vii) $\Sigma a_1^3a_2$.

(viii) $\Sigma(a_2^2-a_3a_1)(a_3^2-a_1a_2)$. (ix) $(a_1^2-a_2a_3)(a_2^2-a_3a_1)(a_3^2-a_1a_2)$.

(x) $\Sigma\dfrac{1}{a_2+a_3}$. (xi) $\Sigma\dfrac{1}{a_2+a_3-a_1}$. (xii) $\Sigma\dfrac{1}{a_1^2+a_2a_3}$.

2. Find the sum (i) of the squares, (ii) of the cubes and (iii) of the fourth powers of the roots of the equation $x^4+px+q=0$.

3. If a, b, c be the roots of the equation $x^3+px^2+qx+r=0$, find the values of

(i) $(b+c-3a)(c+a-3b)(a+b-3c)$.

(ii) $\left(\dfrac{1}{b}+\dfrac{1}{c}-\dfrac{1}{a}\right)\left(\dfrac{1}{c}+\dfrac{1}{a}-\dfrac{1}{b}\right)\left(\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}\right)$.

(iii) $\left(\dfrac{1}{a^2}-\dfrac{1}{bc}\right)\left(\dfrac{1}{b^2}-\dfrac{1}{ca}\right)\left(\dfrac{1}{c^2}-\dfrac{1}{ab}\right)$.

4. Find the sum of the squares and the sum of the cubes of the roots of the equations

(i) $x^3-14x+8=0$. (ii) $x^4-22x^2+84x-49=0$.

5. If $a, b, c, \ldots$ be the roots of the equation

$$x^n + p_1{}^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0,$$

find the values of

(i) Σa^2. (ii) Σa^3. (iii) $\Sigma \frac{1}{a^2}$.

(iv) $\Sigma \frac{a^2}{b}$. (v) $\Sigma \frac{a^3}{b}$. (vi) $\Sigma \frac{a^3}{b^2}$.

6. Find the equation each of whose roots exceeds by 2 a root of the equation

$$x^3 - 4x^2 + 3x - 1 = 0.$$

7. Find the equation whose roots are those of the equation

$$6x^3 - 5x^2 - \tfrac{1}{4} = 0,$$

each multiplied by c, and find the least value of c in order that the resulting equation may have integral coefficients with unity for the coefficient of the highest power.

8. If a, b, c be the roots of the equation $x^3 + px^2 + qx + r = 0$, find the equation whose roots are

(i) bc, ca, ab. (ii) $b+c, c+a, a+b$. (iii) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$.

(iv) $a(b+c), b(c+a), c(a+b)$, (v) $b^2+c^2, c^2+a^2, a^2+b^2$.

(vi) $bc - a^2, ca - b^2, ab - c^2$.

9. If a, b, c, d be the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$, find the equation whose roots are

(i) $b+c+d$, &c. (ii) $b+c+d-2a$, &c.

(iii) $b^2+c^2+d^2-a^2$, &c.

10. Find the equation whose roots are the cubes of the roots of the equation $x^3 + px^2 + qx + r = 0$.

445. *In any equation with real coefficients* **imaginary roots occur in pairs**.

For, if $a + b\sqrt{-1}$ be a root of $f(x) = 0$, $x - a - b\sqrt{-1}$ will be a factor of $f(x)$, and therefore [Art. 193] $x - a + b\sqrt{-1}$ will also be a factor, whence it follows that $a - b\sqrt{-1}$ is also a root of $f(x) = 0$.

Corresponding to the two roots $a \pm b\sqrt{-1}$ of $f(x) = 0$, $f(x)$ will have the *real quadratic factor* $\{(x-a)^2 + b^2\}$.

446. *In any equation with rational coefficients* **quadratic surd roots occur in pairs.**

For, if $a+\sqrt{b}$ be a root of $f(x)=0$, $\sqrt{b}$ being irrational, $x-a-\sqrt{b}$ will be a factor of $f(x)$, and therefore [Art. 179] $x-a+\sqrt{b}$ will also be a factor of $f(x)$, whence it follows that $a-\sqrt{b}$ will also be a root of $f(x)=0$.

Corresponding to the roots $a\pm\sqrt{b}$, $f(x)$ will have the *rational quadratic factor* $\{(x-a)^2-b\}$.

Ex. 1. Solve the equation $x^4-2x^3-22x^2+62x-15=0$, having given that one root is $2+\sqrt{3}$.

Since both $2+\sqrt{3}$ and $2-\sqrt{3}$ are roots of the equation,

$$(x-2-\sqrt{3})(x-2+\sqrt{3}),$$

that is x^2-4x+1, must be a factor of the left-hand member of the equation. Thus we have

$$(x^2-4x+1)(x^2+2x-15)=0.$$

Whence the roots required are $2\pm\sqrt{3}$ and the roots of

$$x^2+2x-15=0.$$

Ex. 2. Solve the equation $2x^3-12x^2+46x-42=0$, having given that one root is $3+\sqrt{-5}$.

Since $3\pm\sqrt{-5}$ are roots of the equation

$$(x-3-\sqrt{-5})(x-3+\sqrt{-5})$$

must be a factor of the left-hand member of the equation, which may be written

$$\{(x-3)^2+5\}(2x-3)=0.$$

Whence the roots required are $3\pm\sqrt{-5}$, $\frac{3}{2}$.

Ex. 3. Solve the equation $x^6-4x^5-11x^4+40x^3+11x^2-4x-1=0$, having given that one root is $\sqrt{2}+\sqrt{3}$.

If $\sqrt{a}+\sqrt{b}$ be a root of any equation with rational coefficients, $\sqrt{a}$ and $\sqrt{b}$ not being similar surds, then $\pm\sqrt{a}\pm\sqrt{b}$ will all four be roots.

Hence in the present case

$$(x-\sqrt{2}-\sqrt{3})(x-\sqrt{2}+\sqrt{3})(x+\sqrt{2}-\sqrt{3})(x+\sqrt{2}+\sqrt{3}),$$

that is x^4-10x^2+1, will be a factor of $f(x)$. The equation may therefore be written

$$(x^4-10x^2+1)(x^2-4x-1)=0,$$

whence the roots are $\pm\sqrt{2}\pm\sqrt{3}$, $2\pm\sqrt{3}$.

Ex. 4. Solve $$x^4 - x^3 - 9x^2 - 14x + 8 = 0,$$
having given that one root is $-1 + \sqrt[3]{3}$.

$x + 1 - \sqrt[3]{3}$ is a factor of $f(x)$; and therefore, as $f(x)$ is rational, the rational expression of lowest degree of which $x + 1 - \sqrt[3]{3}$ is a factor, namely the expression $(x+1)^3 - 3$, must be a factor of $f(x)$. Thus we have
$$\{(x+1)^3 - 3\}(x - 4) = 0.$$

Thus the roots are
$$4, \quad -1 + \sqrt[3]{3}, \quad -1 + \omega\sqrt[3]{3}, \quad -1 + \omega^2\sqrt[3]{3},$$
where ω is an imaginary cube root of unity.

447. **Roots common to two equations.** If the two equations $f(x) = 0$ and $\phi(x) = 0$ have one or more roots in common, $f(x)$ and $\phi(x)$ must have a common factor, which will be found by the process of Art. 98.

Ex. Find the common roots of the equations
$$x^3 - 3x^2 - 10x + 24 = 0 \text{ and } x^3 - 6x^2 - 40x + 192 = 0.$$

The H. C. F. of the left-hand members will be found to be $x - 4$. Hence $x = 4$ gives the common root.

448. When it is known that two roots of an equation are connected by any given relation, these roots can be found.

Ex. 1. Solve the cubic $x^3 - 3x^2 - 10x + 24 = 0$, having given that one root is double another.

Let a and b be the two roots and let $a = 2b$.

Then, since a is a root of the given equation
$$a^3 - 3a^2 - 10a + 24 = 0 \text{ ....(i).}$$

Also, since b is a root,
$$\left(\frac{a}{2}\right)^3 - 3\left(\frac{a}{2}\right)^2 - 10\left(\frac{a}{2}\right) + 24 = 0,$$
or
$$a^3 - 6a^2 - 40a + 192 = 0 \text{ ....(ii).}$$

The factor common to the left-hand members of (i) and (ii) will be found to be $a - 4$. Thus $a = 4$ and $b = 2$; the remaining root of the cubic is then easily found to be -3.

Ex. 2. Solve the cubic $2x^3 - 15x^2 + 37x - 30 = 0$, having given that the roots are in A. P.

The sum of the roots is equal to three times the mean root,

a suppose. Thus $3a = -\frac{15}{2}$, whence $a = -\frac{5}{2}$. Divide $f(x)$ by the factor $2x+5$, and the remaining roots are given by $x^2 - 5x + 6$.

Hence the roots are $2, \frac{5}{2}, 3$.

In the general case suppose that a and b are roots of the equation $f(x) = 0$ connected by the relation $b = \phi(a)$. Then $f(x) = 0$ and $f\{\phi(x)\} = 0$ have a common root, namely a; and this common root can be found as in Art. 447. Thus a and $\phi(a)$ can both be found.

Ex. (i). Find the condition that the roots of $x^3 + px^2 + qx + r = 0$ may be (i) in Arithmetic Progression, (ii) in Geometrical Progression.

Let a, b, c be the roots in order of magnitude;

(i) $a + b + c = 3b$; $\therefore b = -\frac{p}{3}$.

Hence, as b is a root, we have

$$\left(-\frac{p}{3}\right)^3 + p\left(-\frac{p}{3}\right)^2 + q\left(-\frac{p}{3}\right) + r = 0,$$

whence $$2p^3 - 9pq + 27r = 0.$$

(ii) $abc = b^3$; $\therefore b = \sqrt[3]{-r}$.

Hence, as b is a root, we have

$$-r + p(-r)^{\frac{2}{3}} + q(-r)^{\frac{1}{3}} + r = 0,$$

whence $$p^3 r = q^3.$$

449. **Commensurable roots.** When the coefficients of an equation are all rational the commensurable roots can easily be found.

It is at once seen that an equation with integral coefficients and with unity for the coefficient of the first term cannot have a *fractional* root.

For if $\frac{a}{b}$ be a root of $f(x) = 0$, $\frac{a}{b}$ being a fraction in its lowest terms, we have

$$\left(\frac{a}{b}\right)^n + p_1\left(\frac{a}{b}\right)^{n-1} + \dots\dots + p_n = 0.$$

Multiply by b^{n-1}; then all the terms will be integral except the first which will be fractional [for a is prime to

b and therefore a^n is also prime to b], and this is impossible.

Now, from Art. 441, II., any equation can be transformed into another with integral coefficients and with unity for the coefficient of its first term; hence, from the above, we have only to find *integral* roots.

Now it is clear that if a be an integral root of $f(x)=0$, so that $x-a$ is a factor of $f(x)$, a must be a factor of the term which is independent of x. Thus if we apply the test of Art. 88 to all the factors of p_n we shall discover all the integral roots.

Ex. Find the commensurable roots of $x^4-27x^3+42x+8=0$.

Here the commensurable roots, if any, are factors of 8. Hence we have only to test whether any of the numbers ± 8, ± 4, ± 2, ± 1 are roots. It will be found that 4 and 2 are roots. Having found two roots the equation can be completely solved; for we have

$$(x-2)(x-4)(x^2+6x+1)=0.$$

Hence the roots of the equation are 2, 4, $-3\pm 2\sqrt{2}$.

EXAMPLES XLIV.

1. Solve the equation $x^4+2x^3-16x^2-22x+7=0$, having given that $2+\sqrt{3}$ is one root.

2. Solve the equation $3x^3-23x^2+72x-70=0$, having given that $3+\sqrt{-5}$ is one root.

3. One root of the equation

$$3x^5-4x^4-42x^3+56x^2+27x-36=0$$

is $\sqrt{2}-\sqrt{5}$, find the remaining roots.

4. One root of the equation $2x^6-3x^5+5x^4+6x^3-27x+81=0$ is $\sqrt{2}+\sqrt{-1}$. Find the remaining roots.

5. Find the biquadratic equation with rational coefficients one root of which is $\sqrt{3}-\sqrt{5}$.

6. Find the biquadratic equation with rational coefficients one root of which is $\sqrt{2}+\sqrt{-3}$.

7. Shew that $x^3-2x^2-2x+1=0$ and $x^4-7x^2+1=0$ have two roots in common.

8. Solve the equation $x^4-4x^3+11x^2-14x+10=0$ of which two roots are of the form $\alpha+\beta\sqrt{-1}$ and $\alpha+2\beta\sqrt{-1}$.

9. Find the condition that the roots of $x^3+px^2+qx+r=0$ may be in Harmonical Progression.

10. Find the conditions that the roots of $x^4+px^3+qx^2+rx+s=0$ may be in A. P.

11. Find the roots of the equation $x^3-3x^2-13x+15=0$, having given that the roots are in A. P.

12. Solve the equation $x^4+2x^3-21x^2-22x+40=0$, having given that the roots are in A. P.

13. Find the commensurable roots of

(i) $x^3-7x^2+17x-15=0$,

(ii) $x^4-x^3-13x^2+16x-48=0$,

(iii) $3x^3-26x^2+34x-12=0$.

14. Solve the equation $4x^3-32x^2-x+8=0$, having given that the sum of two roots is zero.

15. Solve the equation $x^4+4x^3-5x^2-8x+6=0$, having given that the sum of two roots is zero.

16. Find the condition that the sum of two roots of the equation $x^4+px^3+qx^2+rx+s=0$ may be equal to zero.

17. Solve the equation $x^3-79x-210=0$, having given that two of the roots are connected by the relation $\alpha=2\beta+1$.

18. Solve the equation $3x^3-32x^2+33x+108=0$, having given that one root is the square of another.

19. Shew that, if the roots of the equation

$$x^n+npx^{n-1}+\frac{n(n-1)}{1.2}qx^{n-2}+\ldots=0,$$

be in A. P., they will be obtained from $-p+r\left\{\frac{3(p^2-q)}{n+1}\right\}^{\frac{1}{2}}$ by giving to r the values 1, 3, 5,... when n is even, and the values 2, 4, 6,... when n is odd.

20. Find the condition that the four roots α, β, γ, δ of the equation $x^4+px^3+qx^2+rx+s=0$ may be connected (i) by the relation $\alpha\beta=\gamma\delta$, and (ii) by the relation $\alpha\beta+\gamma\delta=0$.

21. Shew that, if four of the roots of the equation

$$ax^4+bx^3+cx^2+dx+e=0,$$

be connected by the relation $\alpha+\beta=\gamma+\delta$, then will $4abc-b^3-8a^2d=0$.

22. If a, b, c,... be the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0,$$

prove that $\quad (1-a^3)(1-b^3)(1-c^3)\ldots=A^3+B^3+C^3-3ABC,$

where $\quad A=p_n+p_{n-3}+\ldots,\quad B=p_{n-2}+p_{n-4}+\ldots,$

and $\quad C=p_{n-2}+p_{n-5}+\ldots$

450. Derived functions. Let

$$f(x) \equiv p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots\dots + p_n;$$

then, if $x+h$ be put for x, we have

$$f(x+h) = p_0(x+h)^n + p_1(x+h)^{n-1} + p_2(x+h)^{n-2} + \dots + p_n.$$

If now $(x+h)^n$, $(x+h)^{n-1}$, &c. be expanded by the Binomial Theorem, and the result arranged according to powers of h, we shall have

$$\begin{aligned} f(x+h) = f(x) + h\,\{np_0x^{n-1} &+ (n-1)\,p_1x^{n-2} \\ &+ (n-2)\,p_2x^{n-3} + \dots\dots + p_{n-1}\} \\ &+ \text{higher powers of } h. \end{aligned}$$

This expansion is usually written in the form

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\underline{|2}}f''(x) + \dots\dots$$

[The reader who is acquainted with the Differential Calculus will see that the expansion of $f(x+h)$ in powers of h is an example of Taylor's Theorem.]

It will be seen at once that $f'(x)$ is obtained by multiplying every term of $f(x)$ by the index of the power of x it contains and then diminishing that index by unity.

It will also be easily seen that $f''(x)$ can be obtained from $f'(x)$ in a similar manner, and so for $f'''(x)$, &c. in succession. We shall however in what follows only be concerned with $f'(x)$.

Def. The function $f'(x)$ is called the **first derived function of** $f(x)$, the function $f''(x)$ is called the **second derived function of** $f(x)$, and so on.

Thus if
$$\begin{aligned} f(x) &= p_0x^4 + p_1x^3 + p_2x^2 + p_3x + p_4, \\ f'(x) &= 4p_0x^3 + 3p_1x^2 + 2p_2x_1 + p_3, \\ f''(x) &= 12p_0x^2 + 6p_1x + 2p_2, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

451. **Theorem.** *If $f(x)$ be any rational and integral function of x and $f'(x)$ be its first derived function, then will*

$$f'(x) = \frac{f(x)}{x-a_1} + \frac{f(x)}{x-a_2} + \frac{f(x)}{x-a_3} + \ldots\ldots,$$

where $a_1, a_2, a_3, \ldots\ldots$ are the n roots, real or imaginary, of the equation $f(x) = 0$.

We know that

$$f(x) = p_0 (x - a_1)(x - a_2)(x - a_3) \ldots\ldots$$

Hence

$$f(x+h) = p_0 (x - a_1 + h)(x - a_2 + h)(x - a_3 + h) \ldots\ldots$$

The coefficient of h in the expression on the right is by Art. 260 equal to $p_0 \times$ (sum of all the products $n-1$ together of the n quantities $x - a_1, x - a_2, \ldots\ldots, x - a_n$).

But $f(x+h) = f(x) + hf'(x) +$ higher powers of h.

Hence $f'(x) = p_0 \times$ (sum of all the products $n-1$ together of the n quantities $x - a_1, x - a_2, \ldots\ldots, x - a_n$).

Hence $$f'(x) = \frac{f(x)}{x-a_1} + \frac{f(x)}{x-a_2} + \ldots\ldots$$

In the above the quantities $a_1, a_2, \ldots\ldots, a_n$ need not be all different from one another; but if a_1 occur r times, and a_2 occur s times, &c., we shall have

$$f'(x) = \frac{rf(x)}{x-a_1} + \frac{sf(x)}{x-a_2} + \ldots\ldots$$

452. **Equal Roots.** We have seen in the preceding Article that if $a_1, a_2, \ldots\ldots, a_n$ be the n roots of the equation $f(x) = 0$, so that $f(x) = p_0(x - a_1)(x - a_2) \ldots\ldots (x - a_n)$; then will $f'(x) = p_0 \times$ (sum of all the products $n-1$ together of the n quantities $x - a_1, x - a_2, \ldots\ldots, x - a_n$).

Now, if any root, for example a_1, is not repeated, so that the factor $x - a_1$ occurs only once in $f(x)$, then the factor $x - a_1$ will be left out of one of the terms of $f'(x)$ but will occur in all the others; whence it follows that $f'(x)$ is not divisible by $x - a_1$. Thus *a root of $f(x) = 0$ which is not repeated is not a root of $f'(x) = 0$.*

If, however, r roots of the equation $f(x) = 0$ are equal to a_1, the factor $x - a_1$ will occur r times in $f(x)$, and therefore $x - a_1$ will occur at least $r - 1$ times in every term of $f'(x)$, for every term of $f'(x)$ is formed from $f(x)$ by omitting *one* of its factors. Hence *a root of* $f(x) = 0$ *which is repeated* r *times is also a root of* $f'(x) = 0$ *repeated* $r - 1$ *times.*

We can therefore find whether the equation $f(x) = 0$ has any equal roots, by finding the H. C. F. of $f(x)$ and $f'(x)$; and if $f(x)$ be divided by this H. C. F. the quotient when equated to zero will be an equation whose roots are the different roots of $f(x) = 0$, but with each root occurring only once.

Ex. 1. Find the equal roots of the equation

$$x^4 - 5x^3 - 9x^2 + 81x - 108 = 0.$$

Here $$f(x) = x^4 - 5x^3 - 9x^2 + 81x - 108,$$
$$f'(x) = 4x^3 - 15x^2 - 18x + 81.$$

The H. C. F. of $f(x)$ and $f'(x)$ will be found to be $x^2 - 6x + 9$, that is $(x - 3)^2$.

Since $(x - 3)^2$ is a factor of $f'(x)$, $(x - 3)^3$ will be a factor of $f(x)$, and it will be found that $f(x) = (x - 3)^3 (x + 4)$.

Thus the roots of the given equation are $3, 3, 3, -4$.

Ex. 2. Shew that in any cubic equation a multiple root must be commensurable.

This follows from Art. 445 and 446, and from the fact that a *cubic* equation can only have *three* roots.

Ex. 3. Solve the equation $x^5 - 15x^3 - 10x^2 + 60x - 72 = 0$ by testing for equal roots.

Here $$f(x) = x^5 - 15x^3 - 10x^2 + 60x - 72;$$
$\therefore$ $$f'(x) = 5x^5 - 45x^2 - 20x + 60.$$

It will be found that the H.C.F. of $f(x)$ and $f'(x)$ is

$$x^3 - x^2 - 8x + 12.$$

If now we divide $f(x)$ by $x^3 - x^2 - 8x + 12$ the quotient will be $x^2 + x - 6$, and the roots of $x^2 + x - 6 = 0$ are 2 and -3.

Thus the given equation has only two different roots, namely 2 and -3; and it will be found that $f(x) = (x - 2)^3 (x + 3)^2$. Thus the roots of $f(x) = 0$ are $2, 2, 2, -3, -3$.

453. Continuity of any rational and integral function of x.

Let $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_n$ be any rational and integral function of x_1 arranged according to descending powers of x.

Then each term will be finite provided x is finite; and therefore, as the number of the terms is finite, the sum of them all will be finite for any finite value of x.

It can be easily proved that the first (or any other term) can be made to exceed the sum of all the terms which follow it by giving to x a value sufficiently great; and also that the last (or any other term) can be made to exceed the sum of all the terms which precede it by giving to x a value sufficiently small.

For let k be the greatest of the coefficients; then

$$\frac{p_0x^n}{p_1x^{n-1} + \ldots + p_n} > \frac{p_0x^n}{k(x^{n-1} + \ldots 1)} > \frac{p_0x^n(x-1)}{kx^n} > \frac{p_0}{k}(x-1).$$

Now $\frac{p_0}{k}(x-1)$ can be made as great as we please by sufficiently increasing x.

We can prove in a similar manner that $p_n/(p_{n-1}x + \ldots + p_0x^n)$ can be made as small as we please by sufficiently diminishing x.

Now suppose that x is changed into $x+h$; then we shall have

$$f(x+h) - f(x) = hf'(x) + \frac{h^2}{\underline{|2}} f''(x) + \ldots\ldots,$$

where the coefficients $f'(x)$, $f''(x)$, &c. of the different powers of h are finite quantities.

Then by the above, the first term on the right (or if this term vanishes for any particular value of x, then the first term on the right which does not vanish for that value) will exceed the sum of all the terms which follow it, provided h be taken small enough. But the first term will itself become indefinitely small when h is indefinitely small. Therefore *$f(x+h) - f(x)$ can be made as small as we please by taking h sufficiently small.* This shews that as x changes from any value a to another value b

$f(x)$ will change gradually and without any interruption from $f(a)$ to $f(b)$, so that $f(x)$ must pass once at least through every value intermediate to $f(a)$ and $f(b)$.

It must be noticed that it is not proved that $f(x)$ always increases or always diminishes from $f(a)$ to $f(b)$, it may be sometimes increasing and sometimes diminishing as x is changed from a to b; what has been proved is that there is no *sudden change* in the value of $f(x)$.

454. **Theorem.** *If $f(\alpha)$ and $f(\beta)$ have contrary signs one root at least of the equation $f(x) = 0$ must lie between α and β.*

For since $f(x)$ changes continuously from $f(\alpha)$ to $f(\beta)$, it must pass once at least through any value intermediate to $f(\alpha)$ and $f(\beta)$; it therefore follows that for at least *one* value of x intermediate to α and β it must pass through the value zero, which is intermediate to $f(\alpha)$ and $f(\beta)$ since $f(\alpha)$ and $f(\beta)$ are of contrary sign. Thus the equation $f(x) = 0$ is satisfied by at least *one* value of x which lies between α and β.

For example, if $f(x) = x^3 - 4x + 2$, then $f(1) = -1$ and $f(2) = 2$. Hence one root of the equation $x^3 - 4x + 2 = 0$ lies between 1 and 2.

455. **Theorem.** *An equation of an odd degree has at least one real root.*

Let the equation be $f(x) = 0$, where

$$f(x) = x^{2n+1} + p_1 x^{2n} + \dots\dots + p_{2n+1},$$

Then $f(+\infty)$ is positive, $f(0) = p_{2n+1}$, and $f(-\infty)$ is negative.

Thus there must in all cases be one real root, which is positive or negative according as p_{2n+1} is negative or positive.

456. **Theorem.** *An equation of even degree, the coefficient of whose first term is unity and whose last term is negative, has at least two real roots which are of contrary signs.*

Let $x^{2n}+p_1x^{2n-1}+\ldots\ldots+p_{2n}=0$ be the equation, p_{2n} being negative.

Then $f(+\infty)$ is positive, $f(0)=p_{2n}$, and $f(-\infty)$ is positive.

Hence, as p_{2n} is negative, there must be one real root at least between $+\infty$ and 0, and also one at least between 0 and $-\infty$.

457. The following is a very important example.

To prove that if a, b, c, f, g, h be all real the roots of the equation

$$(x-a)(x-b)(x-c)-f^2(x-a)-g^2(x-b)-h^2(x-c)-2fgh=0,$$

will always be real.

We may suppose without loss of generality that $a>b>c$.
Write the equation in the form

$$(x-a)\{(x-b)(x-c)-f^2\}-\{g^2(x-b)+h^2(x-c)+2fgh\}=0.$$

By substituting $+\infty$, b, c, $-\infty$ respectively for x in

$$(x-b)(x-c)-f^2,$$

we see that the roots of the equation $(x-b(x-c)-f^2=0$ are always real; and if α and β be these roots, where $\alpha>\beta$, then $\alpha>b>c>\beta$.

Now substitute $+\infty$, α, β, $-\infty$ for x in the left-hand number of the cubic equation, and we shall have respectively the following results

$$+\infty,\quad -\{g\sqrt{\alpha-b}+h\sqrt{\alpha-c}\}^2,\quad +\{g\sqrt{b-\beta}+h\sqrt{c-\beta}\}^2,\quad -\infty.$$

Hence there is one root of the cubic between $+\infty$ and α, one root between α and β, and one root between β and $-\infty$.

If, however, $\alpha=\beta$ the above proof fails; but if $\alpha=\beta$, then $(x-b)(x-c)-f^2$, must be a perfect square, whence it follows that $b=c$ and $f=0$.

The cubic equation in this case becomes

$$(x-a)(x-b)^2-(g^2+h^2)(x-b)=0,$$

the roots of which are at once seen to be all real.

If α be a root of the cubic equation itself, there will be another real root less than β. Hence all the roots of the cubic must be real, for the equation cannot have *one* imaginary root.

The cubic equation considered above is of great importance in Solid Geometry, and is called the **Discriminating Cubic**.

458. **Theorem.** *If $f(\alpha)$ and $f(\beta)$ are of contrary signs, then an odd number of roots of $f(x)=0$ lie between α and β; also if $f(\alpha)$ and $f(\beta)$ are of the same sign, then*

no roots or an even number of roots of $f(x)=0$ lie between α and β.

Let $a, b, c, \ldots\ldots, k$ be all the roots of the equation $f(x)=0$ which lie between α and β; then

$$f(x)=(x-a)(x-b)(x-c)\ldots\ldots(x-k)\,\phi(x),$$

where $\phi(x)$ is the product of quadratic factors (corresponding to pairs of imaginary roots) which can never change sign, and of real factors which do not change sign while x lies between α and β.

Then

$$f(\alpha)=(\alpha-a)(\alpha-b)(\alpha-c)\ldots\ldots(\alpha-k)\,\phi(\alpha),$$

and $$f(\beta)=(\beta-a)(\beta-b)(\beta-c)\ldots\ldots(\beta-k)\,\phi(\beta).$$

Now, supposing $\alpha>\beta$ all the factors $\alpha-a, \alpha-b, \ldots\ldots, \alpha-k$ are positive; and all the factors $\beta-a, \beta-b, \ldots\ldots, \beta-k$ are negative; also $\phi(\alpha)$ and $\phi(\beta)$ have the same sign. Therefore if $f(\alpha)$ and $f(\beta)$ have contrary signs there must be an *odd* number of the roots $a, b, c, \ldots\ldots, k$. Also, if $f(\alpha)$ and $f(\beta)$ have the same sign there must be no such roots or an *even* number of them.

459. **Rolle's Theorem.** *A real root of the equation $f'(x)=0$ lies between every adjacent two of the real roots of the equation $f(x)=0$.*

Let the real roots of $f(x)=0$, arranged in descending order of magnitude, be $a, b, c, \ldots, k$. Then

$$f(x)=(x-a)(x-b)\ldots(x-k)\,\phi(x),$$

where $\phi(x)$ is the product of real quadratic factors corresponding to pairs of imaginary roots and these quadratic expressions keep their signs unchanged for all real values of x.

Then

$$f(x+\lambda)=(x-a+\lambda)(x-b+\lambda)\ldots(x-k+\lambda)$$
$$\times\{\phi(x)+\lambda\phi'(x)+\text{higher powers of }\lambda\}.$$
$$=f(x)+\lambda\left\{\frac{f(x)}{x-a}+\frac{f(x)}{x-b}+\ldots+\frac{f(x)}{x-k}+\frac{f(x)\,\phi'(x)}{\phi(x)}\right\}+\&c.$$

[See Art. 451.]

$$\text{Hence } f'(x) = \frac{f(x)}{x-a} + \frac{f(x)}{x-b} + \ldots + \frac{f(x)}{x-k} + \frac{f(x)\,\phi'(x)}{\phi(x)}.$$

Now all the terms on the right except the first contain the factor $x-a$, and that term is

$$(x-b)(x-c)\ldots(x-k)\,\phi(x).$$

Hence

$$f'(a) = (a-b)(a-c)\ldots(a-k)\,\phi(a).$$

So

$$\begin{aligned} f'(b) &= (b-a)(b-c)\ldots(b-k)\,\phi(b), \\ f'(c) &= (c-a)(c-b)\ldots(c-k)\,\phi(c), \\ \ldots\ldots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Now $\phi(a)$, $\phi(b)$, $\phi(c)$, &c. have all the same sign. Hence as $a > b > c\ldots$, the signs of $f'(a)$, $f'(b)$, $f'(c)$, &c. are alternately positive and negative. Hence there is at least one root of $f'(x)=0$ between a and b, one root between b and c, &c.

460. **Descartes' Rule of Signs.** *In any equation $f(x)=0$ the number of real positive roots cannot exceed the number of changes in the signs of the coefficients of the terms in $f(x)$, and the number of real negative roots cannot exceed the number of changes in the signs of the coefficients of $f(-x)$.*

We shall first shew that if any polynomial be multiplied by a factor $x-a$, where a is positive, there will be at least *one more change* in the product than in the original polynomial.

Suppose that the signs of any polynomial succeed each other in the order $++-++---+-$, in which there are five changes of sign.

Then writing only the signs which occur we shall have

$$\begin{array}{l} + + - + + - - - + - \\ + - \\ \hline + + - + + - - - + - \end{array}$$

Now we cannot write down the second partial product for we do not know that all the possible terms in the

polynomial are present; but whenever there is a change of sign in the first partial product it is clear that if there is any term in the second row of the same degree in x, so that it would be put under this term which has the changed sign, it must arise from the multiplication of the *next preceding* term so that the two terms would have the *same* sign. Thus whenever there is a change of sign in the first partial product that sign will be retained in the addition of the two lines of partial products. The number of changes of sign, exclusive of the additional one which must be added at the end, cannot therefore be diminished.

Hence the product of any polynomial by the factor $x-a$ will contain at least *one* more change of sign than there are in the original polynomial.

If then we suppose the product of all the factors corresponding to negative and imaginary roots to be first formed, one more change of sign at least is introduced by multiplying by the factor corresponding to each positive root. Therefore the equation $f(x)=0$ cannot have *more* positive roots than there are changes of sign in the coefficients of the terms in $f(x)$.

The second part of the theorem follows at once from the first, for the positive roots of $f(-x)=0$ are the negative roots of $f(x)=0$.

The above proof may be made clearer by taking as a definite example the multiplication of $x^7+2x^6-x^4+4x^3+3x-1$ by $x-1$. The signs of the two lines of partial products will be

$$\begin{array}{ccccccccc}
+ & + & & - & + & & + & - & \\
 & - & - & & + & - & & - & + \\
\hline
+ & & & - & + & & & - & +
\end{array}$$

In the third line the only signs written down are those under the changes in the first line, which changes are all retained in the final product. Hence no matter what has occurred in the intervals the number of changes (exclusive of the one at the end) cannot be diminished.

461. Descartes' Rule of Signs only gives a superior limit to the number of real roots of an equation, but does

not determine the actual number of real roots. The number of the real roots of any equation with numerical coefficients can be found by means of Sturm's Theorem. Before considering Sturm's Theorem we shall shew how to find algebraical solutions of cubic and biquadratic (quartic) equations in their most general forms. Abel has proved that an algebraical solution, that is a solution by radicals, of a general equation of higher degree than the fourth cannot be found, although particular forms of such equations can be solved, for example any reciprocal equation of the fifth degree can always be solved.

EXAMPLES XLV.

1. Solve the following equations each of which has equal roots:

(i) $4x^3 - 12x^2 - 15x - 4 = 0$,

(ii) $x^4 - 6x^3 + 13x^2 - 24x + 36 = 0$,

(iii) $16x^4 - 24x^2 + 16x - 3 = 0$,

(iv) $2x^4 - 23x^3 + 84x^2 - 80x - 64 = 0$.

2. Find the condition that the equation $ax^3 + 3bx^2 + 3cx + d = 0$ may have two equal roots.

3. Shew that, if the equation $ax^3 + 3bx^2 + 3cx + d = 0$ have two equal roots, they are each equal to

$$\frac{1}{2}\,\frac{bc - ad}{ac - b^2}.$$

4. Shew that the roots of the equation

$$\frac{a^2}{x - a'} + \frac{b^2}{x - b'} + \frac{c^2}{x - c'} + \dots + \frac{k^2}{x - k'} - \lambda = 0$$

are all real.

5. Shew that all the roots of the equation

$$\frac{a^2}{x - \alpha} + \frac{b^2}{x - \beta} + \frac{c^2}{x - \gamma} + \dots = m + n^2 x$$

are real.

6. If $a_1, a_2, a_3, \ldots, a_{2n}$ be in descending order of magnitude, and if b be positive, prove that the roots of the equation

$$(x-a_1)(x-a_3)\ldots(x-a_{2n-1})+b(x-a_2)(x-a_4)\ldots(x-a_{2n})=0$$

will all be real, and find their positions.

7. Prove that if a, b, c, d be unequal positive quantities, the roots of the equation

$$\frac{x}{x-a}+\frac{x}{x-b}+\frac{x}{x-c}+x+d=0$$

will all be positive; and that, if roots be $\alpha, \beta, \gamma, \delta$, then will

$$\frac{a^2}{(a-\alpha)(a-\beta)(a-\gamma)(a-\delta)}+\frac{b^2}{(b-\alpha)(b-\beta)(b-\gamma)(b-\delta)}$$
$$+\frac{c^2}{(c-\alpha)(c-\beta)(c-\gamma)(c-\delta)}=0.$$

8. Form the equation whose roots are the values of $p\omega+q\omega^{-1}$, where ω is a fifth root of unity, and shew that the equation is

$$x^5-5pqx^3+5p^2q^2x-p^5-q^5=0.$$

9. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation

$$x^4+4px^3+6qx^2+4rx+s=0,$$

form the equations whose roots are

(i) $\alpha\beta+\gamma\delta,\quad \alpha\gamma+\beta\delta,\quad \alpha\delta+\beta\gamma.$

(ii) $(\alpha+\beta)(\gamma+\delta),\quad (\alpha+\gamma)(\beta+\delta),\quad (\alpha+\delta)(\beta+\gamma).$

10. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation

$$x^4+4px^3+6qx^2+4rx+\delta=0$$

form the equation whose roots are

$$(\alpha+\beta-\gamma-\delta)^2,\quad (\alpha-\beta+\gamma-\delta)^2,\quad (\alpha-\beta-\gamma+\delta)^2.$$

11. If a_1, a_2, a_3 be the roots of

$$x^3+p_1x^2+p_2x+p_3=0,$$

shew that $\quad \Sigma a_1^3a_2^2=2p_1^2p_3-p_1p_2^2+p_2p_3.$

Cubic Equations.

462. The most general form of a cubic equation is

$$x^3 + ax^2 + bx + c = 0.$$

We have however seen [Art. 441, III.] that by increasing each root by $\frac{a}{3}$, the equation will take the simpler form $x^3 + px + q = 0$.

We shall therefore suppose that the equation has already been reduced to this simplified form.

463. *To solve the cubic equation* $x^3 + px + q = 0$.

The solution is at once obtained by comparing the equation with

$$x^3 - 3abx + a^3 + b^3 = 0,$$

i.e. $\quad (x + a + b)(x + \omega a + \omega^2 b)(x + \omega^2 a + \omega b) = 0,$

where ω is an imaginary cube root of unity [Art. 139].

Thus the roots required are

$$-a - b, \quad -\omega a - \omega^2 b, \quad -\omega^2 a - \omega b,$$

where a and b have to be determined from the equations

$$p = -3ab, \quad q = a^3 + b^3.$$

Whence a^3 and b^3 are given by

$$\left\{\frac{q}{2} \pm \sqrt{\left(\frac{q^2}{4} + \frac{p^3}{27}\right)}\right\}.$$

464. The foregoing solution is a slight modification of that called **Cardan's** solution. It is a complete *algebraical* solution of the equation and the values found for x would satisfy the given equation identically. If, however, *numerical values* be given to p and q, the numerical values of a and b cannot be found when $\frac{q^2}{4} + \frac{p^3}{27}$ is *negative*, for we

cannot reduce an expression of the form $(3+5\sqrt{-1})^{\frac{1}{3}}$, for example, to the form $\alpha+\beta\sqrt{-1}$. Thus when p and q are numerical quantities such that $\frac{q^2}{4}+\frac{p^3}{27}$ is *negative*, Carden's solution altogether fails to give a numerical result.

This case is called the **'irreducible case,'** and we shall see further on [Art. 466, Ex. 3] that when $\frac{q^2}{4}+\frac{p^3}{27}$ is negative all the roots of the cubic are real.

It should also be noted that in any case the approximate values of the *real* roots of a cubic can be obtained much more easily by Horner's general process [Art. 473] than by Carden's solution.

Ex. Solve the cubic equation $x^3+4x+5=0$.

Comparing with $\quad x^3-3abx+a^3+b^3=0$,

we have $\quad -3ab=4$ and $a^3+b^3=-5$,

whence a and b are given by

$$\left\{-\frac{5}{2}\pm\frac{1}{18}\sqrt{1257}\right\}^{\frac{1}{3}}.$$

The approximate values of a and b can therefore be found, and then the roots are

$$-a-b,\quad \omega a-\omega^2 b,\quad -\omega^2 a-\omega b.$$

In this example the solution can be obtained in a very simple manner. For, using the test given in Art. 449 for commensurable roots, we are led to find that 1 is a commensurable root, and writing the equation in the form $(x+1)(x^2-x+5)=0$, the roots are at once seen to be

$$-1,\quad \frac{1}{2}(1\pm\sqrt{-19}).$$

BIQUADRATIC EQUATIONS.

465. Several methods of solution of a biquadratic equation have been given. In all of them the solution is shewn to follow from the solution of a *cubic* equation. The simplest method of solution is that due to Ferrari.

To solve the equation $x^4 + px^3 + qx^2 + rx + s = 0$.

Ferrari's Solution. Add $(ax+\beta)^2$ to both sides of the equation; then

$$x^4 + px^3 + (q + a^2)\,x^2 + (r + 2a\beta)\,x + s + \beta^2 = (ax + \beta)^2.$$

Now the left-hand member will be a perfect square, namely $\left(x^2 + \frac{p}{2}\,x + \lambda\right)^2$, provided

$$2\lambda + \frac{p^2}{4} = q + a^2, \quad p\lambda = r + 2a\beta \quad \text{and} \quad \lambda^2 = s + \beta^2.$$

Eliminating a and β, we have a *cubic* equation to determine λ, namely

$$4\,(\lambda^2 - s)\left(2\lambda + \frac{p^2}{4} - q\right) - (p\lambda - r)^2 = 0.$$

One root of this cubic equation is always real, and if this root be found the values of a and β are determined. We then have

$$\left(x^2 + \frac{p}{2}\,x + \lambda\right)^2 = (ax + \beta)^2,$$

whence $$x^2 + \frac{p}{2}\,x + \lambda \pm (ax + \beta) = 0,$$

where a, β and λ are known. Thus the biquadratic equation can be completely solved.

Ex. Solve the equation

$$x^4 + 6x^3 + 14x^2 + 22x + 5 = 0.$$

Add $(ax+\beta)^2$ to both sides; then

$$x^4 + 6x^3 + (14 + a^2)\,x^2 + (22 + 2a\beta)\,x + 5 + \beta^2 = (ax + \beta)^2.$$

The left-hand member is the square of $x^2 + 3x + \lambda$ provided

$$9 + 2\lambda = 14 + a^2, \quad 6\lambda = 22 + 2a\beta \quad \text{and} \quad \lambda^2 = 5 + \beta^2.$$

Whence $$(\lambda^2 - 5)(2\lambda - 5) - (3\lambda - 11)^2 = 0;$$

$\therefore$ $$\lambda^3 - 7\lambda^2 + 28\lambda - 48 = 0.$$

The real root of the cubic is 3.

Then, taking $\lambda=3$, we have

$$a^2=1, \quad 2a\beta=-4, \quad \beta^2=4.$$

Hence $$(x^2+3x+3)^2=(x-2)^2,$$

whence we obtain the roots

$$-2\pm\sqrt{3}, \quad -1\pm2\sqrt{-1}.$$

466. **Sturm's Theorem.** Let $f(x)=0$ be an equation *cleared of equal roots*, and let $f_1(x)$ be the first derived function of $f(x)$. Let the process of finding the highest common factor of $f(x)$ and $f_1(x)$ be performed with the modification that the *sign of every remainder is changed* before using it as a divisor, and let the operation be continued until a remainder is arrived at which does not contain x (this will always happen since $f(x)=0$ has no equal roots and therefore $f(x)$ and $f_1(x)$ have no common measure in x), and change also the sign of this last remainder.

Let $f_2(x)$, $f_3(x)$,..., $f_m(x)$ be the series of modified remainders so obtained, of which the last, $f_m(x)$, does not contain x.

Then *the number of real roots of the equation* $f(x)=0$ between a and β, $[\beta>a]$ *is equal to the excess of the number of changes of sign in the series* $f(x), f_1(x), f_2(x),..., f_m(x)$ *when* $x=a$ *over the number of changes of sign when* $x=\beta$.

For, let q_1, q_2,..., q_{m-1} be the successive quotients; then we have the series of identities

$$\begin{aligned}
f(x)&=q_1f_1(x)-f_2(x),\\
f_1(x)&=q_2f_2(x)-f_3(x),\\
f_2(x)&=q_3f_3(x)-f_4(x),\\
\dots&=\dots\\
f_{m-2}(x)&=q_{m-1}f_{m-1}(x)-f_m(x).
\end{aligned}$$

Now (i) it is clear that no two consecutive functions can vanish for the same value of x, for in that case all the succeeding functions, including $f_m(x)$, would vanish for that value of (x); and, (ii) it is also clear that when any

one of the functions except $f(x)$ vanishes, the two adjacent functions will have contrary signs.

It follows from (i) and (ii) that so long as the increasing value of x does not make $f(x)$ itself vanish, that is *unless we pass through a real root of the equation $f(x)=0$, there can be no alteration in the number of changes of sign in the series of Sturm's functions;* for no function will change sign unless it passes through a zero value, and when this is the case for any function, since the two adjacent functions have opposite signs, there must be one and only one change in the group of three.

Next suppose that a is a real root of the equation $f(x)=0$. Then $f(a \mp \lambda) = f(a) \mp \lambda f'(a) + \&c.$; and as $f(a)=0$, the sign of the series on the right will, if λ be very small, be the same as the sign of $\mp \lambda f'(a)$. Hence, however small λ may be, the sign of $f(a-\lambda)$ must be opposite to that of $f'(a)$, and the sign of $f(a+\lambda)$ must be the same as the sign of $f'(a)$.

Thus *as x increases through a real root of the equation $f(x)=0$, the series of Sturm's functions will lose one change of sign.*

Since we have proved that as x increases the series of Sturm's functions never lose or gain a change of sign except when x passes through a real root of the equation $f(x)=0$, in which case *one* change of sign is always lost, it follows that the excess of the number of changes of sign when $x=\alpha$ over the number of changes when $x=\beta$ must be equal to the number of real roots of the equation which lie between α and β.

To find the total number of real roots of an equation we must substitute $-\infty$ and $+\infty$ in Sturm's functions; then the excess of the number of changes of sign in the series in the former case over that in the latter will give the whole number of real roots.

Ex. 1. Find the number of the real roots of the equation

$$x^4+4x^3-4x-13=0.$$

Here

$$f(x)=x^4+4x^3-4x-13,$$

$$f_1(x)=4(x^3+3x^2-1).$$

N.B. We may clearly multiply or divide by *positive* numerical quantities as in the ordinary process for finding H.C.F. It will be found that

$$f_2(x) = x^2 + x + 4,$$
$$f_3(x) = 2x + 3,$$
$$f_4(x) = -19.$$

Substitute $-\infty$, 0, $+\infty$ in the above functions, and the series of signs will be

$$+ - + - -; \quad - - + + -; \quad + + + + -.$$

Thus there is one real root between $-\infty$ and 0, and one real root between 0 and $+\infty$.

Ex. 2. Find the number and the position of the real roots of the equation $x^5 - 5x + 1 = 0$.

Here $\qquad f(x) = x^5 - 5x + 1,$

and $\qquad f_1(x) = 5(x^4 - 1).$

It will be found that

$$f_2(x) = 4x - 1,$$
$$f_3(x) = +255.$$

The following are the series of signs corresponding to the values of x written in the same line

$-\infty$,	$-$	$+$	$-$	$+$
-2,	$-$	$+$	$-$	$+$
-1,	$+$	0	$-$	$+$
0,	$+$	$-$	$-$	$+$
1,	$-$	0	$+$	$+$
2,	$+$	$+$	$+$	$+$

Hence there is one real negative root between -2 and -1, one positive root between 0 and 1 and another between 1 and 2, the remaining two roots being imaginary.

Ex. 3. Find the condition that all the roots of the equation

$$x^3 + px + q = 0$$

may be real.

$$f(x) = x^3 + px + q,$$
$$f_1(x) = 3x^2 + p.$$

The other functions will be found to be

$$f_2(x) = -2px - 3q,$$
$$f_3(x) = -(27q^2 + 4p^3).$$

The signs for $-\infty$ and $+\infty$ are

$$-,\ +,\ +2p,\ -(27q^2+4p^3),$$

and

$$+,\ +,\ -2p,\ -(27q^2+4p^3).$$

In order that the roots may be all real, it is necessary and sufficient that there shall be three changes of sign in the first line and none in the second, the conditions for which are that p and $27q^2+4p^3$ must both be negative, the second of which implies the first.

467. Although Sturm's Theorem completely solves the problem of determining the number and the position of the real roots of an equation, it is often a very laborious process. In some cases the position of the real roots can be determined without difficulty by actual substitution; and sometimes the necessity for using Sturm's Theorem can be obviated by some special device.

Ex. 1. Find the number and position of the real roots of the equation

$$x^4-41x^2+40x+126=0.$$

Substitute in $f(x)$ the values 1, 2, 3, 4, 5, 6 in succession, and the signs will be $+$, $+$, $-$, $-$, $-$, $+$. Hence there is one root (at least) between 2 and 3, and one (at least) between 5 and 6; but by Descartes' Rule of Signs there cannot be *more* than two positive roots.

Hence there are two positive roots which lie between 2 and 3 and between 5 and 6 respectively.

We can find in a similar manner that there are two negative roots which lie between -1 and -2 and between -6 and -7 respectively.

Ex. 2. Find the number and position of the real roots of the equation

$$x^4-14x^2+16x+9=0.$$

In this case we should easily find the two negative roots which lie between 0 and -1 and between -4 and -5 respectively. The positive roots would, however, probably escape notice (unless Sturm's Theorem were used) as they *both* lie between 2 and 3; it will in fact be found that $f(2)$ is $+$, $f(2\frac{1}{2})$ is $-$, and $f(3)$ is $+$.

Ex. 3. Find in any manner the number and position of the real roots of the equation

$$x^6-5x^5-7x^2+8x+20=0.$$

By Descartes' Rule of Signs we see by inspection that there cannot be *more* than two positive roots and there cannot be *more* than two negative roots.

Now $f(1)$ is $+$, $f(2)$ is $-$; thus *one* real root lies between 1 and 2. Since $f(\infty)$ is $+$, there must be another positive root which is easily found to lie between 5 and 6.

Change x into $-x$, then the negative roots of the given equation are positive roots of

$$x^6+5x^5-7x^2-8x+20=0.$$

Now $f(x)$ must clearly be positive for all positive values between 0 and 1; and if $x>1$,

$$f(x)>6x^4-15x^2+20,$$

which is always positive since

$$4\times 6\times 20-15^2>0.$$

Hence there can be no real negative roots.

468. We shall conclude by shewing how to find the approximate values of the real roots of any equation. This can be done in various ways; we shall, however, only give **Horner's** method. We must first give the explanations of the separate processes which are employed.

469. **Synthetic Division.** Suppose that when

$$f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_n$$

is divided by $x-\lambda$ the quotient is

$$Q\equiv b_0x^{n-1}+b_1x^{n-2}+b_2x^{n-3}+\dots+b_{n-1},$$

and that the remainder is R, where R does not contain x.

Then $$f(x)\equiv Q\times(x-\lambda)+R.$$

But $Q\times(x-\lambda)+R$ is at once seen to be

$$b_0x^n+(b_1-\lambda b_0)\,x^{n-1}+(b_2-\lambda b_1)\,x^{n-2}+\dots$$
$$+(b_{n-1}-\lambda b_{n-2})\,x+R-\lambda b_{n-1}.$$

Equating coefficients of the different powers of x in $f(x)$ and in the expression last written, we have

$$b_0=a_0,\ b_1-\lambda b_0=a_1,\ b_2-\lambda b_1=a_2,\ \dots\dots$$
$$b_{n-1}-\lambda b_{n-2}=a_{n-1},\ R-\lambda b_{n-2}=a_n.$$

From the above relations it will be seen that the values of b_0, b_1, b_2, &c. can be obtained at once by the process indicated below:

$$\begin{array}{cccccccc} a_0 & a_1 & a_2 & a_3 & \dots\dots & a_{n-1} & a_n & \\ & \lambda b_0 & \lambda b_1 & \lambda b_2 & & \lambda b_{n-2} & \lambda b_{n-1} & . \\ \hline b_0 & b_1 & b_2 & b_3 & & b_{n-1} & R & \end{array}$$

First $b_0 = a_0$; multiply b_0 by λ and add to a_1, the sum is b_1; multiply b_1 by λ and add to a_2, the sum is b_2; proceed in this way to the end.

Ex. **Find the quotient and the remainder when**

$$x^5 - 6x^4 + 2x^3 + 15x^2 + 7$$

is divided by $x - 2$.

$$\begin{array}{r} 1 - 6 + 2 + 15 + 0 + \ \ 7 \\ 2 - 8 - 12 + 6 + 12 \\ \hline 1 - 4 - 6 + \ \ 3 + 6 + 19 \end{array}$$

Thus the required quotient is

$$x^4 - 4x^3 - 6x^2 + 3x + 6,$$

the remainder being 19.

The above process is called the method of **Synthetic Division.** The method can easily be extended to the case when the divisor is a multinomial expression, but this extension is not needed for our present purposes.

470. The actual values of b_0, b_1, b_2, &c. in terms of a_0, a_1, a_2, &c. and λ can be at once written down; they are

$$b_0 = a_0,\ b_1 = a_1 + \lambda a_0,\ b_2 = a_2 + \lambda a_1 + \lambda^2 a_0,$$

$$b_3 = a_3 + \lambda a_2 + \lambda^2 a_1 + \lambda^3 a_0,\ \ldots\ldots,$$

$$b_{n-1} = a_{n-1} + \lambda a_{n-2} + \lambda^2 a_{n-3} + \lambda^3 a_{n-4} + \ldots,\ + \lambda^{n-1} a_0,$$

and $$R = a_n + \lambda a_{n-1} + \ldots = f(\lambda).$$

Thus $$\frac{f(x)}{x - \lambda} = a_0 x^{n-1} + (a_1 + \lambda a_0) x^{n-2} + \ldots$$

From the above we can obtain the formula of Art. 439.

For, if $a, b, c, \ldots$ be the roots of the equation $f(x) = 0$; then

$$f'(x) = \frac{f(x)}{x - a} + \frac{f(x)}{x - b} + \ldots$$

$$= \{a_0 x^{n-1} + (a_1 + a a_0) x^{n-2} + (a_2 + a a_1 + a^2 a_0) x^{n-3} + \ldots\}$$

$$+ \{a_0 x^{n-1} + (a_1 + b a_0) x^{n-2} + (a_2 + b a_1 + b^2 a_0) x^{n-3} + \ldots\}$$

$$+ \ldots$$

$$= n a_0 x^{n-1} + (n a_1 + a_0 \Sigma a) x^{n-2}$$

$$+ (n a_2 + a_1 \Sigma a + a_0 \Sigma a^2) x^{n-3} + \ldots$$

But $$f'(x) = n a_0 x^{n-1} + (n - 1) a_1 x^{n-2} + (a - 2) a_2 x^{n-3} + \ldots$$

Equating the coefficients of like powers in the two expansions, we have

$$\begin{aligned} na_0 &= na_0, \\ (n-1)\,a_1 &= na_1 + a_0\Sigma a, \\ (n-2)\,a_2 &= na_2 + a_1\Sigma a + a_0\Sigma a^2, \\ \cdots\cdots &= \cdots\cdots\cdots\cdots\cdots\cdot \end{aligned}$$

Whence the required result follows at once.

471. We have already seen [Art. 441, III.] that in order to diminish each of the roots of the equation $f(x) = 0$ by λ, we have only to substitute $y + \lambda$ for x in $f(x)$.

Let the equation whose roots are those of

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0,$$

each diminished by λ, be

$$b_0y^n + b_1y^{n-1} + b_2y^{n-2} + \ldots + b_n = 0.$$

Then, since $y = x - \lambda$, the last equation is equivalent to

$$b_0(x-\lambda)^n + b_1(x-\lambda)^{n-1} + \ldots + b_{n-1}(x-\lambda) + b_n = 0.$$

The equation last written must be *identical* with $f(x) = 0$.

Hence we have identically

$$f(x) = b_0(x-\lambda)^n + b_1(x-\lambda)^{n-1} + \ldots + b_{n-1}(x-\lambda) + b_n.$$

From the form of the right-hand member of the above identity, it follows that if we divide $f(x)$ by $x - \lambda$, and then divide the quotient by $(x - \lambda)$, and so on, the successive remainders will be the quantities $b_n, b_{n-1}, \ldots, b_1, b_0$.

Ex. 1. Find the equation whose roots are those of

$$x^4 - 2x^3 + 3x - 5 = 0,$$

each diminished by 2.

Using the method of Art. 469 to perform the successive divisions, the whole operation is indicated below, the successive remainders being printed in black type.

$$\begin{array}{rrrrr} 1 & -2+ & 0+ & 3 & -5 \\ & 2 & 0 & 0 & 6 \\ \hline 1 & 0 & 0 & 3 & \mathbf{+1} \\ & 2 & 4 & 8 & \\ \hline 1 & 2 & 4 & \mathbf{11} & \\ & 2 & 8 & & \\ \hline 1 & 4 & \mathbf{12} & & \\ & 2 & & & \\ \hline \mathbf{1} & \mathbf{6} & & & \end{array}$$

The first division gives the quotient x^3+3 with remainder 1; the second division gives the quotient x^2+2x+4 with remainder 11; the third division gives the quotient $x+4$ with remainder 12, and the last division gives the quotient 1, and remainder 6.

Ex. 2. Find the equation whose roots are those of

$$x^3-x^2-x+4=0,$$

each increased by 3.

The divisor is here $x+3$, and the work is as under.

$$\begin{array}{rrrr}
1 & -1 & -1 & +4 \\
 & -3 & +12 & -33 \\
\hline
1 & -4 & +11 & -29 \\
 & -3 & +21 & \\
\hline
1 & -7 & +32 & \\
 & -3 & & \\
\hline
1 & -10 & &
\end{array}
\qquad\qquad
\begin{array}{rrrr}
1 & -1 & -1 & +4 \\
1 & -4 & +11 & \vert -29 \\
1 & -7 & \vert +32 & \\
\hline
1 & -10 & &
\end{array}$$

Thus the transformed equation is

$$x^3-10x^2+32x-29=0.$$

We shall in future write the operation as on the right, the multiplication and addition being performed mentally, and the result only being written down.

472. In order to multiply all the roots of the equation

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n=0$$

by ten, we must substitute $\dfrac{y}{10}$ for x in its left-hand member. If we then multiply throughout by 10^n, the transformed equation will be

$$a_0x^n+10a_1x^{n-1}+10^2a_2x^{n-2}+\ldots+10^na^n=0.$$

Thus in an equation with numerical coefficients the roots will be multiplied by 10 by affixing one nought to the coefficient of x^{n-1}, two noughts to the coefficient of x^{n-2}, and so on.

For example, the equation whose roots are those of

$$x^4-2x^3+5x+8=0,$$

each multiplied by ten, is

$$x^4-20x^3+5000x+80000=0.$$

473. **Horner's method of approximating to the real roots of equations with numerical coefficients.**

Having found (by trial or by Sturm's Theorem) two consecutive integers between which a real positive root of

the given equation must lie, the first step is to *diminish all the roots of the given equation by the smaller of those integers.* Then, by supposition, the transformed equation will have a root between 0 and 1. We now *multiply all the roots of the last equation by* 10 by the process of Art. 472, so that this new equation has by supposition a root between 0 and 10; now find by trial between what two integers less than 10 the root lies, and diminish the roots of the equation by the smaller of these integers. Then again multiply the roots by 10, and continue the process until the required degree of accuracy is attained.

After the roots of the given equation have been diminished by the integral part of the required root, the roots are multiplied by 10 in order to avoid decimals in the work, the next integral root found must therefore originally have been so many *tenths.* After again multiplying the roots by ten, the next integral root found corresponds to *hundredths* in the original equation; and so on.

By the above process it is clear that we are continually approximating to the root sought; care must, however, be taken that we do not pass beyond the root, which would be shewn by the change in sign of the constant term.

The negative roots can be found approximately in a similar manner after changing x into $-x$.

Ex. 1. Find to two places of decimals, the positive root of the equation

$$x^3-3x-4=0.$$

There can only be one positive root, and by trial this must lie between 2 and 3. First diminish the roots by 2, and the transformed equation will be found to be $x^3+6x^2+9x-2=0$. Multiply the roots by 10 and we have the equation $x^3+60x^2+900x-2000=0$, which will be found to have a root between 1 and 2. Diminish the roots of this last equation by 1, and the transformed equation will be $x^3+63x^2+1023x-1039=0$. Multiply the roots of this equation by 10, and the resulting equation will be found to have a root between 9 and 10. Diminish the roots of the last equation by 9, and the resulting equation is $x^3+657x^2+113883x-66541=0$, which could be used to obtain a more accurate result.

The work is written as under, lines being drawn to indicate the completion of each stage of the process.

1	0	−3	−4	(2·19...
	2	+1	−2000	
	4	900	−1039000	
	60	961		
	61	102300	− 66541	
	62	108051		
	630	113883		
	639			
	648			
	657			

Ex. 2. Find the cube root of 30.

We have to find the positive root of the equation $x^3 - 30 = 0$. We proceed as under

1	0	0	−30	(3·107
	3	9	−3000	
	6	2700		
	90	2791	− 209000000	
	91	28830000		
	92			
	9300	28895149	− 6733957	
	9307	28960347		
	9314			
	9321			

It will be seen that after two or three multiplications of the roots by 10, the numbers in the two last columns will become very much greater than in the others; a contracted process can then be employed, namely, instead of affixing one, two, three, &c. zeros to the coefficients after the first in order from left to right, we may cut off one, two, three, &c. figures from the coefficients after the first in order from right to left. Proceeding in this way with the above example after the stage at which it was left, we have

$\cancel{1}$	$93\cancel{2}\cancel{1}$	$2896034\cancel{7}$	− 6733957	(3·1072325
	$\cancel{9}\cancel{3}$	$28962\cancel{2}\cancel{0}$	− 941517	
			− ·72651	
			− 16727	
			− 2247	

The first of the new figures is 2; and after finding 2, the numbers standing in the columns will be 93, 2896220, −941517, the original first column having previously disappeared. We then cut off one figure from the second column and both from the first; we then have only to divide 941517 by 289622, cutting off one figure from the divisor

at each successive stage, as in the ordinary method of contracted division.

474. **Imaginary roots.** The numerical values of the imaginary roots of an equation can theoretically be obtained in the following manner, but the work would, except in very simple cases, be very laborious.

Ex. Find the numerical values of the imaginary roots of the equation $x^3+3x-1=0$.

Put $\alpha+\iota\beta$ for x in $f(x)$, and equate the real and imaginary expressions separately to zero; then we shall have

$$\alpha^3+3\alpha-1-3\alpha\beta^2=0 \text{ and } 3\alpha^2\beta-\beta^3+3\beta=0.$$

Rejecting the factor $\beta=0$, which corresponds to a real root of the given equation, we have by eliminating β the equation

$$8\alpha^3+6\alpha-1=0.$$

Now α must be a *real* root of the equation last written, and this real root will be found to be $-\cdot16109......$

Then $\beta^2=3(\alpha^2+1)$, whence β is found to be $1\cdot75438....$ Thus the required roots are $-\cdot16109...\pm1\cdot75438...\sqrt{-1}$.

EXAMPLES XLVI.

1. Solve the following equations:

(i) $x^3-12x+65=0$. (ii) $x^3-9x+28=0$.

(iii) $x^3-48x-520=0$. (iv) $x^3-21x-344=0$.

(v) $x^3-2x+5=0$. (vi) $x^3-6x-11=0$.

2. Solve the following equations:

(i) $x^4+2x^3+14x+15=0$. (ii) $x^4-12x-5=0$.

(iii) $x^4-12x^2+24x+140=0$. (iv) $4x^4+4x^3-7x^2-4x-12=0$.

3. Apply Sturm's Theorem to find the number and position of the real roots of the following equations:

(i) $x^3-3x+6=0$. (ii) $x^3-x^2-33x+61=0$.

(iii) $2x^4-x^2-10x+3=0$. (iv) $x^4-14x^2+16x+9=0$.

(v) $x^4-7x^2+3x-20=0$.

4. Find all Sturm's functions for the equation $x^3+3px^2+3qx+r=0$, and hence shew that, if $p^2<q$, there must be two imaginary roots.

5. Prove that the roots of $x^3+px^2+r=0$ are all real if either p or r be negative, and $-4p^3r$ be greater than $27r^2$.

6. The coefficients of the algebraical equation $f(x)=0$ are all integers. Shew that, if $f(0)$ and $f(1)$ are both odd numbers, the equation can have no integral roots.

7. Shew that one root of the equation $x^3-2x-5=0$ is $2\cdot09455148$.

8. Find the real positive roots of the following equations, each to 4 places of decimals:

(i) $x^3-7x+7=0$. (iv) $x^4+x^3-4x^2-16=0$.

(ii) $x^3-8x-40=0$. (v) $x^4-14x^2+16x+9=0$.

(iii) $x^3-6x^2+9x-3=0$. (vi) $x^5-2=0$.

9. Find the number and position of the real roots of

(i) $x^4+2x^3-23x^2-24x+144=0$,

and (ii) $x^4-26x^2+48x+9=0$.

10. Prove that the equation

$$x^6-7x^3+15x^2+3x-4=0$$

cannot have more than 4 real roots; prove also that these roots must lie between 8 and -5.

11. Solve the equation

$$2x^5-7x^4+6x^3-11x^2+4x+6=0,$$

having given that the real roots are commensurable.

12. Find the equation whose roots are the square of the roots of

$$x^3-3px^2-3(1-p)x+1=0,$$

hence shew that the given equation has three real roots for all real values of p.

13. Prove that the equation

$$x^3-3px^2-3(1-p)x+1=0$$

has three real roots for all real values of p.

Prove also that, if these roots be α, β, γ then

$$\beta(1-\gamma)=\gamma(1-\alpha)=\alpha(1-\beta)=1,$$

or

$$\beta(1-\alpha)=\gamma(1-\beta)=\alpha(1-\gamma)=1.$$

14. Shew that the equation whose roots are the sum of pairs of roots of the quintic $x^5+px+q=0$ is

$$x^{10}-3px^6-11qx^5-4p^2x^2+4pqx-q^2=0.$$

15. Prove that the equation

$$x^4+4ax^3+6a^2x^2+4ax+1=0$$

has no real roots unless

$$1>a^2>\frac{1}{9},$$

and that the equation has two real roots if a^2 is between these limits.

16. Prove that, if a be the root of the equation

$$x^4 + ax^3 - 6x^2 - ax + 1 = 0,$$

so also is

$$\frac{1+a}{1-a}.$$

Prove also that the other two roots are

$$-\frac{1}{a} \text{ and } \frac{1+a}{1-a}.$$

17. Prove that, if $\alpha, \beta, \gamma, \ldots$ be the roots of

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0,$$

then

$$\Sigma\alpha^2\beta^2\gamma = -p_2p_3 + 3p_1p_4 - 5p_5.$$

18. Shew that, if $a_1, a_2, \ldots$ be the roots of

$$x^5 - 5px^3 + 5p^2x - q = 0,$$

then

$$\Sigma a_1^4 a_2^3 a_3^2 a_4 + 5q^2 + 500p^5 = 0.$$

ANSWERS TO THE EXAMPLES.

EXAMPLES I.

1. $4x$. **2.** $-2x-6y-4z$.

3. $\frac{5}{12}a+\frac{5}{12}b+\frac{5}{12}c$. **4.** a^4. **5.** $-xy-4y^2$.

6. $2m^2+2mn+2n^2$. **7.** $3a^2+2b^2+c^2+ab-4ac+bc$.

8. a^2b+10b^3. **9.** $-2a+5b-4c$.

10. $-a-\frac{13}{6}b+\frac{8}{3}c$. **11.** x^2-x-9.

12. $-5a^4+3a^3b-3ab^3+5b^4$. **13.** $2x^2-7xy+7y^2$.

14. $-bc+4ca+4ab$. **15.** $-3a^2+2b^2-3c^2+bc+ca+ab$.

16. $x-y$. **17.** $-5y-3z$. **18.** $-2x+2y$.

19. $b+d$. **20.** y. **21.** $4a$. **22.** $-3x+3y$.

23. $-4n+4m$. **24.** 20. **25.** -20.

EXAMPLES II.

1. $2x^2-5ax+2a^2$. **2.** $x^2-\frac{82}{9}x+1$. **3.** x^3-1.

4. x^3+y^3. **5.** x^4-1. **6.** y^5-x^5.

7. x^4-x^2+4x-4. **8.** $1+a^2x^2+a^4x^4$.

9. x^8+x^4+1. **10.** $6x^4-5x^3y+14x^2y^2-5xy^3+6y^4$.

11. $2x^6-10x^5+5x^4-22x^3-5x^2+5x+1$.

12. $4x^6-10x^5y+10x^4y^2-21x^3y^3-5x^2y^4+5xy^5+y^6$.

13. $6a^6+11a^5b-16a^4b^2+20a^3b^3-29a^2b^4+15ab^5-3b^6$.

14. $2a^6x^6-3a^5x^5y^2+8a^4x^4y^4-11a^3x^3y^6+6a^2x^2y^8+20axy^{10}-10y^{12}$.

15. $2a-3a^2+a^3+6a^4-5a^5-18a^6+44a^7-42a^9$.

16. $a^3+b^3+c^3-3abc$. **17.** $x^3+y^3+z^3-3xyz$.

18. $8a^3+27b^3-c^3+18abc$. **19.** x^8-1.

20. x^8-256y^8. **21.** $x^8-2x^4y^4+y^8$.

22. $x^{12}-3x^8+3x^4-1$. **23.** $x^8+2x^6+3x^4+2x^2+1$.

24. $a^8+8a^6b^2+48a^4b^4+128a^2b^6+256b^8$.

25. (i) $a^2+4b^2+9c^2+4ab-6ac-12bc$,

(ii) $a^4-2a^3b+3a^2b^2-2ab^3+b^4$,

(iii) $b^2c^2+c^2a^2+a^2b^2+2a^2bc+2ab^2c+2abc^2$.

(iv) $1-4x+10x^2-12x^3+9x^4$.

(v) $x^6+2x^5+3x^4+4x^3+3x^2+2x+1$.

26. (i) $a^3+b^3+c^3+3(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2)+6abc$.

(ii) $8a^3-27b^3-8c^3-36a^2b+54ab^2-24a^2c+24ac^2-54b^2c$
$-36bc^2+72abc$.

(iii) $1+3x+6x^2+7x^3+6x^4+3x^5+x^6$. **27.** $8xz$.

30. $x^4-2a^3x-2ax^3-a^4$. **41.** 0. **50.** $\Sigma a^3-3\Sigma abc$.

EXAMPLES III.

1. $x-3y$. **2.** x^2+4y^2. **3.** $9x^2-12xy+16y^2$.

4. $-3x-2y$. **5.** $1+x+x^2+x^3-4x^4$.

6. $x^4+x^3y+x^2y^2+xy^3-4y^4$. **7.** $1+2x+3x^2+4x^3+5x^4$.

8. $m^4+2m^3n+3m^2n^2+4mn^3+5n^4$.

9. $1+2x+3x^2+4x^3+5x^4+6x^5$.

10. $1+x^2+x^4+x^6$. **11.** $1-2x+3x^2$.

12. $2-3x+2x^2$. **13.** $2x^2-3xy+y^2$.

14. $x^2-xy-2x+y^2+y+1$. **15.** x^3+y^3.

16. $x^2-2xy-2y^2$. **17.** $a+2b-3c$.

18. $a+2b+3c$. **19.** $3a^2+4ab+b^2$.

20. $x^2+y^2+z^2-1$. **21.** $a^2-2ab+ac+b^2-bc+c^2$.

22. $a^2+4b^2+c^2-2ab+ac+2bc$. **23.** $a+2b+3c$.

24. $9a^2+4b^2+9c^2-6bc+9ca+6ab$.

25. cx^2+dx-c. **26.** $2ax-(3b-4c)y$. **27.** $a^2-3ab+b^2$.

28. x^2-xy+y^2, $(x+y)^2-z(x+y)+z^2$.

29. x^2+xy+y^2, $(x+y)^2+2z(x+y)+4z^2$.

EXAMPLES IV.

1. $(a-2b)(a+2b)(a^2+4b^2)$. **2.** $(2x-3ab)(2x+3ab)(4x^2+9a^2b^2)$.

3. $(4+3a-2b)(4-3a+2b)$. **4.** $(2y+2z-x)(2y-2z+x)$.

5. $5ax(2ax+3y)(2ax-3y)$. **6.** $4a^2x^2(3x^2+y^2)(3x^2-y^2)$.

7. $8(a-b)(a+b)(a^2+b^2)$. **8.** $16(a-b)(a+b)(a^2+b^2)$.

9. $24x(x-1)(x+1)^2$. **10.** $16x(2-3x^2)$.

11. $4b^3(2a-b^2)(4a^2+2ab^2+b^4)$. **12.** $(a^2-4bc)(a^4-2a^2bc+4b^2c^2)$.

13. $(a-4)(a+2)$. **14.** $(4-x)(3+x)$.

15. $(1-21x)(1+3x)$. **16.** $-4(a-1)^2$.

17. $ab(a-b)(a-3b)$. **18.** $a^2b(a+b)(a+4b)$.

19. $(b+c-a)(b+c-5a)$. **20.** $(3a+3b-c-d)^2$.

21. $(x-2)(x+2)(x-5)(x+5)$. **22.** $(5x-y)(5x+y)(2x-y)(2x+y)$.

23. $(x^2-4y^2z^2)^2=(x+2yz)^2(x-2yz)^2$.

24. $a^2(a+b)(a-b)(3a+b)(3a-b)$.

25. $(x-b)(x+b-2a)$. **26.** $(x-a)(x+2y+a)$.

27. $(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d)$.

28. $(x+y+a+b)(x+y-a-b)(x-y+a-b)(-x+y+a-b)$.

EXAMPLES V.

1. $(x+1)(x-1)(x+a)$.

2. $(a+b)(c-d)$.

3. $(a-b)(c+d)(c-d)$.

4. $(ax+by)(cx+dy)$.

5. $(ax+b)(cx^2+d)$.

6. $2(a-d)(a+b+c+d)$.

7. $(a+b)(a-b)(a^2+ab+b^2)$.

8. $(a-b)^2(a^2+ab+b^2)$.

9. $(a+1)(a-1)(b+1)(b-1)$.

10. $(x+z)(x-z)(y+z)(y-z)$.

11. $(x^2z-1)(y^2z-1)$.

12. $(x+y)(x+z)(x^2-xz+z^2)$.

13. $(x-y)(x+y+z)$.

14. $(x+3)(x-3)(x^2+2)$.

15. $(x^2+5x+1)(x^2-5x+1)$.

16. $(x^2+4xy+y^2)(x^2-4xy+y^2)$.

17. $(x^2+x+1)(x^2-x+1)(x^4-x^2+1)$.

18. $(x+a+b)(x-a-b)(x+a-b)(x-a+b)$.

19. $(x^2-2y^2z^2)^2$.

20. $(x-2b+ab)(x-2a-ab)$.

21. $(x+b)(x^2+a)$.

22. $(1-x^2)\{1+y+x(1-y)\}\{1+y-x(1-y)\}$.

23. $(x+y-3z)(x-y+z)$.

24. $(2y-x+a)(y-2x-a)$.

25. $(a-3b+c)(a+b-3c)$.

26. $(2a-11b+1)(a+2b-3)$.

27. $(1-ax)(1+ax+bx^2)$.

28. $(1-ax)(1-ax-cx^2)$.

29. $-(b-c)(c-a)(a-b)$.

30. $(b+c)(c+a)(a+b)$.

31. $(a-b)(a-c)(b+c)$.

32. $(x^2-xy+y^2)\{x(a+1)+y(b+1)\}$.

33. $(xy+ab)(ay^2+b^2x)$.

34. $(2x-z)(x-y)^2$.

35. $(x^2-yz)(y^2-zx)(z^2-xy)$.

36. $(x+4)(x+2)(x-1)(x-3)$.

37. $(x+4)(x+2)(x^2+5x+8)$.

38. $x(x+5)(x^2+5x+10)$.

39. $(x+2)(x+6)(x^2+8x+10)$.

40. $(x+8)(2x+15)(2x^2+35x+120)$.

EXAMPLES VI.

1. $3(y-z)(z-x)(x-y)$.

2. $5(y-z)(z-x)(x-y)(x^2+y^2+z^2-yz-zx-xy)$.

3. $(b+c)(b-c)(c+a)(c-a)(a+b)(a-b)$.

4. $(b-c)(c-a)(a-b)(a+b+c)$.

5. $(b-c)(c-a)(a-b)(a^3+b^3+c^3+b^2c+bc^2+c^2a+ca^2+a^2b+ab^2-9abc)$.

6. $-(b-c)(c-a)(a-b)$.

7. $-(b-c)(c-a)(a-b)[b^2c^2+c^2a^2+a^2b^2+abc(a+b+c)]$.

8. $-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$.

9. $-(b-c)(c-a)(a-b)(a^3+b^3+c^3+b^2c+bc^2+c^2a+ca^2+a^2b+ab^2+abc)$.

10. $24abc$. **11.** $80abc(a^2+b^2+c^2)$. **12.** $4abc$.

13. $2abc$. **14.** $4abc$. **15.** $-4(b-c)(c-a)(a-b)$.

16. $3(y+z)(z+x)(x+y)$.

17. $5(y+z)(z+x)(x+y)(x^2+y^2+z^2+yz+zx+xy)$.

18. $-(b-c)(c-a)(a-b)$. **19.** $-2(b-c)(c-a)(a-b)(a+b+c)$.

20. $-(b-c)(c-a)(a-b)(3a^2+3b^2+3c^2+5bc+5ca+5ab)$.

21. $(b+c)(c+a)(a+b)$. **22.** $-(b-c)(c-a)(a-b)(a+b+c)^2$.

23. $(x+y+z)(yz+zx+xy)$. **24.** $(b+c)(c+a)(a+b)(a+b+c)$.

25. $12xyz(x+y+z)$. **26.** $-3(b-c)(c-a)(a-b)$.

27. $16(b-c)(c-a)(a-b)(d-a)(d-b)(d-c)$.

32. $27a^2b^2(a+b)^2$. **34.** $(a^2+b^2)^2(c^2+d^2)^2$.

36. $(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)$.

37. $-(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)(bcd+cda+dab+abc)$.

EXAMPLES VII.

1. $a-b$.
2. $2x-1$.
3. x^2-y^2.
4. $2x-y$.
5. $x-2y+3z$.
6. $4a^2-3ab+b^2$.
7. $a+2b$.
8. $2x^2-3x+1$.
9. $x-a$.
10. x^2+x-6.
11. x^2-x+3.
12. x^2-3x+7.

EXAMPLES VIII.

1. $12x^4+2ax^3-4a^2x^2-27a^3x-18a^4$.
2. $(4a-b)(a-b)(3a^2+b^2)$.
3. $(x^2-2x+7)(6x^3+x^2-44x+21)$.
4. $(x^2+5x+7)(7x^4-40x^3+75x^2-40x+7)$.
5. $x(x+1)(x+2)(x-2)(x+3)$.
6. $x(x-1)(x+2)(x+6)(x^2-2x+4)$.
7. $2a(2a-b)(2a-3b)(2a+3b)$.
8. $6x(x+1)(x-3)(x-4)$.
9. $(3x+2)(8x^3+27)(8x^3-27)$.
10. $3(x-3y)^2(x^2-4y^2)$.
11. $(x-2y)(x-3y)(x-4y)$.

14. $(ac'-a'c)^3=(ba'-b'a)^2(b'c-bc')$.

EXAMPLES IX.

1. $\dfrac{5b^2c^3y^3z^2}{6a^3x^3}$.
2. $\dfrac{3ab^2c^6x^5z^4}{y^5}$.
3. $\dfrac{a-b}{a+4b}$.
4. $\dfrac{x^2y^2-1}{4x^2y^2+1}$.
5. $\dfrac{x^2+xy+y^2}{x^2-xy+y^2}$.
6. $\dfrac{x^2-xy+y^2}{x^2+y^2}$.
7. $\dfrac{2x-1}{x^2+1}$.
8. $\dfrac{(x-1)^2}{x^2-3x+1}$.
9. $\dfrac{2x+3y}{3x^2-y^2}$.
10. $\dfrac{9x^3-3x-2}{6x^3+3x^2-1}$.

11. $\dfrac{a+b}{(a-b+c)(-a+b+c)}$. **12.** $yz+zx+xy$.

13. $-\frac{1}{2}(y-z)(z-x)(x-y)$. **14.** $\dfrac{a-c}{d-b}$.

15. $\dfrac{yz+zx+xy}{x+y+z}$. **16.** $\dfrac{a^2+b^2}{a^2-b^2}$.

17. $\dfrac{1}{1-9x^2}$. **18.** $\dfrac{xy+y^2}{x^2-4y^2}$.

19. $\dfrac{2x+4a}{x-2a}$. **20.** $\dfrac{48}{(x+2)(x+4)(x+6)(x+8)}$.

21. $\dfrac{48a^3}{(x+a)(x+3a)(x+5a)(x+7a)}$. **22.** $\dfrac{24a^4}{x(x^2-a^2)(x^2-4a^2)}$.

23. 0. **24.** 0. **25.** 1. **26.** -1.

27. d. **28.** 0. **29.** 1. **30.** 2.

31. $a+b+c$. **32.** $a^2+b^2+c^2+bc+ca+ab$.

33. $(a+b+c)^2$. **34.** $a+b+c$.

35. $\dfrac{a+b+c}{(-a+b+c)(a-b+c)(a+b-c)}$. **36.** 0.

37. $16\left(\dfrac{x^4+a^4}{x^4-a^4}\right)^2$. **38.** $4\dfrac{a^4x^2-b^4y^2}{a^4x^4-b^4y^4}$.

40. $\dfrac{(x-p)(x-q)}{(x+a)(x+b)(x+c)}$. **41.** -2. **42.** 4.

43. $\dfrac{2abc(a+b+c)}{(-a+b+c)(a-b+c)(a+b-c)}$. **47.** $2(x+y+z)$.

EXAMPLES X.

1. $2a-b,\ a-2b$. **2.** $1,\ \dfrac{b+c-2a}{c+a-2b}$. **3.** $0,\ \dfrac{2ab}{b-a}$.

4. $a-2b,\ b-2a$. **5.** ± 1. **6.** ± 1.

7. $1,\ -3$. **8.** 1.

9. $0,\ \pm 5\sqrt{2}$. **10.** $6,\ -8\frac{1}{2}$.

11. $\frac{50}{29}$.

12. $\frac{a^2c+b^2a+c^2b-3abc}{a^2+b^2+c^2-bc-ca-ab}$.

13. $0, \frac{1}{3}\{a+b+c\pm\sqrt{(a^2+b^2+c^2-bc-ca-ab)}\}$.

14. $[bc+ca+ab\pm\sqrt{\{b^2c^2+c^2a^2+a^2b^2-abc(a+b+c)\}}]\div(a+b+c)$.

15. $5, -\frac{5}{4}$.

16. $\pm\sqrt{6}$.

17. $\pm\sqrt{ab}, \pm\sqrt{-ab}$.

18. $0, -\frac{5}{2}$.

19. $\pm\sqrt{\left(\frac{b}{a}+b^2\right)}$.

20. $\{a+b+c\pm\sqrt{(a^2+b^2+c^2-bc-ca-ab)}\}$.

21. $a, -\frac{b^2+c^2}{b+c}$.

22. $-2(a+b+c)$.

23. $\frac{a^2+b^2}{a+b}$.

24. $\frac{cd(a+b)-ab(c+d)}{ab-cd}$.

25. $\frac{ab-cd}{c+d-a-b}$.

26. $0, a+b, \frac{a^2+b^2}{a+b}$.

27. $\frac{3(b-c)(c-a)(a-b)}{(b-c)^2+(c-a)^2+(a-b)^2}$.

28. $0, \frac{1}{26}(-19\pm\sqrt{-3})$.

29. $0, \pm\sqrt{mab}$.

30. $8, -5$.

31. $2, 3$.

32. 1.

33. $0, a^2-b^2$.

34. a, b.

35. $\frac{a}{b}, \frac{c}{d}$.

36. $0, 4(a+b)$.

37. $-a, -b$.

38. $\frac{1}{2}(a-b)$.

39. $\pm\sqrt{ab}$.

40. $0, \pm 2\sqrt{ab}$.

41. $\pm\frac{2}{3}\sqrt{3a^2-3b^2}$.

42. $\pm\sqrt{\frac{b^4-a^4}{a^2-2b^2}}$.

43. 0.

44. $\pm a\pm b$.

45. $-\frac{1}{3}\{a+b+c\pm 2\sqrt{(a^2+b^2+c^2-bc-ca-ab)}\}$.

46. $\frac{ab(a+b)}{a^2+ab+b^2}, -\frac{3ab(a+b)}{a^2+ab+b^2}$.

47. $\pm\frac{b^2c^2+c^2a^2+a^2b^2}{2abc}$.

48. $\pm\frac{1}{2abc}\sqrt{\{2a^2b^2c^2(a^2+b^2+c^2)-b^4c^4-c^4a^4-a^4b^4\}}$

$=\pm\frac{1}{2abc}\sqrt{\{(bc+ca+ab)(-bc+ca+ab)(bc-ca+ab)(bc+ca-ab)\}}.$

49. Values between 3 and $-\frac{4}{3}$.

52. Values between 3 and $\frac{1}{3}$.

53. x lies between -2 and 8, and y between -9 and 1.

54. x between -2 and 10, and y between -1 and 5.

55. $\frac{a}{2}$.

59. (i) $a^2x^2+(b^3-3abc)x+c^3=0.$

(ii) $a^2cx^2+xb(b^3-3ac)+ac^2=0.$

(iii) $x^2-bx+ac=0.$

60. (i) $-2\frac{c^2}{a}$. (ii) $\frac{b^2c^2}{a^2}-2\frac{c^3}{a}$.

65. $\frac{6}{5}$.

EXAMPLES XI.

1. $\pm 2,\ \pm\sqrt{-2}.$

2. $a,\ a\omega,\ a\omega^2,\ -2a,\ -2a\omega,\ -2a\omega^2.$

3. $-a,\ -a\omega,\ -a\omega^2,\ 2a,\ 2a\omega,\ 2a\omega^2.$

4. $1,\ \frac{1}{4}(1\pm\sqrt{-15}).$

5. $0,\ 1,\ 3,\ -8.$

6. $1,\ -2,\ \frac{1}{2}(-1\pm\sqrt{-19}).$

7. $-1,\ -6,\ \frac{1}{2}(-7\pm 3\sqrt{5}).$

8. $\frac{1}{2},\ -\frac{15}{2},\ \frac{1}{2}(-7\pm 4\sqrt{2}).$

9. $3,\ -1,\ 1\pm 2\sqrt{19}.$

10. $\frac{1}{2}(-1\pm\sqrt{-3}),\ \frac{1}{2}(a\pm\sqrt{a^2-4}).$

11. $0, \ -5, \ \frac{1}{2}(-5 \pm \sqrt{-15})$. **12.** $a, \ -9a, \ -4a \pm a\sqrt{-15}$.

13. $7a, \ -8a, \ \frac{a}{2}(-1 \pm \sqrt{-167})$. **14.** $-4, \ -6, \ \frac{1}{2}(-15 \pm \sqrt{129})$.

15. $3, \ -\frac{21}{13}$. **16.** $\pm\sqrt{(a+b)}$. **17.** $\pm a \pm b$.

18. $2, \frac{1}{2}, \ \frac{1}{4}(-3 \pm \sqrt{-7})$. **19.** $3, \frac{1}{3}, \ \frac{1}{3}(-1 \pm \sqrt{-8})$.

20. $-1, \ \frac{1}{4}[1+\sqrt{5} \pm \sqrt{\{+2\sqrt{5}-10\}}], \ \frac{1}{4}[1-\sqrt{5} \pm \sqrt{(-2\sqrt{5}-10)}]$.

21. $\pm 1, \ \pm\sqrt{-1}; \ \pm\frac{1}{\sqrt{2}} \pm \sqrt{-\frac{1}{2}}$. **22.** $2, \ 2, \ \frac{2}{3}$.

23. $-1, \ 2, \ 3, \ -4$. **24.** $\pm 1, \ \frac{1}{2}(-7 \pm 3\sqrt{5})$.

25. $a, \ b, \ c$. **26.** $9, \ -3 \pm \sqrt{-47}$.

27. $9, \ -6, \ \frac{1}{2}(3 \pm \sqrt{-215})$. **28.** $a, \ b, \ \frac{1}{2}(a+b)$.

29. $a, \ b; \ \frac{1}{2}(a+b) \pm \frac{1}{14}(a-b)\sqrt{-7}$.

30. $a, \ b, \ \frac{1}{2}(a+b), \ \frac{1}{2}(a+b) \pm \frac{1}{6}(a-b)\sqrt{-3}$.

31. $a, \ b, \ \frac{1}{2}(a+b)$. **32.** $a, \ b, \ \frac{1}{2}\{a+b \pm \frac{1}{63}(a-b)\sqrt{-63}\}$.

33. $a, \ b, \ \frac{1}{2}\{a+b \pm (a-b)\sqrt{-3}\}$.

34. $a, \ b$. **35.** $a, \ b$, and roots of $(a-x)(x-b) = 16(a-b)^2$.

36. $\frac{a}{2} \pm \sqrt{\frac{-3a^2}{4} \pm \sqrt{\frac{1}{2}a^4 + \frac{1}{2}b^4}}$.

37. $a-2b, \ b-2a, \ -\frac{1}{2}\{a+b \pm (a-b)\sqrt{-15}\}$.

38. Roots of $x(a-x) = \left(\sqrt{b} \pm \sqrt{\frac{a}{2} + \frac{b}{2}}\right)^4$.

39. $\frac{a^2}{b}$, $\frac{b^2}{a} \pm \sqrt{ab}$. **40.** $\frac{bc}{a}$, $\frac{ca}{b}$, $\frac{ab}{c}$.

41. 0, $a+b$, $\frac{a^2+b^2}{a+b}$, $\frac{2ab}{a+b}$.

42. $\frac{b}{2}\{-b \pm \sqrt{(b^2+4)}\}$, $\frac{b}{2}\{-a \pm \sqrt{(a^2+4)}\}$. **43.** $\frac{1}{2}\{a \pm \sqrt{(a^2-4b^2)}\}$.

44. $-(a+b+c)$, $-\frac{1}{2}(a+b+c) \pm \frac{1}{2}\sqrt{(\Sigma a^2 - 2\Sigma bc)}$.

45. $a+b+c$, $\frac{2}{3}(a+b+c) \pm \frac{1}{3}\sqrt{(\Sigma a^2 - 7\Sigma bc)}$. **46.** a, b, c.

47. 0, $\pm\sqrt{\left\{-\frac{cd(a+b)+ab(c+d)}{a+b+c+d}\right\}}$, $\pm\sqrt{\left\{\frac{ab(c+d)-cd(a+b)}{c+d-a-b}\right\}}$.

EXAMPLES XII.

1. $x=1$, $y=-1$. **2.** $x=\frac{18}{7}$, $y=\frac{8}{3}$.

3. $x=3$, $y=6$. **4.** $x=\frac{2}{3}$, $y=3$.

5. $x=b$, $y=a$. **6.** $x=ab$, $y=-a-b$.

7. $x=a+b$, $y=a-b$. **8.** $x=y=a$.

9. $x=a$, $y=b$. **10.** $x=a(a-b)$, $y=b(a-b)$.

11. $x=-3$, $y=3$, $z=1$. **12.** $x=\frac{1}{2}$, $y=\frac{1}{3}$, $z=\frac{1}{6}$.

13. $x=y=z=1$.

14. $x=b+c-a$, $y=c+a-b$, $z=a+b-c$.

15. $x=b+c$, $y=c+a$, $z=a+b$.

16. $x=-\frac{1}{2}(2a+b+c)$, $y=-\frac{1}{2}(a+2b+c)$, $z=-\frac{1}{2}(a+b+2c)$.

17. $x=y=z=\frac{1}{a+b+c}$.

18. $x=\frac{1}{2}(2a+b+c)$, $y=\frac{1}{2}(a+2b+c)$, $z=\frac{1}{2}(a+b+2c)$.

19. $x=\frac{a}{(a-b)(a-c)}$, $y=\frac{b}{(b-c)(b-a)}$, $z=\frac{c}{(c-a)(c-b)}$.

20. $x=a$, $y=b$, $z=c$.

21. $x=-a+b+c$, $y=a-b+c$, $z=a+b-c$.

22. $x=a(b-c)$, $y=b(c-a)$, $z=c(a-b)$.

23. $x=1$, $y=0$, $z=0$.

24. $x=abc$, $y=bc+ca+ab$, $z=a+b+c$.

25. $x=\frac{m(m-b)(m-c)}{a(a-b)(a-c)}$, &c.

26. $x=a$, $y=b$, $z=c$.

27. $x=b+c$, $y=c+a$, $z=a+b$.

28. $x=\frac{a(a+b+c)}{(a-b)(a-c)}$, &c.

29. $x=\frac{1}{2}(m+n)$, $y=\frac{1}{2}(n+l)$, $z=\frac{1}{2}(l+m)$.

30. $x=y=z=l^2+m^2+n^2-mn-nl-lm$.

31. $lx=my=nz=1$.

32. $x=\frac{(a+\alpha)(a+b)(a+c)}{(\alpha-\beta)(\alpha-\gamma)}$, &c.

33. $x=\frac{1}{3}(b+c+d-2a)$, &c.

34. $x=abcd$, $y=-(bcd+cda+dab+abc)$,
$z=bc+ca+ab+ad+bd+cd$,
$\omega=-(a+b+c+d)$.

EXAMPLES XIII.

1. 12, 11.

2. 1, 1, $\frac{8}{15}$, $\frac{1}{15}$.

3. ± 3, ± 1; $\pm 4\sqrt{\frac{2}{3}}$, $\pm\frac{1}{2}\sqrt{\frac{2}{3}}$.

4. ± 2, ± 3; $\pm\frac{16}{9}\sqrt{3}$, $\mp\frac{13}{9}\sqrt{3}$.

5. 12, 7; -7, -12.

6. a, b; $2a-b$, $2b-a$.

7. $\frac{2ab}{b-a}$, $\frac{2ab}{a+b}$.

8. $\pm\frac{a}{b}\sqrt{a^2+b^2}$, $\pm\frac{b}{a}\sqrt{a^2+b^2}$.

9. $\pm\frac{a}{3}$, $\pm b$; $\pm a$, $\pm\frac{b}{3}$.

10. $a\pm\sqrt{\frac{b^3-a^3}{3a}}$, $a\mp\sqrt{\frac{b^3-a^3}{3a}}$.

11. ± 7, ∓ 5; ± 5, ∓ 7.

12. 9, 4; 4, 9.

13. 64, 8; 8, 64.

14. $-6\pm\sqrt{30}$, $6\mp\sqrt{30}$.

15. $\frac{1}{2}(1\pm\sqrt{-11})$, $\frac{1}{2}(1\mp\sqrt{-11})$; 2, -1; -1, 2.

16. 1, 1; $2\pm\sqrt{7}$, $2\mp\sqrt{7}$.

17. 2, 4; $-\frac{7}{3}$, $-\frac{14}{3}$.

18. 2, 1; 1, 2.

19. $2a$, $2b$; $-a$, $-b$.

20. $\frac{1}{2}$, 2.

21. b, a.

22. 6, 6; $-\frac{3}{2}(1\pm\sqrt{5})$, $-\frac{3}{2}(1\mp\sqrt{5})$.

23. $\pm a\sqrt{\frac{\pm ab}{a^2+b^2}}$, $\pm b\sqrt{\frac{\pm ab}{a^2+b^2}}$.

24. $\pm\frac{1}{a}\sqrt{1+a^2}$, $\pm\sqrt{1+a^2}$.

25. 8, 4; 2, 4.

26. 4, 2; 2, 4; $3\pm\sqrt{-13}$, $3\mp\sqrt{-13}$.

27. $\frac{1}{2}$, $\frac{1}{6}$; $\frac{1}{6}$, $\frac{1}{2}$; 0, 0.

28. 3, -6; $-\frac{1}{3}$, $\frac{2}{3}$.

29. 1, $\frac{1}{2}$; -1, -2.

30. 0, 0; $\frac{a+b}{2ab}$, $\frac{b-a}{2ab}$.

31. b, a; $\frac{a^2}{b}$, $\frac{b^2}{a}$.

32. b, a; b, $\frac{b^2}{a}$; $\frac{a^2}{b}$, a; $\sqrt[3]{a^2b}$, $\sqrt[3]{ab^2}$.

EXAMPLES XIV.

1. $\frac{bc}{a}$, $\frac{ca}{b}$, $\frac{ab}{c}$; $-\frac{bc}{a}$, $-\frac{ca}{b}$, $-\frac{ab}{c}$.

2. $\frac{x}{a^2}=\frac{y}{b^2}=\frac{z}{c^2}=\pm\frac{1}{\sqrt{(a^2+b^2+c^2)}}$.

3. $x(b+c)=y(c+a)=z(a+b)=\pm\sqrt{\left\{\frac{1}{2}(b+c)(c+a)(a+b)\right\}}$.

4. $0, 0, 0;\ \frac{2abc}{-bc+ca+ab},\ \frac{2abc}{bc-ca+ab},\ \frac{2abc}{bc+ca-ab}$.

5. $0, 0, 0;\ \frac{2bc}{b+c-a},\ \frac{2ac}{c+a-b},\ \frac{2ab}{a+b-c}$.

6. $x=y=z=\pm 2$.

7. $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}=\pm\frac{1}{\sqrt{(2a+2b+2c)}}$.

8. $-a\pm\frac{bc}{a},\ -b\pm\frac{ca}{b},\ -c\pm\frac{ab}{c}$.

9. $\frac{x}{a^2(b^2+c^2)}=\frac{y}{b^2(c^2+a^2)}=\frac{z}{c^2(a^2+b^2)}=\pm\frac{1}{2abc}$.

10. $\frac{x}{a(-a+b+c)}=\frac{y}{b(a-b+c)}=\frac{z}{c(a+b-c)}$

$$=\pm\frac{1}{\sqrt{\{(-a+b+c)(a-b+c)(a+b-c)\}}}.$$

11. $x=y=z=0;\ \frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$

$$=\pm\frac{1}{\sqrt{\{(b+c-a)(c+a-b)(a+b-c)\}}}.$$

12. $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}=\frac{2(a^2+b^2+c^2)}{(b+c-a)^2+(c+a-b)^2+(a+b-c)^2}$;

$$x=y=z=0.$$

13. $x=1\pm\sqrt{\frac{(b+1)(c+1)}{a+1}}$, &c.

14. $x=a\pm\sqrt{\frac{(a^2+\beta)(a^2+\gamma)}{a^2+a}}$, &c.

15. $x=a\pm\sqrt{\frac{(g^2+ca)(h^2+ab)}{f^2+bc}}$, &c.

16. $1, 2, 3;\ \frac{3}{10},\ \frac{5}{6},\ \frac{2}{3}$.

17. 1, 2, 3. **18.** 3, 5, 7. **19.** 0, 4, 5.

20. 3, 3, 4. **21.** $0, 0, 0;\ \frac{a}{2},\ \frac{b}{2},\ \frac{c}{2}$.

22. $-a, b, c;\ a, -b, c;\ a, b, -c;\ \frac{a}{4}(1\pm\sqrt{-7}),\ \frac{b}{4}(1\pm\sqrt{-7}),$

$\frac{c}{4}(1\pm\sqrt{-7}).$

23. $x=\frac{2}{\sqrt{\{(a-b+c)(a+b-c)\}}}$, &c.

24. $x=\frac{b^4+c^4-a^2(b^2+c^2)}{\sqrt{\{2a^6+2b^6+2c^6-6a^2b^2c^2\}}}$.

25. $x=\frac{a}{\sqrt{2}}\left(\frac{b}{c}+\frac{c}{b}\right)$, &c.

26. $$\frac{x}{(b+c)^2-(c+a)(a+b)}=\frac{y}{(c+a)^2-(a+b)(b+c)}$$
$$=\frac{z}{(a+b)^2-(b+c)(c+a)}=\pm\frac{1}{2\sqrt{(3abc-a^3-b^3-c^3)}}.$$

27. $a, b, c;\ \frac{1}{3}(2b+2c-a),\ \frac{1}{3}(2c+2a-b),\ \frac{1}{3}(2a+2b-c).$

28. $x=\pm\frac{1}{4abc}(bc-ca+ab)(bc+ca-ab)$, &c.

29. $a, 0, 0;\ 0, b, 0;\ 0, 0, c;\ \frac{(ca-b^2)(ab-c^2)}{3abc-a^3-b^3-c^3}.$

30. $0, 0, 0;\ -\frac{3}{2}a, \frac{a}{2}, \frac{a}{2};\ \frac{b}{2}, -\frac{3}{2}b, \frac{b}{2};\ \frac{c}{2}, \frac{c}{2}, -\frac{3}{2}c;$

$\frac{1}{2}(-a+b+c),\ \frac{1}{2}(a-b+c),\ \frac{1}{2}(a+b-c).$

32. $$\frac{a^2x}{b-c}=\frac{b^2y}{c-a}=\frac{c^2z}{a-b}=\sqrt{(-abc)},$$
$$\frac{ax}{b-c}=\frac{by}{c-a}=\frac{cz}{a-b}=\sqrt{\frac{-abc}{bc+ca+ab}}.$$

EXAMPLES XV.

1. 20, 30. **2.** *A* £10, *B* £15, *C* £25. **3.** 20 years ago.

4. 2*s*. **5.** 5, 15, 30. **6.** 5 days.

7. 1800. **8.** 58. **9.** 30 miles.

10. 120 lbs. **11.** 4 days. **12.** 36, 9, 12, 15.

13. 48 miles. **14.** 15 miles. **15.** 54, 81, 108.

17. A £450, B £225, C £237. 10*s*., D £87. 10*s*.

18. ± 5. **19.** 38, 83. **20.** 18 miles.

21. At 1 o'clock, 15 miles from Cambridge. **22.** A £10, B £5, C £1000.

23. 25. **24.** 9, 7; $8\sqrt{2}$, $\sqrt{2}$. **25.** 50 miles.

26. 576. **28.** 3 miles an hour. **29.** 3 hours.

30. 253. **31.** 2 gals. from the first, and 12 gals. from the second.

32. 15 minutes past 10. **33.** 9 o'clock, 30 miles from Cambridge.

34. 45 and $22\frac{1}{2}$ miles an hour. **35.** £3.

36. 450 miles. **37.** 30 miles.

EXAMPLES XVI.

3. $a+b+c+abc=0$. **4.** $(b+c-a)(c+a-b)(a+b-c)=8$.

6. $\dfrac{c^2}{(b-d)^2}-\dfrac{a^2}{(b+d)^2}=1$. **7.** $a^3+b^3+c^3-3abc=d^3$.

8. $a^3+2c^3-3ab^2=0$. **15.** $l^2+m^2+n^2-lmn-4=0$.

16. $al^2+bm^2+cn^2+lmn=4abc$. **17.** $a^2+b^2+c^2-abc-4=0$.

18. $a^3+b^3+c^3-5abc=0$, $y^3z^3+z^3x^3+x^3y^3+x^2y^2z^2=0$.

19. $\Sigma b^3c^3=5a^2b^2c^2$.

20. $a^3+b^3+c^3-bc(b+c)-ca(c+a)-ab(a+b)=0$.

EXAMPLES XVII.

1. $a^{\frac{1}{6}}b^{\frac{1}{6}}$. **2.** 1. **3.** $\dfrac{a^{\frac{3}{4}}c^{\frac{1}{2}}}{b^{\frac{1}{3}}}$. **4.** 1.

5. $x-y$. **6.** x^4+1+x^{-4}. **7.** $x+y+z-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$.

8. $x^{\frac{2}{3}}-x^{-\frac{2}{3}}$. **9.** $a^{\frac{4}{10}}+a^{\frac{3}{10}}x^{\frac{1}{5}}+a^{\frac{2}{10}}x^{\frac{2}{5}}+a^{\frac{1}{10}}x^{\frac{3}{5}}+x^{\frac{4}{5}}$. **10.** $x+y$.

12. $4x^2+3x+2-3x^{-1}$. **13.** $x^{\frac{2}{3}}+2x^{\frac{1}{2}}+1+2x^{-\frac{1}{2}}+x^{-\frac{2}{3}}$.

14. $a^{\frac{2}{3}}x^{-\frac{2}{3}}+a^{\frac{1}{3}}x^{-\frac{1}{3}}+a^{-\frac{1}{3}}x^{\frac{1}{3}}+a^{-\frac{2}{3}}x^{\frac{2}{3}}$.

15. $x^2y^{-\frac{12}{5}}-x^{\frac{4}{3}}y^{-\frac{8}{5}}+x^{\frac{2}{3}}y^{-\frac{4}{5}}-1+x^{-\frac{2}{3}}y^{\frac{4}{5}}-x^{-\frac{4}{3}}y^{\frac{8}{5}}+x^{-2}y^{\frac{12}{5}}$.

20. (i) $a^{\frac{5}{2}}-a^2b^{\frac{4}{3}}+a^{\frac{3}{2}}b^{\frac{8}{3}}-ab^4+a^{\frac{1}{2}}b^{\frac{16}{3}}-b^{\frac{20}{3}}$.

(ii) $a^{\frac{10}{3}}x^{\frac{25}{6}}-a^{\frac{8}{3}}x^{\frac{10}{3}}y^{\frac{1}{2}}+a^2x^{\frac{5}{2}}y-a^{\frac{4}{3}}x^{\frac{5}{3}}y^{\frac{3}{2}}+a^{\frac{2}{3}}x^{\frac{5}{6}}y^2-y^{\frac{5}{2}}$.

(iii) $a^2+b^2x^{\frac{2}{3}}+c^2x^{\frac{4}{3}}-bcx-cax^{\frac{2}{3}}-abx^{\frac{1}{3}}$.

(iv) $\left\{x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}-y^{\frac{1}{3}}z^{\frac{1}{3}}-z^{\frac{1}{3}}x^{\frac{1}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}\right\}\left\{(x+y+z)^2\right.$
$\left.+3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}(x+y+z)\right\}+9x^{\frac{2}{3}}y^{\frac{2}{3}}z^{\frac{2}{3}}$.

EXAMPLES XVIII.

1. $2-\sqrt{3}$. **2.** $5-\sqrt{15}$. **3.** $\frac{4}{5}\sqrt{2}$. **4.** 52.

5. 0. **6.** $14\frac{1}{2}$. **7.** $\dfrac{2\sqrt{3}+3\sqrt{2}-\sqrt{30}}{12}$. **8.** $\dfrac{\sqrt{30}+2\sqrt{3}-3\sqrt{2}}{12}$.

9. $\frac{1}{2}(\sqrt{21}+\sqrt{10}-\sqrt{14}-\sqrt{15})$. **10.** $\frac{1}{10}(\sqrt{6}+\sqrt{10}-\sqrt{21}-\sqrt{35})$.

11. $\dfrac{4\sqrt[3]{4}+2\sqrt[3]{2}+4}{3}$. **12.** $3\sqrt[3]{3}+1$. **13.** $\sqrt[3]{2}-1$.

14. $\dfrac{\sqrt[3]{12}-\sqrt[3]{4}}{4}$. **15.** $7-2\sqrt{13}$. **16.** $5-\sqrt{3}$.

17. $1+\sqrt{5}+\sqrt{7}$. **18.** $\sqrt{3}-1$. **19.** $2-\sqrt{3}$. **20.** 1.

21. $\dfrac{15+\sqrt{10}}{5}$. **22.** $\sqrt{2}+\sqrt{6}-\frac{4}{3}\sqrt{3}$. **23.** $\sqrt{\dfrac{2}{3}}$.

24. $1+\sqrt{2}+\sqrt{3}$. **25.** $\sqrt{3}+\sqrt{2}+\sqrt{6}$. **26.** $2+\sqrt{2}-\sqrt{5}-\sqrt{6}$.

EXAMPLES XIX.

1. $2x^5-3y^3$. **2.** $x^4-3x^2y^6$. **3.** $a-2b-3c$.

4. $5a^2-3b^2-2c^2$. **5.** x^3+x^2+x+1. **6.** $2x^2-2xy^2-y^4$.

7. $7+8x^2+5x^3$. **8.** $x^2-x+2-x^{-1}+x^{-2}$. **9.** $\dfrac{5x}{y}-2-\dfrac{y}{5x}$.

10. $x^{\frac{5}{6}} - 2x^{\frac{1}{2}} - x^{\frac{1}{6}}$. **11.** $x^{\frac{1}{3}} - 2x^{\frac{1}{2}} + x^{\frac{5}{6}}$. **12.** $a^{-\frac{3}{5}} x^{\frac{7}{5}} - x^{\frac{4}{5}} - a^{\frac{4}{5}}$.

13. $x - 8$. **14.** $x^2 - xy + y^2$. **15.** $1 - 3x^2 + 2x^4$.

16. $2(bc + ca + ab)$. **17.** $x^2 - x(y + z) - yz$. **18.** $a^2 + b^2$.

21. $A = 20$, $B = 68$, $C = -44$; or $A = 52$, $B = -68$, $C = 76$.

23. $af = gh$, $bg = hf$, and $ch = fg$.

EXAMPLES XX.

5. $x = 3$. **6.** $a : b : c = 2 : 3 : 4$.

17. (i) $\frac{3}{5}$, 1. (ii) 0, 1. (iii) ∞, 0.

EXAMPLES XXI.

2. 2, 4, 6, 8; -2, -4, -6, -8. **4.** 2, 4, 8.

14. 6, ± 12, 24, &c. **17.** 3, 9, 15.

EXAMPLES XXII.

1. 31. **2.** $(r-1)(r-1)$......, 1000.... **4.** 1 lb., 2 lb., 1024 lb.

7. 46. **8.** 6. **13.** 502 or 361. **15.** 288, 289 or 290.

16. 2775 or 2525. **17.** 135. **18.** $a = 8$, $b = 0$, $c = 6$.

19. 7. **20.** 1089. **23.** 142857, 285714. **24.** 166, 199.

EXAMPLES XXIII.

1. $\lfloor 20 / \{\lfloor 4\}^5$. **2.** 185. **3.** 3^{12}; $\lfloor 12 / (\lfloor 4)^3$.

5. 260. **7.** $\frac{1}{2} n(n-1)$. **8.** $\frac{1}{2} n(n-1) - \frac{1}{2} m(m-1) + 1$.

9. $\frac{1}{6} \{n(n-1)(n-2) - m(m-1)(m-2)\}$.

11. $\frac{1}{8}n(n-1)(n-2)(n-3)$. **12.** $3m^2$. **13.** $\dfrac{\underline{|m+n-2}}{\underline{|m-1}\,\underline{|n-1}}$.

14. $\frac{1}{6}n(n-4)(n-5)$. **17.** $\dfrac{\underline{|p+1}}{\underline{|n}\,\underline{|p-n+1}}$. **20.** $\dfrac{\underline{|mn}}{\underline{|m}\,(\underline{|n})^m}$.

23. $2(mn+m+n-1)$.

24. $2\Sigma a+2\Sigma ab-2(n-1)$, where n is the number of given diameters.

EXAMPLES XXIV.

1. $x^5+5ax^4+10a^2x^3+10a^3x^2+5a^4x+a^5$.

2. $32a^5-80a^4x+80a^3x^2-40a^2x^3+10ax^4-x^5$.

3. $1-6x^2+15x^4-20x^6+15x^8-6x^{10}+x^{12}$.

4. $16a^4-96a^5+216a^6-216a^7+81a^8$.

5. $16x^8-96x^6+216x^4-216x^2+81$.

6. $x^{10}-10x^8y^3+40x^6y^6-80x^4y^9+80x^2y^{12}-32y^{15}$.

7. $405x^8y^2$. **8.** $\dfrac{\underline{|20}}{\underline{|16}\,\underline{|4}}3^{16}4^4x^{16}$. **9.** $924x^{30}$.

10. $-\dfrac{\underline{|42}}{\underline{|3}\,\underline{|39}}x^3y^{39}$. **11.** $70x^4$. **12.** $\dfrac{\underline{|21}}{\underline{|10}\,\underline{|11}}x^{10}$ and $\dfrac{\underline{|21}}{\underline{|11}\,\underline{|10}}x^{11}$.

13. $(-1)^r\dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}3^rx^{n-r}y^r$. **14.** $\dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}x^{2n-2r}y^{3r}$.

15. $(3x)^{15}-30(3x)^{14}y+420(3x)^{13}y^2\ldots-945x^2(2y)^{13}+45x(2y)^{14}-(2y)^{15}$.

16. $924x^6$. **17.** $6435x^7$, $6435x^8$.

22. $(-1)^n\,\underline{|2n}\,/\,\underline{|n}\,\underline{|n}$. **23.** 7 or 14. **24.** 7.

EXAMPLES XXVI.

1. Convergent. **2.** Convergent. **3.** Convergent.

4. Convergent. **5.** Convergent. **6.** Divergent.

7. Divergent. **8.** Convergent if $x>1$; Divergent if $x \not> 1$.

9. Divergent. **10.** Convergent if $x>1$; Divergent if $x \not> 1$.

11. Convergent if $x \not> 1$; Divergent if $x > 1$.

12. Convergent if $x \not> 1$; Divergent if $x > 1$. **13.** Divergent.

14. Divergent. **15.** Convergent if $m < 1$; Divergent if $m \not< 1$.

16. Convergent if $m < 1$; Divergent if $m \not< 1$. **17.** Divergent.

18. Divergent. **19.** Divergent. **20.** Divergent.

21. Divergent. **22.** Convergent if $x < 1$, Divergent if $x > 1$. If $x = 1$, then Convergent if $k > 1$ and Divergent if $k \not> 1$.

23. Convergent if $x < 1$; Divergent if $x \not< 1$.

24. Convergent if $x \not> 1$; Divergent if $x > 1$.

26. Convergent if $x < 1$, Divergent if $x > 1$. If $x = 1$, then Convergent if $m < \frac{1}{2}$ and Divergent if $m \not< \frac{1}{2}$.

27. Convergent if $x < 1$, Divergent if $x > 1$. If $x = 1$, then Convergent if $k < \frac{1}{2}$ and Divergent if $k \not< \frac{1}{2}$.

EXAMPLES XXVII.

1. (i) $(r+1)x^r$. (ii) $\frac{1}{2}(r+1)(r+2)x^r$. (iii) $\dfrac{(r+1)(r+2)\dots(r+n-1)}{\underline{|n-1}}x^r$.

(iv) $(-1)^r \dfrac{2.5.8\dots(3r-1)}{3.6.9\dots3r}x^r$. (v) $(-1)^{r-1}\dfrac{2.1.4\dots(3r-5)}{3.6.9\dots3r}x^r$.

(vi) $(-1)^r \dfrac{5.2.1.4.7\dots(3r-8)}{3.6.9\dots3r}x^r$. (vii) $\dfrac{3.8.13\dots(5r-2)}{1.2.3\dots r}x^r$.

(viii) $-\dfrac{2.3.8.13\dots(5r-7)}{1.2.3\dots r}x^r$. (ix) $\dfrac{q(q+p)(q+2p)\dots(q+\overline{r-1}.p)}{p^r\,\underline{|r}}x^r$.

(x) $(2a)^{-\frac{3}{5}}\dfrac{2.7.12\dots(5r-3)}{\underline{|r}}\left(\dfrac{3x}{10a}\right)^r$.

(xi) $-\dfrac{5.3.1.1.3.5\dots(2r-7)}{\underline{|r}}x^r a^{5-r}$, $r > 3$.

(xii) $-\dfrac{2.5.12\dots(7r-9)}{4.8.12\dots4r}x^r 4^{\frac{2}{7}}$.

2. (i) The ninth term, (ii) the eighth term.

3. The 39th term. **4.** The first and second terms.

5. After the 12th term. **8.** (i) $\dfrac{1.4.7...(3r-2)}{3.6.9...3r} a^{\frac{1}{3}-2r} x^{2r}$.

(ii) $2a^{-r}x^r$. (iii) $4ra^{-r}x^r$. (iv) $\dfrac{1.3.5...(n-1)}{2.4.6...n} a^{-n}x^n$ when n is even, $\dfrac{1.3.5...(n-2)}{2.4.6...(n-1)} a^{-n}x^n$ when n is odd.

(v) $(2r^2+2r+1)\,a^{-r-1}x^r$. (vi) $(-1)^r 16\,(r-1)\,a^{2-r}x^r$.

EXAMPLES XXVIII.

1. (i) 2. (ii) $\sqrt{\frac{2}{3}}$. (iii) $4\sqrt[3]{4}$. (iv) $\sqrt{27}-2$.

(v) 1. (vi) $\sqrt[3]{4}$. (vii) $\sqrt{\frac{8}{27}}$. (viii) $\frac{1}{2}$.

(ix) $\sqrt[3]{\frac{9}{4}}$. (x) $\frac{10}{9}-\frac{1}{3}\sqrt{11}$. (xi) $\frac{1}{24}$. (xii) $\frac{37}{245}$.

11. $(-1)^{r-1}\dfrac{(n-2)(n-3)...(n-r)}{\lfloor r-1}$. **23.** 1. **25.** 141.

31. $-245/8$. **32.** 246,792. **33.** 462. **34.** 35.

35. Coefficient of x^{3r} is $3^{3r}2^{-3r-2}a^{-3r-2}$, of x^{3r+1} is $-3^{3r+1}2^{-3r-3}a^{-3r-3}$ and of x^{3r+2} is 0.

38. $\frac{1}{30}(n+1)(n+2)(n+3)(n+4)(n+5)$.

39. $2^{n+r}\lfloor 3n+r-1 / \lfloor r \lfloor 3n-1$.

EXAMPLES XXIX.

1. $\dfrac{18}{5(x+6)}-\dfrac{3}{5(x+1)}$. **2.** $\dfrac{4}{x-3}-\dfrac{3}{x-2}$.

3. $\dfrac{3}{4(x+3)}-\dfrac{5}{8(x+5)}-\dfrac{1}{8(x+1)}$.

4. $\dfrac{1}{x}-\dfrac{2}{(x+1)^2}$. **5.** $\dfrac{2}{(x-2)^2}-\dfrac{1}{x-2}+\dfrac{1}{x+1}$.

6. $\dfrac{1}{12(x+1)}-\dfrac{7}{3(x-2)}+\dfrac{13}{4(x-3)}$. **7.** $\dfrac{1}{5(x+2)}+\dfrac{4x-8}{5(x^2+1)}$.

8. $\dfrac{1}{1-10x}+\dfrac{1}{3(1+3x)}-\dfrac{1}{3(1+3x)^2}$. **9.** $\dfrac{1}{2(x^2+1)}+\dfrac{1}{2(x-1)^2}$.

10. $\dfrac{3}{2(1-3x)^3}+\dfrac{21}{8(1-3x)^2}+\dfrac{21}{32(1-3x)}+\dfrac{7}{32(1+x)}$.

11. $\dfrac{1}{x^2+1}+\dfrac{1}{x-2}-\dfrac{1}{x+3}$. **12.** $\dfrac{2}{(x-2)^2}+\dfrac{4}{5(x-2)}+\dfrac{x+2}{5(x^2+1)}$.

13. $\dfrac{2}{5(x-1)^2}+\dfrac{11}{25(x-1)}-\dfrac{11x-4}{25(x^2+4)}$.

14. $\dfrac{3}{5(x-2)}-\dfrac{1}{2(x-1)^2}-\dfrac{1}{2(x-1)}-\dfrac{x+2}{10(x^2+1)}$.

15. $\dfrac{1}{8x^2}-\dfrac{1}{16x}-\dfrac{1}{x+1}+\dfrac{17}{16(x+2)}+\dfrac{1}{(x+2)^2}+\dfrac{3}{4(x+2)^3}$.

16. $\dfrac{1}{4(x+2)^3}+\dfrac{1}{6(x+2)^2}+\dfrac{11}{144(x+2)}+\dfrac{1}{9(x-1)}-\dfrac{1}{8x^2}-\dfrac{3}{16x}$.

17. $(-1)^n\{2^{-n}-3^{-n-1}\}$. **18.** $\dfrac{4}{9}\left(-\dfrac{1}{2}\right)^{n+1}-\dfrac{1}{9}(3n+7)$.

20. $\dfrac{7}{8}(3^n-1)-\dfrac{5}{8}\{(-1)^n-1\}$. **21.** $\dfrac{1}{24}\{9+5^{n+2}-2.3^{n+2}-2^{n+4}\}$.

22. $(n^2+7n+8)\,2^{n-3}$; $\dfrac{1}{3}(n^3+9n^2+14n)\,2^{n-4}$.

EXAMPLES XXXII.

1. 1·262. **2.** 1·48169. **3.** £1146·74.

5. £742. 19*s.* 6*d.* **7.** £785. 10*s.*

8. £1979. 5*s.* 6*d.* **9.** £1735 nearly. **10.** £122·58.

EXAMPLES XXXIII.

1. $\dfrac{1}{12}\{(3n+1)(3n+4)(3n+7)(3n+10)-1.4.7.10\}$.

2. $\dfrac{1}{8}\left\{\dfrac{1}{3.7}-\dfrac{1}{(4n+3)(4n+7)}\right\}$; $S_\infty=\dfrac{1}{168}$.

3. $\dfrac{1}{12}n(n+1)(3n^2+23n+46)$. **4.** $\dfrac{4}{3}n(n+1)(n+2)-3n$.

5. $\frac{1}{2}n(n+1)^2(n+2)$. **6.** $\frac{1}{12}n(n+1)(n+2)(3n+5)$.

7. $\frac{1}{24}(2n-1)(2n+1)(2n+3)(6n+7)+\frac{7}{8}$.

8. $\frac{11}{180}-\frac{6n+11}{12(2n+1)(2n+3)(2n+5)}$; $S_\infty=\frac{11}{180}$.

9. $\frac{5}{36}-\frac{3n+5}{6(n+1)(n+2)(n+3)}$; $S_\infty=\frac{5}{36}$.

10. $\frac{5}{4}-\frac{2n+5}{2(n+1)(n+2)}$; $S_\infty=\frac{5}{4}$.

11. $\frac{1}{8}-\frac{4n+3}{8(2n+1)(2n+3)}$; $S_\infty=\frac{1}{8}$.

12. $\frac{29}{36}-\frac{6n^2+27n+29}{6(n+1)(n+2)(n+3)}$; $S_\infty=\frac{29}{36}$.

13. $S_n=\frac{2n}{n+1}$, $S_\infty=2$. **14.** $\frac{1}{36}(n+1)(n+2)(4n+3)-\frac{1}{6}$.

15. $\frac{1}{120}n(n+1)(n+2)(8n^2+11n+1)$.

16. $na^2+n(n-1)ab+\frac{1}{6}(n-1)n(2n-1)b^2$.

17. $na^3+\frac{3}{2}n(n-1)a^2b+\frac{1}{2}(n-1)n(2n-1)ab^2+\frac{1}{4}n^2(n-1)^2b^3$.

18. $\frac{1}{3}n(4n^2-1)$. **19.** $\frac{1}{3}n(16n^2-12n-1)$.

23. $\frac{1}{6}n(n+1)(n+2)$. **24.** $\frac{1}{6}n(n+1)(4n-1)$.

25. $nab-\frac{1}{2}n(n-1)(a+b)+\frac{1}{6}n(n-1)(2n-1)$.

27. (i) $\frac{2^{n+1}}{n+2}-1$. (ii) $1-\frac{1}{(n+1)2^n}$.

(iii) $2-\frac{2}{n+1}\left(\frac{2}{3}\right)^n$. (iv) $\frac{5}{4}-\frac{5}{2(n+1)(n+2)}\left(\frac{5}{7}\right)^n$.

(v) $\frac{3}{2}-\frac{3}{(n+1)(n+2)}\left(\frac{3}{4}\right)^n$. (vi) $3-\frac{6}{(n+1)(n+2)}\left(\frac{6}{7}\right)^n$.

EXAMPLES XXXIV.

1. (i) $\dfrac{4\,.\,7\,.\,10\ldots(3n+4)}{2\,.\,5\,.\,8\ldots(3n+2)}-2.$ (ii) $\dfrac{2\,.\,5\,.\,8\ldots(3n+2)}{4\,.\,7\,.\,10\ldots(3n+1)}-2.$

(iii) $1-\dfrac{5\,.\,7\ldots(2n+3)}{8\,.\,10\ldots(2n+6)}$; 1. (iv) $11\left\{1-\dfrac{13\,.\,15\ldots(2n+11)}{14\,.\,16\ldots(2n+12)}\right\}$; 11.

2. (i) $2+3(n-1)(n-2)$; $2n+n(n-1)(n-2)$.

(ii) $7n-(n-1)(n-2)$; $\dfrac{7}{2}n(n+1)-\dfrac{1}{3}n(n-1)(n-2)$.

(iii) $2^{n+1}-n-2$; $2^{n+2}-1-\dfrac{1}{2}(n+2)(n+3)$.

(iv) $2^{n+1}-n(n+1)-n$; $2^{n+2}-4-\dfrac{1}{3}n(n+1)(n+2)-\dfrac{1}{2}n(n+1)$.

(v) $\dfrac{1}{24}n(n+1)(n+2)(n+3)$; $\dfrac{1}{120}n(n+1)(n+2)(n+3)(n+4)$.

(vi) $(n-2)(n-1)n(n+1)+(n-1)n-n+2$;

$\dfrac{1}{5}(n-2)(n-1)n(n+1)(n+2)+\dfrac{1}{3}(n-1)n(n+1)-\dfrac{1}{2}n(n+1)+2n.$

3. (i) $\dfrac{2-4x}{1-4x+x^2}$. (ii) $\dfrac{1-2x}{1-5x+4x^2}$. (iii) $\dfrac{1-6x}{1-12x+32x^2}$.

(iv) $\dfrac{15+x-19x^2}{15-14x-35x^2-42x^3}$. (v) $\dfrac{1+x}{(1-x)^3}$.

4. (i) $2^{n+1}-2$; $2^{n+2}-2n-4$.

(ii) $\dfrac{1}{7}\{3^n+11(-4)^{n-1}\}$; $\dfrac{1}{10}+\dfrac{3^{n+1}}{14}-\dfrac{11}{35}(-4)^n$.

(iii) $\dfrac{1}{4}\{3^n-(-1)^n\}$; $\dfrac{1}{8}\{3^{n+1}-3\}$ when n is even, and $\dfrac{1}{8}\{3^{n+1}-1\}$ when n is odd.

5. $\dfrac{1}{2^n}\{(1+\sqrt{5})^n+(1-\sqrt{5})^n\}$. **6** $a=1,\ b=4,\ c=1,\ d=0.$

8. $\dfrac{2-3x-x^2}{(1-x)^2(1-2x)}$. **9.** $\dfrac{1}{(1-x)^2(1-2x)}$.

10. $\dfrac{(x+a)(x+b)\ldots(x+l)}{abc\ldots l}$.

22. $\dfrac{n^2+n}{4(n+2)}$. **23.** $\dfrac{1}{2}\left(x+\dfrac{1}{x}\right)\log(1+x)+\dfrac{x}{4}-\dfrac{1}{2}$.

24. $1-\left(1-\frac{1}{x}\right)\log(1-x)$. **25.** $\frac{6+36x+6x^2}{(1-x)^4}$. **30.** $x<1$.

32. (i) Divergent. (ii) Divergent. (iii) Convergent. (iv) Convergent. (v) Convergent if $\gamma>\alpha+\beta$, Divergent if $\gamma<\alpha+\beta$.

EXAMPLES XXXVI.

7. $\frac{n+1}{n+2}$.

10. $b_{n-1}p^2_n-(b_na_n+b^2_nb_{n-1})p^2_{n-1}-a_nb_{n-1}(b_nb_{n-1}+a_n)p^2_{n-2}$
$+a_nb_na^2_{n-1}p^2_{n-3}=0$.

EXAMPLES XXXVII.

1. (i) $4+\frac{1}{8+}\dots\dots$ (ii) $11+\frac{1}{1+}\frac{1}{4+}\frac{1}{1+}\frac{1}{22+}\dots\dots$

(iii) $5+\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{10+}\dots\dots$

(iv) $6+\frac{1}{1+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{5+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{1+}\frac{1}{12+}\dots\dots$

8. (i) $\sqrt{\frac{5}{3}}$. (ii) $\frac{1}{7}(4+\sqrt{37})$. (iii) $\frac{1}{52}(28-\sqrt{30})$.

16. $\frac{1}{2}(n^2+3n)$. **26.** $e^{-\frac{1}{2}}$.

EXAMPLES XXXVIII.

6. 733. **10.** 3, 7, 9, 11, 13, 19. **13.** $504n-6$.

EXAMPLES XL.

1. (i) $x=2,\ y=3$. (ii) $x=1,\ y=10;\ x=14,\ y=2$.

(iii) $x=4,\ y=8;\ x=13,\ y=1$.

(iv) 696, 3; 625, 18; 554, 33; ......; 57, 138.

2. 22, 30.

3. (i) $x=4+13m,\ y=1+7m$. (ii) $x=9+11m,\ y=7+9m$.
(iii) $x=15m-7,\ y=17m-10$. (iv) $x=64+69m,\ y=44+49m$.

4. (i) 3, 1, 2; 5, 2, 1; 2, 4, 1.
(ii) 1, 21, 1; 5, 14, 1; 9, 7, 1; 3, 13, 2; 7, 6, 2; 5, 5, 3; 3, 4, 4; 1, 12, 3; 1, 3, 5.
(iii) 2, 8, 3. (iv) 8, 38, 50; 19, 44, 35; 30, 50, 20; 41, 56, 5.

5. (i) 1325, 2; 441, 3; 101, 8; 77, 10; 33, 21; 25, 27; 5, 112; 1, 333.
(ii) 5, 3. (iii) 8, 5. (iv) 6, 1; 13, 14.

7. 195, 121; 52, 264.

8. 3. **9.** 20. **10.** 3. **11.** £3. 14*s*. 6*d*., £4. 5*s*. 6*d*.

12. 2*s*. 7*d*., 2*s*. 10*d*., 2*s*. 11*d*., 3*s*. 1*d*., 3*s*. 2*d*., 3*s*. 3*d*., 3*s*. 4*d*., 3*s*. 5*d*., 3*s*. 6*d*., 3*s*. 8*d*., 3*s*. 9*d*. and 4*s*.

13. 11, 12, 15, 24, 36. **14.** 15, 55; 25, 65; 35, 75. **15.** 21.

EXAMPLES XLII.

1. $\frac{36}{67}, \frac{31}{67}$. **2.** $\frac{11664}{33397}, \frac{11124}{33397}, \frac{10609}{33397}$. **4.** 8.

8. $3n^2+5n+2$ pence. **15.** $\frac{1}{4}$. **16.** $\frac{1}{2}$. **21.** $\frac{7}{9}$.

EXAMPLES XLIII.

1. (i) q. (ii) $27q$. (iii) $-2p$. (iv) $-3q$. (v) $2p^2$. (vi) $3q$.
(vii) $-2p^2$. (viii) $3p^2$. (ix) $-p^3$. (x) p/q. (xi) $p/2q$.
(xii) $-p^2/(14q^2+p^3$. **2.** (i) 0. (ii) $-3p$. (iii) $-4q$.

3. (i) $3p^3-16pq+64r$. (ii) $(q^3-4pqr+8r^2)/r^3$. (iii) $(q^3-p^2r)/r^4$.

4. (i) 28, -24. (ii) 44, -168.

5. (i) $p_1^2-2p_2$. (ii) $3p_1p_2-p_1^3-3p_3$. (iii) $(p_{n-1}^2-2p_{n-2}p_n)/p_n^2$.
(iv) $p_1+p_{n-1}(2p_2-p_1^2)/p_n$. (v) $(p_1^3+3p_3-3p_1p_2)\,p_{n-1}/p_n+2p_2-p_1^2$.
(vi) $(3p_1p_2-p_1^3-3p_3)(p_{n-1}^2-2p_{n-2}p_n)/p_n^2+p_1$.

6. $x^3-10x^2+31x-31=0.$ **7.** 6. **8.** (i) $x^3-qx^2+prx-r^2=0.$
(ii) $x^3+2px^2+(p^2+q)y-r+pq=0.$
(iii) $x^3(r-pq)+x^2(3r-2pq+p^3)+x(3r-pq)+r=0.$
(iv) $x^3-2qx^2+(q^2+pr)x+r^2-pqr=0.$
(v) Eliminate x between given equation and $y=(p+x)^2+\frac{2r}{x}.$
(vi) $y^3-(3q-p^2)y^2+(3q^2-qp^2)y+rp^3-q^3=0.$
9. (i) Substitute $-(y+p)$ for x.
(ii) Substitute $-\frac{1}{3}(y+p)$ for x.
(iii) Substitute $\frac{1}{2}(p^2-2q-y)$ for x^2.
10. $(y+r)^3+q^3y+p^3y^2-3pqy(y+r)=0.$

EXAMPLES XLIV.

1. $2\pm\sqrt{3},\ -3\pm\sqrt{2}.$ **2.** $\frac{5}{3},\ 3\pm\sqrt{-5}.$
3. $\frac{4}{3},\ \pm\sqrt{2}\pm\sqrt{5}.$ **4.** $\pm\sqrt{2}\pm\sqrt{-1},\ \frac{3}{4}(1\pm\sqrt{-7}).$
5. $x^4-16x^2+4=0.$ **6.** $x^4+2x^2+25=0.$
8. $1\pm\sqrt{-1},\ 1\pm2\sqrt{-1}.$ **9.** $2q^3-9pqr+27r^2=0.$
10. $p^3-4pq+8r=0$ and $(p^2+4q)(36q-11p^2)-1600s.=0.$
11. 5, 1, −3. **12.** 4, 1, −2, −5.
13. (i) +4, −4. (ii) 3. (iii) $\frac{2}{3}$. **14.** 8, $\pm\frac{1}{2}$.
15. $\pm\sqrt{2},\ -2\pm\sqrt{7}.$ **16.** $r^2-pqr+p^2s=0.$
17. 3, 7, −10. **18.** 3, 9, $-\frac{4}{3}$.
20. (i) $r^2-p^2s=0.$ (ii) $p^2s+q^2=4qs.$

EXAMPLES XLV.

1. (i) $-\frac{1}{2},\ -\frac{1}{2},\ 4.$ (ii) $3,\ 3,\ \pm2\sqrt{-1}.$ (iii) $\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ -\frac{3}{2}.$
(iv) $4,\ 4,\ 4,\ -\frac{1}{2}.$ **2.** $(bc-ad)^2=4(b^2-ac)(c^2-bd).$
9. (i) $y^3-6qy^2+4(4pr-s)y-8(2p^2s-3qs+2r^2).$
(ii) $(y-6q)^3+6q(y-6q)^2+4(4pr-s)(y-6q)+8(2p^2s-3qs+2r^2)=0.$
10. $(y-16p^2+24q)^3-24q(y-16p^2+24q)^2$
$+64(4pr-s)(y-16p^2+24q)-512(2p^2s-3qs+2r^2)=0.$

EXAMPLES XLVI.

1. (i) $-5,\ -\omega-4\omega^2,\ -\omega^2-4\omega$, *i.e.* $-5,\ \frac{5}{2}\pm\frac{3}{2}\sqrt{-3}$.

(ii) $-4,\ -\omega-3\omega^2,\ -\omega^2-3\omega$. (iii) $10,\ 2\omega+8\omega^2,\ 2\omega^2+8\omega$.

(iv) $8,\ \omega+7\omega^2,\ \omega^2+7\omega$.

(v) $-2{\cdot}094\ldots,\ -1{\cdot}703\ldots\omega-{\cdot}391\ldots\omega^2,\ -1{\cdot}703\ldots\omega^2-{\cdot}391\ldots\omega$.

(vi) $3{\cdot}0913,\ 2{\cdot}1699\omega+{\cdot}9214\omega^2,\ 2{\cdot}1699\omega^2+{\cdot}9214\omega$.

2. (i) $-1,\ -3,\ 1\pm 2i$. (ii) $1\pm\sqrt{2},\ -1\pm\sqrt{-5}$.

(iii) $3\pm\sqrt{-5},\ -3\pm\sqrt{-1}$. (iv) $-2,\ \frac{3}{2},\ \frac{1}{4}(-1\pm\sqrt{-15})$.

3. (i) One real root between -3 and -4.

(ii) One between -7 and -6, one between 1 and 2, and one between 5 and 6.

(iii) One between 0 and 1, and one between 1 and 2.

(iv) One between 2 and 3, one between 0 and -1, and one between -4 and -5.

(v) One between 2 and 3, and one between -3 and -4.

8. (i) 1·3570, 1·6139. (ii) 4·1891. (iii) ·4679, 1·6527, 3·8793.

(iv) 2·2318. (v) 2·1624, 2·4142. (vi) 1·1487.

9. (i) $3,\ 3,\ -4,\ -4$. (ii) $3,\ 3,\ -3\pm\sqrt{8}$.

11. $1,\ 3,\ -\frac{1}{2},\ \pm\sqrt{-2}$.